QUATERNIONIC BERTRAND CURVES IN EUCLIDEAN 4-SPACE

(COMMUNICATED BY UDAY CHAND DE)

OSMAN KEÇİLIÖĞLU AND KAZIM İLARSLAN

ABSTRACT. In this paper, by using the similar idea of Matsuda and Yorozu [12], we prove that if bitorsion of a quaternionic curve is no vanish, then there is no quaternionic curve in $E^4$ is a Bertrand curve. Then we define $(1, 3)$ type Bertrand curves for quaternionic curve in Euclidean 4-space. We give some characterizations for a $(1, 3)$ type quaternionic Bertrand curves in Euclidean 4-space by means of the curvature functions of the curve.

1. Introduction

Characterization of a regular curve is one of the important and interesting problems in the theory of curves in Euclidean space. There are two ways widely used to solve these problems: to figure out the relationship between the Frenet vectors of the curves (see [11]), and to determine the shape and size of a regular curve by using its curvatures. $k_1$ (or $\kappa$) and $k_2$ (or $\tau$), the curvature functions of a regular curve, have an effective role. For example: if $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, the curve is a circle with radius $(1/k_1)$, etc.

In 1845, Saint Venant (see [14]) proposed the question whether the principal normal of a curve is the principal normal of another’s on the surface generated by the principal normal of the given one. Bertrand answered this question in (3) published in 1850. He proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by $k_1$ and $k_2$ respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in \mathbb{R}$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves (see [11]).

There are many important papers on Bertrand curves in Euclidean space (see: 4, 8, 13).

When we investigate the properties of Bertrand curves in Euclidean n-space, it is easy to see that either $k_2$ or $k_3$ is zero which means that Bertrand curves in
\( E^n \) \((n > 3)\), are degenerate curves (see [13]). This result is restated by Matsuda and Yorozu [12]. They proved that there was not any special Bertrand curves in \( E^n \) \((n > 3)\) and defined a new kind, which is called \((1, 3)\)-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1, 7, 10]) as well as in Euclidean space.

K. Bharathi and M. Nagaraj (in [2]) studied a quaternionic curve in Euclidean 3-space \( E^3 \) and Euclidean 4-space \( E^4 \) and gave the Frenet formula for quaternionic curve. For the newest results for quaternionic curves, we refer the papers [8, 9, 16].

In this paper, by using the similar idea of Matsuda and Yorozu [12], we prove that if bitorsion of a quaternionic curve \( \alpha \) is no vanish, then there is no quaternionic curve in \( E^4 \) is a Bertrand curve. Then we define \((1, 3)\)-type quaternionic Bertrand curves for quaternionic curves in Euclidean 4-space. We give some characterizations for a \((1, 3)\)-type quaternionic Bertrand curves in Euclidean 4-space by means of the curvature functions of the curve.

2. Preliminaries

Let \( Q_H \) denote a four dimensional vector space over a field \( H \) whose characteristic grater than 2. Let \( e_i \) \((1 \leq i \leq 4)\) denote a basis for the vector space. Let the rule of multiplication on \( Q_H \) be defined on \( e_i \) \((1 \leq i \leq 4)\) and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined by \( q = a e_1 + b e_2 + c e_3 + d e_4 \) where \( a, b, c, d \) are ordinary numbers. Such that

\[
\begin{align*}
e_4 &= 1 \\
e_1^2 &= e_2^2 = e_3^2 = -1 \\
e_1 e_2 &= -e_2 e_1 = e_3, \\
e_2 e_3 &= -e_3 e_2 = e_1, \\
e_3 e_1 &= -e_1 e_3 = e_2.
\end{align*}
\]

If we denote \( S_q = d \) and \( V_q = a e_1 + b e_2 + c e_3 \), we can rewrite a real quaternion whose basic algebric form is \( q = S_q + V_q \) where \( S_q \) is scalar part and \( V_q \) is vectorial part of \( q \). Using these basic products we can now expand the product of two quaternions as

\[
p \times q = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \land V_q \quad \text{for every } p, q \in Q_H
\]

where we have used the inner and cross products in Euclidean space \( E^3 \). There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol \( \gamma \) and defined as follows:

\[
\gamma q = -a e_1 - b e_2 - c e_3 + d e_4 \quad \text{for every } a e_1 + b e_2 + c e_3 + d e_4 \in Q_H
\]

which is called the "Hamilton conjugation". This defines the symmetric, real valued, non-degenerate, bilinear form \( h \) as follows:

\[
h(p, q) = \frac{1}{2} |p \times q + q \times p| \quad \text{for every } p, q \in Q_H.
\]

And then, the norm of any \( q \) real quaternion denoted

\[
\|q\|_2^2 = h(q, q) = q \times \gamma q.
\]

\( q \) is called a spatial quaternion whenever \( q + \gamma q = 0 \) (6).

