AN ESTIMATE OF THE DOUBLE GAMMA FUNCTION

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CRISTINEL MORTICI AND SORINEL DUMITRESCU

Abstract. The object of the present paper is to establish some bounds for the double gamma function.

1. Introduction

The double gamma function $G$, or the $G$-function satisfies

$$\ln G(x+1) = \left(-\frac{1}{2} + \ln \sqrt{2\pi}\right) x - \frac{\gamma + 1}{2} x^2 + S(x)$$

for $x > 0$ where

$$S(x) = \sum_{k=1}^{\infty} \left[k \ln \left(1 + \frac{x}{k}\right) - x + \frac{x^2}{2k}\right].$$

See, e.g., [3]. The $G$-function is closely related to the Euler gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-tx} dt, \quad x > 0,$$

since $G(1) = 1$ and $G(x+1) = \Gamma(x) G(x)$, for $x > 0$ and $G(n+2) = 1!2! \cdots n!$, for all positive integers $n$. The double gamma function is also called the Barnes $G$-function since it was introduced by Barnes [1, 3]. Batir [4, Theorem 2.2] estimated $S(x)$ from (1.2) via some convexity arguments and obtained some double inequalities for the $G$-function.

The aim of this note is to give a different method for estimating $S(x)$ and consequently to establish the error estimate made in the approximation formula

$$\ln G(x+1) \approx \left(-\frac{1}{2} + \ln \sqrt{2\pi}\right) x - \frac{\gamma + 1}{2} x^2 + S_n(x),$$

where

$$S_n(x) = \sum_{k=1}^{n} \left[k \ln \left(1 + \frac{x}{k}\right) - x + \frac{x^2}{2k}\right].$$

Precisely, we give the following

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Theorem 1.1. Let
\[
\varepsilon_n (x) = \ln G (x + 1) - \left\{ \left( -\frac{1}{2} + \ln \sqrt{2\pi} \right) x - \frac{\gamma + 1}{2} x^2 + S_n (x) \right\}.
\]
Then for every \( x > \sqrt[3]{3} \), there exists a positive integer \( n (x) \) such that
\[
\frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} \leq \varepsilon_n (x) \leq \frac{x^3}{3(n + 1) + \frac{x^{12}(3x+4)}{216}},
\]
(2.1)
then for every \( x > \sqrt[3]{3} \), there exists a positive integer \( n (x) \) such that
\[
\frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} \leq \varepsilon_n (x) \leq \frac{x^3}{3(n + 1) + \frac{x^{12}(3x+4)}{216}},
\]
(2.1)
the right-hand side inequality holds for all \( x > 0 \) and integers \( n \geq 1 \).

2. The Proofs

We first give the following

Lemma 2.1. For every \( x > \sqrt[3]{3} \), there exists a positive integer \( n (x) \) such that for all \( n \geq n (x) \), it holds
\[
\frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} \leq \varepsilon_n (x) \leq \frac{x^3}{3(n + 1) + \frac{x^{12}(3x+4)}{216}},
\]
(2.1)
the right-hand side inequality holds for all \( x > 0 \) and integers \( n \geq 1 \).

Proof. Let
\[
f (t) = (t + 1) \ln \left( 1 + \frac{x}{t + 1} \right) - x + \frac{x^2}{2(t + 1)} - \left( \frac{x^3}{3t} - \frac{x^3}{3(t + 1)} \right),
\]
with
\[
f'' (t) = -\frac{x^3 \left( 10t + 4x + 16tx + 20t^2 + 12t^3 + 2x^2 + 6tx^2 + 24t^2 x + 9t^3 x + 6t^2 x^2 + 2 \right)}{3t^3 (t + 1)^3 (t + x + 1)^2}.
\]
\[
< 0.
\]
Now \( f \) is strictly concave, with \( f \left( \infty \right) = 0 \), so \( f \left( t \right) < 0 \), for all \( t > 0 \). This completely justifies the right-hand side inequality \( \left( 2.1 \right) \).

Let
\[
g (t) = (t + 1) \ln \left( 1 + \frac{x}{t + 1} \right) - x + \frac{x^2}{2(t + 1)} - \left( \frac{x^3}{3t + \frac{x^{12}(3x+4)}{216}} - \frac{x^3}{3(t + 1) + \frac{x^{12}(3x+4)}{216}} \right),
\]
with
\[
g'' (t) = \frac{x^3 P \left( t \right)}{(t + x + 1)^2 (t + 1)^3 (648t + 4x^{12} + 3x^{13})^3 (648t + 4x^{12} + 3x^{13} + 648)^3},
\]
where \( P \left( t \right) = \sum_{k=0}^{6} a_k \left( x \right) t^k \), having the leading coefficient
\[
a_6 \left( x \right) = 914039610015744 \left( 3x + 4 \right) \left( x^3 + 3 \right) \left( x^3 - 3 \right) \left( x^6 + 9 \right).
\]
For \( x > \sqrt[3]{3} \), we have \( a_6 \left( x \right) > 0 \), so we can find a positive integer \( n (x) \) such that
\( P \left( t \right) > 0 \), for all \( t \geq n (x) \).

