HYPERGEOMETRIC REPRESENTATION FOR BASKAKOV-DURRMeyer-STANCU TYPE OPERATORS

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Abstract. In the present paper, we introduce and study the mixed summation-integral type operators having Baskakov and Beta basis functions in summation and integration, respectively. First, we estimate moments of these operators using hypergeometric series. Next, we obtain an error estimation in simultaneous approximation for Baskakov-Durrmeyer-Stancu operators.

1. Introduction

Khan [4] and Mishra [5] have proved some results dealing with the degree of approximation of functions in $L_p$-spaces using different types of operators. Baskakov-Durrmeyer operators were first considered by Sahai-Prasad [8] in 1985. Sinha et al. [9] improved and corrected the results of [8]. In 2005, Finta [1] introduced a new type of Baskakov-Durrmeyer operator by taking the weight function of Beta operators on $L[0, \infty)$ as

$$D_n(f, x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t)f(t)dt + p_{n,0}(x)f(0),$$

where $p_{n,k}(x) = \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}}$ and $b_{n,k}(t) = \frac{(n+1)(n+2)_k}{k!} \frac{t^k}{(1+t)^{n+k+2}}$.

Wafi and Khatoon [11] have proved inverse theorem for generalized Baskakov operators. Recently, Gupta and Yadav [3] introduced the Baskakov-Beta-Stancu operator and investigated like asymptotic formula, moments of these operators using hypergeometric series and errors estimation in simultaneous approximation. We
write the operators (1) as
\[ D_n(f, x) = (n + 1) \sum_{k=1}^{\infty} \frac{(n)_k}{k!} \frac{x^k}{(1 + x)^{n+k}} \int_0^\infty \frac{(n + 2)_k}{k!} \frac{t^k}{(1 + t)^{n+k+2}} f(t)dt + \frac{f(0)}{(1 + x)^n}. \]

By hypergeometric series \( {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k \) and the Pochhammer symbol \((n)_k\) is
\( (n)_k = n(n+1)(n+2)(n+3)...(n+k-1), \) using the equality \((1)_k = k!\), we can write
\[ D_n(f, x) = (n + 1) \int_0^\infty \frac{f(t)(1 + x)^2}{(1 + x)(1 + t)^{n+2}} \left[ {}_2F_1 \left( n, n + 2; 1; \frac{xt}{(1 + x)(1 + t)} \right) - 1 \right] dt + \frac{f(0)}{(1 + x)^n}. \]

Now using \( {}_2F_1(a, b; c; x) = {}_2F_1(b, a; c; x) \) and applying Pfaff-Kummer transformation
\[ {}_2F_1(a, b; c; x) = (1 - x)^{-a} {}_2F_1 \left( a, c - b; \frac{x}{x - 1} \right) \]
we have
\[ D_n(f, x) = (n + 1) \int_0^\infty f(t)(1 + x)^2 \left[ \frac{{}_2F_1 \left( n + 2, 1 - n; 1; \frac{-xt}{1 + x + t} \right)}{(1 + x)(1 + t)^{n+2}} - \frac{1}{(1 + x)(1 + t)^{n+2}} \right] dt + \frac{f(0)}{(1 + x)^n}. \]

This is the form of the operators (1) in terms of hypergeometric functions.

Verma et al. [10] considered Baskakov-Durrmeyer-Stancu operators and studied some approximation properties of these operators. Very recently, Mishra and Patel [9] introduced a simple Stancu generalization of q-analogue of well-known Durrmeyer operators. We first estimate moments of q-Durrmeyer-Stancu operators. Here, we introduce Baskakov-Durrmeyer-Stancu type operators in terms of hypergeometric functions, for \( 0 \leq \alpha \leq \beta \)
\[ D_{n,\alpha,\beta}(f, x) = (n + 1) \int_0^\infty f \left( \frac{nt + \alpha}{n + \beta} \right)(1 + x)^2 \left[ \frac{{}_2F_1 \left( n + 2, 1 - n; 1; \frac{-xt}{1 + x + t} \right)}{(1 + x + t)^{n+2}} - \frac{1}{(1 + x)(1 + t)^{n+2}} \right] dt + \frac{f(0)}{(1 + x)^n}. \]

For \( \alpha = \beta = 0 \) the operators (3) reduces to the operators (1).

we know that
\[ \sum_{k=0}^{\infty} p_{n,k}(x) = 1, \quad \int_0^\infty p_{n,k}(x)dx = \frac{1}{n - 1}, \quad \sum_{k=1}^{\infty} b_{n,k}(t) = n + 1, \quad \int_0^\infty b_{n,k}(t)dt = 1. \]

We take
\[ C_{\nu}[0, \infty) = \{ f \in C[0, \infty) : f(t) = O(t)^{\nu}, \nu > 0 \}. \]
The operators $D_{n,a,b}(f, x)$ are well defined for $f \in C[0, \infty)$. The behavior of these operators is very similar to the operators recently introduced in [7] by Mishra et al.