The Serret-Frenet formulae for quaternionic curves in \( E^3 \) and \( E^4 \) are as follows (2):
The three-dimensional Euclidean space $E^3$ is identified with the space of spatial quaternions $\{ p \in Q_H \mid p + \gamma p = 0 \}$ in an obvious manner. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$. Let

$$\alpha : I \subset \mathbb{R} \rightarrow Q_H$$

$$s \rightarrow \alpha (s) = \sum_{i=1}^{3} \alpha_i (s) e_i \quad (1 \leq i \leq 3)$$

be an arc-lenghted curve with nonzero curvatures $\{ k, r \}$ and $\{ t (s), n (s), b (s) \}$ denote the Frenet frame of the curve $\alpha$. Then Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where $k$ is principal curvature, $r$ is torsion of $\alpha$.

Theorem 2.2. The four-dimensional Euclidean space $E^4$ are identified with the space of unique quaternions. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and

$$\alpha : I \subset \mathbb{R} \rightarrow Q_H$$

$$s \rightarrow \alpha (s) = \sum_{i=1}^{4} \alpha_i (s) e_i,$$

be a smooth curve in $E^4$ with nonzero curvatures $\{ K, k, r - K \}$ and $\{ T (s), N (s), B_1 (s), B_2 (s) \}$ denotes the Frenet frame of the curve $\alpha$. Then the frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B'_1 \\ B'_2 \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r - K) \\ 0 & 0 & -(r - K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where $K$ is the principal curvature, $k$ is torsion of $\beta$ and $(r - K)$ is bitorsion of $\alpha$.

Definition 2.3. Let $\alpha (s)$ and $\beta (s^*)$ be two quaternionic curves in $E^4$, $\{ T (s), T (s), N (s), B_1 (s), B_2 (s) \}$ and $\{ T^* (s^*), N^* (s^*) , B^*_1 (s^*), B^*_2 (s^*) \}$ are Frenet frames, respectively, on these curves. $\alpha$ and $\beta$ are called Bertrand curves if there exist a bijection

$$\varphi : I \rightarrow I^*$$

$$s \rightarrow \varphi (s) = s^*, \quad \frac{ds^*}{ds} \neq 0$$

and the principal normal lines of $\alpha$ and $\beta$ at corresponding points coincide.

3. Main Results

Theorem 3.1. Let $\alpha$ be a quaternionic curve in $E^4$. If bitorsion of $\alpha$ is no vanish, then there is no quaternionic curve in $E^4$ is a Bertrand curve.

Proof. Let $\alpha$ be a Bertrand curve in $E^4$ and $\beta$ a Bertrand mate of $\beta$. $\beta$ distinct from $\beta$. Let the pair of $\alpha (s)$ and $\beta (s^*) = \beta (\varphi (s))$ be of corresponding points of $\alpha$ and $\beta$. Then the curve $\beta$ is given by

$$\beta (s^*) = \beta (\varphi (s)) = \alpha (s) + \lambda (s) N (s) \quad (1)$$
where $\lambda$ is a $C^\infty$ function on $I$. Differentiating (11) with respect to $s$, we obtain
\[ \varphi'(s) \frac{d\beta(s^*)}{ds} = \alpha'(s) + \lambda'(s) N(s) + \lambda(s) N'(s). \]

Here and hereafter, the prime denotes the derivative with respect to $s$. By the Frenet equations, it holds that
\[ \varphi'(s) T^*(\varphi(s)) = (1 - \lambda(s) K(s)) T(s) + \lambda'(s) N(s) + \lambda(s) k(s) B_1(s). \]