Now \( g'' \left( t \right) > 0 \), for all \( t \geq n (x) \), so \( g \) is strictly convex on \( \left[ n (x), \infty \right) \). But \( g \left( \infty \right) = 0 \), so \( g \left( t \right) > 0 \), for all \( t \geq n (x) \) and the left-hand side of \( \left( 2.1 \right) \) follows. \( \square \)
Proof of Theorem 1. Inequality (2.1) can be written as
\[ \frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} < S_{n+1} (x) - S_n (x) < \frac{x^3}{3n} - \frac{x^3}{3(n+1)}. \]

By adding these telescoping inequalities from \( n \geq n(x) \) to \( n + p - 1 \), we deduce
\[ \frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} < S_{n+p} (x) - S_n (x) < \frac{x^3}{3n} - \frac{x^3}{3(n+p)}, \]
then taking the limit as \( p \to \infty \), we get
\[ \frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} \leq S (x) - S_n (x) \leq \frac{x^3}{3n}. \]

Now the conclusion follows since \( \varepsilon_n (x) = S (x) - S_n (x) \).

\[ \blacksquare \]

3. A POWER SERIES PROOF

In this concluding section we give an alternative proof of (2.1). In fact, we show how increasingly better estimates of
\[ \phi_x (n) = (n + 1) \ln \left( 1 + \frac{x}{n+1} \right) - x + \frac{x^2}{2(n+1)} \]
can be obtained by truncation of the associated power series. As before, we assume in this section that \( x \) is arbitrary, but fixed positive number. By standard computations, or better by using a computer software for symbolic computations such as Maple, we deduce that
\[
\phi_x (n) = \frac{1}{3n^2} x^3 - \frac{1}{12n^3} x^3 (3x + 8) + \frac{1}{20n^4} x^3 (15x + 4x^2 + 20) \\
- \frac{1}{30n^5} x^3 (45x + 24x^2 + 5x^3 + 40) + \frac{1}{42n^6} x^3 (105x + 84x^2 + 35x^3 + 6x^4 + 70) \\
+ O \left( \frac{1}{n^7} \right).
\]

Evidently,
\[ \lim_{n \to \infty} n^3 \left( \phi_x (n) - \frac{1}{3n^2} x^3 \right) = - \frac{1}{12} x^3 (3x + 8) < 0, \]
so there is a positive integer \( m = m(x) \) such that
\[ \phi_x (n) < \frac{1}{3n^2} x^3, \]
for every \( n \geq m \). By similar arguments, we can state the following inequality
\[
\frac{1}{3n^2} x^3 - \frac{1}{12n^3} x^3 (3x + 8) + \frac{1}{20n^4} x^3 (15x + 4x^2 + 20) - \frac{1}{30n^5} x^3 (45x + 24x^2 + 5x^3 + 40) \\
< \phi_x (n) \\
< \frac{1}{3n^2} x^3 - \frac{1}{12n^3} x^3 (3x + 8) + \frac{1}{20n^4} x^3 (15x + 4x^2 + 20)
\]
for values of $n$ greater than an initial value $n_0$, which is a stronger inequality than (2.1). For the lower term, we have

$$
\left( \frac{x^3}{3n + \frac{x^{12}(3x+4)}{216}} - \frac{x^3}{3(n + 1) + \frac{x^{12}(3x+4)}{216}} \right)
- \left( \frac{1}{3n^2} x^3 - \frac{1}{12n^3} x^3 (3x + 8) + \frac{1}{20n^4} x^3 (15x + 4x^2 + 20) - \frac{1}{30n^5} x^3 (45x + 24x^2 + 5x^3 + 40) \right)
= - \frac{x^3 A(x)}{60n^5 (648n + 4x^{12} + 3x^{13}) (648n + 4x^{12} + 3x^{13} + 648)} < 0,
$$

where $A(x) = (77,760x^{13} + 103,680x^{12} - 6298,560x - 8,398,080) n^4 + \cdots$ is a fourth degree polynomial in $n$, with positive leading coefficient when $x \geq 2$.

For the upper term in (2.1), we have

$$
\left( \frac{1}{3n^2} x^3 - \frac{1}{12n^3} x^3 (3x + 8) + \frac{1}{20n^4} x^3 (15x + 4x^2 + 20) \right)
- \left( \frac{x^3}{3n} - \frac{x^3}{3(n + 1)} \right)
= - \frac{x^3 B(x)}{60n^4 (n + 1)} < 0,
$$

where $B(x) = (15x + 20) n^2 + (-12x^2 - 30x - 20) n - (12x^2 + 45x + 60)$.

Our assertion is now completely proved.

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References


Valahia University of Târgoviște, Department of Mathematics, Bd. Unirii 18, 130082 Târgoviște/România,

E-mail address: cristinel.mortici@hotmail.com

Ph. D. Student, University Politehnica of Bucharest, Splaiul Independenței 313, Bucharest/România,

E-mail address: sorineldumitrescu@yahoo.com