Since $(T^*(\varphi(s)), N^*(\varphi(s))) = 0$ and $N^*(\varphi(s)) = \mp N(s)$, we obtain, for all $s \in I$,
\[ \lambda'(s) = 0, \]
that is, $\lambda$ is a constant function on $I$. Thus (11) are rewritten as
\[ \beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda N(s) \]
and we obtain
\[ \varphi'(s) T^*(\varphi(s)) = (1 - \lambda K(s)) T(s) + \lambda k(s) B_1(s) \]
for all $s \in I$. By (12), we can set
\[ T^*(\varphi(s)) = (\cos \theta(s)) T(s) + (\sin \theta(s)) B_1(s), \]
where $\theta$ is a $C^\infty$-function on $I$ and
\[ \cos \theta(s) = \frac{1 - \lambda K(s)}{\varphi'(s)} \]
\[ \sin \theta(s) = \frac{\lambda k(s)}{\varphi'(s)}. \]

Differentiating (11) and using the Frenet equations, we obtain
\[ \bar{K}(\varphi(s)) \varphi'(s) N^*(\varphi(s)) = \frac{d \cos \theta(s)}{ds} T(s) \]
\[ + (K(s) \cos \theta(s) - k(s) \sin \theta(s)) N(s) \]
\[ + \frac{d \sin \theta(s)}{ds} B_1(s) \]
\[ + (r - K(s)) \sin \theta(s) B_2(s). \]

Since $N^*(\varphi(s)) = \mp N(s)$ for all $s \in I$, we obtain
\[ (r - K(s)) \sin \theta(s) = 0. \]

By $(r - K(s)) \neq 0$ and (12), we obtain that $\sin \theta(s) = 0$. Thus, by $k(s) \neq 0$ and (11), we obtain that $\lambda = 0$. This completes the proof of theorem. \qed

**Definition 3.2.** Let $\alpha(s)$ and $\beta(s^*)$ be two quaternionic curves in $E^4$. \{ $T(s), N(s), B_1(s), B_2(s)$ \} and \{ $T^*(s^*), N^*(s^*), B_1^*(s^*), B_2^*(s^*)$ \} are Frenet frames, respectively, on these curves. $\alpha$ and $\beta$ are called quaternionic $(1,3)$-Bertrand curves if there exist a bijection
\[ \varphi : I \rightarrow I^* \]
\[ s \rightarrow \varphi(s) = s^*, \frac{ds^*}{ds} \neq 0 \]
and the plane spanned by $N(s), B_2(s)$ at the each point $\alpha(s)$ of $\alpha$ coincides with the plane spanned by $N^*(s^*), B_2^*(s^*)$ corresponding point $\beta(s^*) = \beta(\varphi(s))$ of $\beta$. 

Theorem 3.3. Let \( \alpha \) be a quaternionic curve in \( \mathbb{E}^4 \) with curvature functions \( K, k, r - K \) and \( r - K \neq 0 \). Then \( \alpha \) is a \((1,3)\)-Bertrand curve if and only if there exist constant real numbers \( \lambda, \mu, \gamma, \delta \) satisfying

(i) \( \lambda k(s) - \mu (r - K)(s) \neq 0 \)

(ii) \( \lambda K(s) + \gamma (\lambda k(s) - \mu(r - K)(s)) = 1 \)

(iii) \( \gamma K(s) - k(s) = \delta(r - K)(s) \)

(iv) \( \left\{ \gamma^2 - 1 \right\} k(s) K(s) + \gamma \left[ (K(s))^2 - (k(s))^2 - (r - K)^2(s) \right] \neq 0 \)

for all \( s \in I \).

Proof. We assume that \( \alpha \) is a \((1,3)\)-Bertrand curve parametrized by arclength \( s \). The \((1,3)\)-Bertrand mate \( \beta \) is given by

\[
\beta(s) = \beta(\varphi(s)) = \alpha(s) + \lambda(s)N(s) + \mu(s)B_2(s)
\]

for all \( s \in I \). Here \( \lambda \) and \( \mu \) are \( C^\infty \)-function on \( I \), and \( s^* \) is the arclength parameter of \( \beta \). Differentiating (8) with respect to \( s \), and using the Frenet equations, we obtain

\[
\varphi'(s)T^*(\varphi(s)) = (1 - \lambda(s)K(s))T(s) + \lambda'(s)N(s) + \mu(s)(r - K(s))B_1(s) + \mu'(s)B_2(s)
\]

for all \( s \in I \).

Since the plane spanned by \( N(s) \) and \( B_2(s) \) coincides with the plane spanned by \( N^*(\varphi(s)) \) and \( B_2^*(\varphi(s)) \), we can put

\[
N^*(\varphi(s)) = (\cos\theta(s))N(s) + (\sin\theta(s))B_2(s)
\]

\[
B_2^*(\varphi(s)) = (-\sin\theta(s))N(s) + (\cos\theta(s))B_2(s)
\]

and we notice that \( \sin\theta(s) \neq 0 \) for all \( s \in I \). By the following facts

\[
0 = \langle \varphi'(s)T^*(\varphi(s)), N^*(\varphi(s)) \rangle = \lambda'(s)\cos\theta(s) + \mu'(s)\sin\theta(s)
\]

\[
0 = \langle \varphi'(s)T^*(\varphi(s)), B_2^*(\varphi(s)) \rangle = -\lambda'(s)\sin\theta(s) + \mu'(s)\cos\theta(s)
\]

we obtain

\[
\lambda'(s) = 0, \mu'(s) = 0
\]

that is, \( \lambda \) and \( \mu \) are constant function on \( I \) with values \( \lambda \) and \( \mu \), respectively. Therefore, for all \( s \in I \), (8) is rewritten as

\[
\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda N(s) + \mu B_2(s)
\]

and we obtain

\[
\varphi'(s)T^*(\varphi(s)) = (1 - \lambda K(s))T(s) + (\lambda k(s) - \mu(r - K)(s))B_1(s).
\]

Here we notice that

\[
(\varphi'(s))^2 = (1 - \lambda K(s))^2 + (\lambda k(s) - \mu(r - K)(s))^2 \neq 0
\]

for all \( s \in I \). Thus we can set

\[
T^*(\varphi(s)) = (\cos\tau(s))T(s) + (\sin\tau(s))B_1(s)
\]

and

\[
\cos\tau(s) = \frac{1 - \lambda K(s)}{\varphi'(s)}
\]

\[
\sin\tau(s) = \frac{\lambda k(s) - \mu(r - K)(s)}{\varphi'(s)}
\]
where $\tau$ is a $C^\infty$-function on $I$. Differentiating (14) with respect to $s$ and using the Frenet equations, we obtain

$$
\overline{K}(\varphi(s)) \varphi'(s) N^*(\varphi(s)) = \frac{d}{ds} \cos \tau(s) T(s) + (K(s) \cos \tau(s) - k(s) \sin \tau(s)) N(s) + \frac{d}{ds} B_1(s) + (r - K(s) \sin \tau(s)) B_2(s).
$$

Since $N^*(\varphi(s))$ is expressed by linear combination of $N(s)$ and $B_2(s)$, it holds that

$$
\frac{d}{ds} \cos \tau(s) = 0, \quad \frac{d}{ds} \tau(s) = 0,
$$

that is, $\tau$ is a constant function on $I$ with value $\tau_0$. Thus we obtain

$$
T^*(\varphi(s)) = (\cos \tau_0) T(s) + (\sin \tau_0) B_1(s) \tag{17}
$$

$$
\varphi'(s) \cos \tau_0 = 1 - \lambda K(s) \tag{18}
$$

$$
\varphi'(s) \sin \tau_0 = \lambda k(s) - \mu (r - K(s)) \tag{19}
$$

for all $s \in I$. There fore we obtain

$$
(1 - \lambda K(s)) \sin \tau_0 = (\lambda k(s) - \mu (r - K(s))) \cos \tau_0 \tag{20}
$$

for all $s \in I$.

If $\sin \tau_0 = 0$, then it holds $\cos \tau_0 = \mp 1$. Thus (17) implies that $T^*(\varphi(s)) = \mp T(s)$. Differentiating this equality, we obtain

$$
\overline{K}(\varphi(s)) \varphi'(s) N^*(\varphi(s)) = \mp K(s) N(s),
$$

that is,

$$
N^*(\varphi(s)) = \mp N(s),
$$

for all $s \in I$. By Theorem 3.1, this fact is a contradiction. Thus we must consider only the case of $\sin \tau_0 \neq 0$. Then (19) implies

$$
\lambda k(s) - \mu (r - K(s)) \neq 0
$$

that is, we obtain the relation (i).

The fact $\sin \tau_0 \neq 0$ and (20) imply

$$
\lambda K(s) + \sin^{-1} \tau_0 \cos \tau_0 (\lambda k(s) - \mu (r - K(s))) = 1.
$$

From this, we obtain

$$
\lambda K(s) + \gamma (\lambda k(s) - \mu (r - K(s))) = 1
$$

for all $s \in I$, where $\gamma = \sin^{-1} \tau_0 \cos \tau_0$ is a constant number. Thus we obtain the relation (ii).

Differentiating (17) with respect to $s$ and using the Frenet equations, we obtain

$$
\overline{K}(\varphi(s)) \varphi'(s) N^*(\varphi(s)) = (K(s) \cos \tau_0 - k(s) \sin \tau_0) N(s) + (r - K(s) \sin \tau_0) B_2(s)
$$

for all $s \in I$. From the above equality, (18), (19) and (b), we obtain

$$
[\overline{K}(\varphi(s)) \varphi'(s)]^2 = [K(s) \cos \tau_0 - k(s) \sin \tau_0]^2 + [(r - K(s) \sin \tau_0]^2
$$

$$
= (\lambda k(s) - \mu (r - K(s))^2
$$

$$
\times \left[ (\gamma K(s) - k(s))^2 + ((r - K(s)) \right)] (\varphi'(s))^{-2}
$$
for all \( s \in I \). From (13) and (ii), it holds
\[
(\varphi'(s))^2 = (\gamma^2 + 1) \left( \lambda k(s) - \mu (r - K)(s) \right)^2.
\]
Thus we obtain
\[
\left[ K(\varphi(s)) \varphi'(s) \right]^2 = \frac{1}{\gamma^2 + 1} \left[ (\gamma K(s) - k(s))^2 + ((r - K)(s))^2 \right].
\]
By (18), (19) and (ii), we can set
\[
N^*(\varphi(s)) = (\cos \eta(s)) N(s) + (\sin \eta(s)) B_2(s),
\]
where
\[
\cos \eta(s) = \frac{(\lambda k(s) - \mu (r - K)(s)) (\gamma K(s) - k(s))}{K(\varphi(s))(\varphi'(s))^2}
\]
\[
\sin \eta(s) = \frac{(r - K)(s) (\lambda k(s) - \mu (r - K)(s))}{K(\varphi(s))(\varphi'(s))^2}
\]
for all \( s \in I \). Here, \( \eta \) is a \( C^\infty \)-function on \( I \).

Differentiating (22) with respect to \( s \) and using Frenet equations, we get
\[
-\varphi'(s) K(\varphi(s)) T^*(\varphi(s)) + \varphi'(s) \bar{k}(\varphi(s)) B_1^*(\varphi(s)) = -K(s) \cos \eta(s) T(s) + \frac{d}{ds}(\cos \eta(s)) N(s)
\]
\[
+ \{ k(s) \cos \eta(s) - (r - K)(s) \sin \eta(s) \} B_1(s) + \frac{d}{ds}(\sin \eta(s)) B_2(s)
\]
for all \( s \in I \). From the above fact, it holds
\[
\frac{d \cos \eta(s)}{ds} = 0, \quad \frac{d \sin \eta(s)}{ds} = 0
\]
that is, \( \eta \) is a constant function on \( I \) with value \( \eta_0 \). Let \( \delta = (\cos \eta_0)(\sin \eta_0)^{-1} \) be a constant number. Then (23) and (24) imply
\[
\gamma K(s) - k(s) = \delta (r - K)(s)
\]
that is we obtain the relation (iii).

Moreover, we obtain
\[
-\varphi'(s) K(\varphi(s)) T^*(\varphi(s)) + \varphi'(s) \bar{k}(\varphi(s)) B_1^*(\varphi(s)) = -K(s) \cos \eta_0 T(s)
\]
\[
+ \{ k(s) \cos \eta_0 - (r - K)(s) \sin \eta_0 \} B_1(s)
\]
By the above equality and (12), we obtain
\[
\varphi'(s) \bar{k}(\varphi(s)) B_1^*(\varphi(s)) = \varphi'(s) K(\varphi(s)) T^*(\varphi(s))
\]
\[
- K(s) \cos \eta_0 T(s)
\]
\[
+ \{ k(s) \cos \eta_0 - (r - K)(s) \sin \eta_0 \} B_1(s)
\]
\[
= \left\{ (\varphi'(s))^2 \bar{K}(\varphi(s)) \right\}^{-1}
\]
\[
\times \{ A(s) T(s) + B(s) B_1(s) \},
\]
constant numbers \( \lambda, \mu, \gamma \) with respect to curvature functions \( K, k \). Since \( \overline{s} \) for all \( s \) where

\[
\begin{align*}
\phi &\text{ exists a regular map where } \overline{s} = 0 \text{ if } \lambda k (s) - \mu (r - K) (s) > 0 \text{ and } \lambda k (s) - \mu (r - K) (s) < 0.
\end{align*}
\]

From (ii) and (21), \( A(s) \) and \( B(s) \) are rewritten as:

\[
A(s) = (\lambda k (s) - \mu (r - K) (s)) (\gamma^2 + 1)^{-1} \times \left\{ (1 - \gamma^2) K(s) k(s) - \gamma \left[ (K(s))^2 - (k(s))^2 - ((r - K)(s))^2 \right] \right\}
\]

\[
B(s) = \gamma (\gamma^2 + 1)^{-1} (\lambda k(s) - \mu (r - K) (s)) \times \left\{ (\gamma^2 - 1) k(s) K(s) + \gamma \left[ (K(s))^2 - (k(s))^2 - (r - K)^2 (s) \right] \right\}.
\]

Since \( \overline{K}(\varphi(s)) \varphi'(s) N^*(\varphi(s)) \neq 0 \) for all \( s \in I \), it holds

\[
\left\{ (\gamma^2 - 1) k(s) K(s) + \gamma \left[ (K(s))^2 - (k(s))^2 - (r - K)^2 (s) \right] \right\} \neq 0
\]

for all \( s \in I \). Thus we obtain the relation (iv).

We assume that \( \alpha : I \to Q_H \) is a \( C^\infty \)-special Frenet curve in \( Q_H \) with curvature functions \( K, k \) and \( (r - K) \) satisfying the relation (i), (ii), (iii) and (iv) for constant numbers \( \lambda, \mu, \gamma \) and \( \delta \). Then we define a \( C^\infty \)-curve \( \beta \) by

\[
\beta(s) = \alpha(s) + \lambda N(s) + \mu B_2(s)
\]

for all \( s \in I \), where \( s \) is arclength parameter of \( \alpha \). Differentiating (25) with respect to \( s \) and using the Frenet equations, we obtain

\[
\frac{d\beta(s)}{ds} = (1 - \lambda K(s)) T(s) + (\lambda k(s) - \mu (r - K)) B_1(s)
\]

for all \( s \in I \). Thus, by the relation (ii), we obtain

\[
\frac{d\beta(s)}{ds} = (\lambda k(s) - \mu (r - K)(s)) \left[ \gamma T(s) + B_1(s) \right]
\]

for all \( s \in I \). Since the relation (i) holds, the curve \( \beta \) is a regular curve. Then there exists a regular map \( \varphi : I \to \overline{I} \) defined by

\[
s^* = \varphi(s) = \int_0^s \left\| \frac{d\beta(t)}{dt} \right\| dt
\]

where \( s^* \) denotes the arclength parameter of \( \beta \), and we obtain

\[
\varphi'(s) = \varepsilon \sqrt{2 + 1 (\lambda k(s) - \mu (r - K)(s))} > 0,
\]

(26)

where \( \varepsilon = 1 \) if \( \lambda k(s) - \mu (r - K)(s) > 0 \), and \( \varepsilon = -1 \) if \( \lambda k(s) - \mu (r - K)(s) < 0 \). Thus the curve \( \beta \) is rewritten as

\[
\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda N(s) + \mu B_2(s)
\]
for all \( s \in I \). Differentiating the above equality with respect to \( s \), we obtain
\[
\phi'(s) \frac{d\beta(s^*)}{ds^*} = (\lambda k(s) - \mu (r-K)(s)) \left[ \gamma T(s) + B_1(s) \right].
\] (27)

We can define the unit vector field \( T^* \) along \( \beta \) by \( T^*(s^*) = \frac{d\beta(s^*)}{ds^*} \) for all \( s^* \in I_1 \). By (26) and (27), we obtain
\[
T^*(\phi(s)) = \varepsilon \left( \gamma^2 + 1 \right)^{-\frac{1}{2}} \left[ \gamma T(s) + B_1(s) \right]
\] (28)

for all \( s \in I \). Differentiating (28) with respect to \( s \) and using the Frenet equations, we obtain
\[
\phi'(s) \frac{dT^*(s^*)}{ds^*} \bigg|_{s^* = \phi(s)} = \varepsilon \left( \gamma^2 + 1 \right)^{-\frac{1}{2}} \left\{ [\gamma K(s) - k(s)] N(s) + (r-K)(s) B_2(s) \right\}
\]
and
\[
\left\| \frac{dT^*(s^*)}{ds^*} \bigg|_{s^* = \phi(s)} \right\| = \frac{1}{\sqrt{\left[ \gamma K(s) - k(s) \right]^2 + ((r-K)(s))^2}} \left[ K(\phi(s)) \right] \left( \gamma K(s) - k(s) \right)
\]

By the fact that \((r-K)(s) \neq 0 \) for all \( s \in I \), we obtain
\[
K(\phi(s)) = \left\| \frac{dT^*(s^*)}{ds^*} \bigg|_{s^* = \phi(s)} \right\| > 0
\] (29)

for all \( s \in I \). Then we can define a unit vector field \( N^* \) along \( \beta \) by
\[
N^*(s^*) = N^*(\phi(s))
\]
and
\[
N^*(s^*) = \frac{1}{K(\phi(s))} \frac{dT^*(s^*)}{ds^*} \bigg|_{s^* = \phi(s)}
\]

for all \( s \in I \). Thus we can put
\[
N^*(\phi(s)) = \cos \xi(s) N(s) + \sin \xi(s) B_2(s)
\] (30)

where
\[
\cos \xi(s) = \frac{\gamma K(s) - k(s)}{\sqrt{\left[ \gamma K(s) - k(s) \right]^2 + ((r-K)(s))^2}}
\] (31)
and
\[
\sin \xi(s) = \frac{(r-K)(s)}{\sqrt{\left[ \gamma K(s) - k(s) \right]^2 + ((r-K)(s))^2}}
\] (32)

for all \( s \in I \). Here \( \xi \) is a \( C^\infty \) function on \( I \). Differentiating (30) with respect to \( s \) and using the Frenet equations, we obtain
\[
\phi'(s) \frac{dN^*(s^*)}{ds^*} \bigg|_{s^* = \phi(s)} = - \cos \xi(s) K(s) T(s)
\]
\[
+ \frac{d \cos \xi(s)}{ds} N(s)
\]
\[
+ (k(s) \cos \xi(s) - (r-K)(s) \sin \xi(s)) B_1(s)
\]
\[
+ \frac{d \sin \xi(s)}{ds} B_2(s)
\].
Differentiating \((iii)\) with respect to \(s\), we obtain
\[(\gamma K'(s) - k'(s))(r - K)(s) - (\gamma K(s) - k(s))(r - K)'(s) = 0.\] (33)
Differentiating (31) and (32) with respect to \(s\) and using (33), we obtain
\[
\frac{d}{ds} \cos \xi(s) = 0, \quad \frac{d}{ds} \sin \xi(s) = 0
\]
that is, \(\xi\) is a constant function on \(I\) with value \(\xi_0\). Thus we obtain
\[
\cos \xi_0 = \frac{\gamma K(s) - k(s)}{\sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}}
\]
(34)
\[
\sin \xi_0 = \frac{(r - K)(s)}{\sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}} \neq 0.
\] (35)
From (30), it holds
\[
N^*(\varphi(s)) = \cos \xi_0 N(s) + \sin \xi_0 B_2(s).
\] (36)
Thus we obtain, by (28) and (29),
\[
\tilde{K}(\varphi(s)) T^*(\varphi(s)) = \frac{\gamma K(s) - k(s)}{\sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}} T(s)
\]
and by (34), (35) and (36)
\[
\frac{dN^*(s^*)}{ds^*}_{s^* = \varphi(s)} = - \frac{K(s) (\gamma K(s) - k(s))}{\sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}} T(s)
\]
\[+ \left(\frac{k(s) (\gamma K(s) - k(s)) - ((r - K)(s))^2}{\sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}}\right) B_1(s)
\]
for all \(s \in I\). By the above equalities, we obtain
\[
\frac{dN^*(s^*)}{ds^*}_{s^* = \varphi(s)} + \tilde{K}(\varphi(s)) T^*(\varphi(s)) = \frac{P(s)}{R(s)} T(s) + \frac{Q(s)}{R(s)} B_1(s),
\]
where
\[
P(s) = - \left[\gamma \left\{(K(s))^2 - (k(s))^2 - ((r - K)(s))^2\right\} + (\gamma^2 - 1) K(s) k(s)\right]
\]
\[
Q(s) = \gamma \left[\gamma \left\{(K(s))^2 - (k(s))^2 - ((r - K)(s))^2\right\} + (\gamma^2 - 1) K(s) k(s)\right]
\]
\[
R(s) = \varepsilon \varphi'(s) (\gamma^2 + 1) \sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2} \neq 0
\]
for all \(s \in I\). We notice that, by \((iii)\), \(P(s) \neq 0\) for all \(s \in I\). Thus we obtain
\[
\tilde{k}(\varphi(s)) = \frac{\left\|\frac{dN^*(s^*)}{ds^*}_{s^* = \varphi(s)} + \tilde{K}(\varphi(s)) T^*(\varphi(s))\right\|}{\varepsilon \varphi'(s) \sqrt{\gamma^2 + 1} \sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}}
\]
\[
= \gamma \left\{(K(s))^2 - (k(s))^2 - ((r - K)(s))^2\right\} + (\gamma^2 - 1) K(s) k(s)
\]
\[
\varphi'(s) \sqrt{\gamma^2 + 1} \sqrt{[\gamma K(s) - k(s)]^2 + ((r - K)(s))^2}
\]
for all $s \in I$. Thus we can define a unit vector field $B^*_1(s^*)$ along $\beta$ by

\[
B^*_1(s^*) = \frac{1}{k(\varphi(s))} \left( \frac{dN^*(s^*)}{ds^*}_{s^*=\varphi(s)} + \bar{K}(\varphi(s)) T^*(\varphi(s)) \right)
\]

that is,

\[
B^*_1(\varphi(s)) = \frac{1}{\varepsilon \sqrt{\gamma^2 + 1}} (-T(s) + \gamma B_2(s)) \tag{37}
\]

for all $s \in I$. Next we can define a unit vector field $B^*_2$ along $\beta$ by

\[
B^*_2(s^*) = B^*_2(\varphi(s)) = \frac{1}{\varepsilon \sqrt{\gamma^2 + 1}} (-\gamma N(s) + \gamma K(s) - k(s)) B_2(s) \tag{38}
\]

for all $s \in I$. Now we obtain, by (28), (36), (37) and (38),

\[
\det [T^*(\varphi(s)), N^*(\varphi(s)), B^*_1(\varphi(s)), B^*_2(\varphi(s))] = 1
\]

and $\{T^*(\varphi(s)), N^*(\varphi(s)), B^*_1(\varphi(s)), B^*_2(\varphi(s))\}$ is orthonormal for all $s \in I$. Thus the frame $\{T^*, N^*, B^*_1, B^*_2\}$ along $\beta$ is of orthonormal and of positive. And we obtain

\[
\langle r-K \rangle(s) = \frac{\sqrt{\gamma^2 + 1}}{\sqrt{\gamma^2 + 1}} \frac{1}{\varepsilon} \frac{d}{ds^*}_{s^*=\varphi(s)} B^*_2(\varphi(s)) \neq 0
\]

for all $s \in I$. Thus the curve $\beta$ is a Frenet curve in $Q_H$. And it is trivial that he plane spanned by $N(s), B_2(s)$ at the each point $\alpha(s)$ of $\alpha$ coincides with the plane spanned by $N^*(s^*), B^*_2(s^*)$ corresponding point $\beta(s^*) = \beta(\varphi(s))$ of $\beta$. Therefore $\alpha$ is a $(1,3)$-Bertrand curve in $Q_H$.

\[
\square
\]

References


Osman Kecilioglu
Department of Statistics, Faculty of Science and Arts, Kirikkale University 71450, Kirikkale, Turkey.

E-mail address: okecilioglu@yahoo.com

Kazim Ilarslan
Department of Mathematics, Faculty of Science and Arts, Kirikkale University 71450, Kirikkale, Turkey.

E-mail address: kilarslan@yahoo.com