In the present note, first, we estimate moments of Baskakov-Durrmeyer-Stancu operators with the help of hypergeometric series. Next, we give an error estimation in simultaneous approximation for the operators [9].

2. Auxiliary results

In the sequel we shall need several lemmas.

**Lemma 2.1.** For $n > 0$ and $s > -1$, we have

$$D_n(t^s, x) = \frac{\Gamma(n - s + 1)\Gamma(s + 1)}{\Gamma(n + 1)} \left[ (1 + x)^s \binom{n}{s} \right]_2 F_1 \left( 1 - n, -s; 1; \frac{x}{1 + x} \right) - (1 + x)^{-n}. $$

Moreover,

$$D_n(t^s, x) = \frac{(n + s - 1)!(n - s)!}{n!(n - 1)!} x^s + \frac{s(n - 1)(n + s - 2)!(n - s)!}{n!(n - 1)!} x^{s-1} + O(n^{-2}).$$

**Proof.** Taking $f(t) = t^s$, $t = (1 + x)u$ and using Pfaff-Kummer transformation the right-hand side of (2), we get

$$D_n(t^s, x) = (n + 1) \int_0^\infty \frac{(1 + x)^{s+3}u^s}{(1 + x)(1 + u)^n+2} \sum_{k=0}^\infty \frac{(1 - n)_k(n + 2)_k}{(k!)^2} \frac{(-x(1 + x)u)_k}{(1 + x)(1 + u)^k} du$$

$$+ \frac{\Gamma(n - s + 1)\Gamma(s + 1)}{\Gamma(n + 1)}(1 + x)^{-n} = Q_1 + Q_2(say).$$

$$Q_1 = (n + 1) \sum_{k=0}^\infty \frac{(n + 2)_k(1 - n)_k}{(k!)^2} (-x)^k (1 + x)^{s-n+1} \int_0^\infty \frac{u^{s+k}}{(1 + u)^{n+k+2}} du$$

$$= (n + 1) \sum_{k=0}^\infty \frac{(n + 2)_k(1 - n)_k}{(k!)^2} (-x)^k (1 + x)^{s-n+1} \Gamma(s + k + 1, n - s + 1)$$

$$= (n + 1) \sum_{k=0}^\infty \frac{(n + 2)_k(1 - n)_k}{(k!)^2} (-x)^k (1 + x)^{s-n+1} \frac{\Gamma(s + k + 1)\Gamma(n - s + 1)}{\Gamma(n + k + 2)}.$$
Combining $Q_1$ and $Q_2$, we get

$$D_n(t^s, x) = \frac{\Gamma(n - s + 1)\Gamma(s + 1)}{\Gamma(n + 1)} \left[ (1 + x)^s \right] F_1 \left( 1 - n, -s; 1; \frac{x}{1 + x} \right) - (1 + x)^{-n} \right].$$

The other consequence follows from the above equation by writting the expansion of hypergeometric series.

**Lemma 2.2.** For $0 \leq \alpha \leq \beta$ and $m > 0$, we have

$$D_{n,\alpha,\beta}(t^s, x) = x^n \frac{n^s (n + s - 1)!(n - s)!}{(n + \beta)^s n!(n - 1)!} + \sum_{j=0}^{\infty} \frac{s(s - 1)(s - 2)\alpha n^{s-1} (n + s - 2)!(n - s + 1)!}{(n + \beta)^s n!(n - 1)!} \right).$$

**Proof:** Using binomial theorem, the relation between operators (2) and (3) can be defined as

$$D_{n,\alpha,\beta}(t^s, x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t) \left( \frac{nt + \alpha}{n + \beta} \right)^s dt + (1 + x)^{-n} \left( \frac{\alpha}{n + \beta} \right)^s.$$

Using (15), we get Lemma 2.2.

**Lemma 2.3.** For $m \in \mathbb{N} \cup \{0\}$, if

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left( \frac{k}{n} - x \right)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and we have the recurrence relation:

$$nU_{n,m+1}(x) = x(1 + x) \left[ U'_{n,m}(x) + mU_{n,m-1}(x) \right].$$

Consequently, $U_{n,m}(x) = O\left(n^{-\lfloor (m+1)/2 \rfloor}\right)$, where $[m]$ is integral part of $m$.

**Lemma 2.4.** For $m \in \mathbb{N} \cup \{0\}$, if

$$\mu_{n,m}(x) = D_{n,\alpha,\beta}((t - x)^m, x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t) \left( \frac{nt + \alpha}{n + \beta} - x \right)^m dt + p_{n,0}(x) \left( \frac{\alpha}{n + \beta} - x \right)^m$$

then

$$\mu_{n,0}(x) = 1, \ \mu_{n,1}(x) = \frac{\alpha - \beta x}{n + \beta}.$$
and for \( n > m \) we have recurrence relation:

\[
(n - m) \left( \frac{n + \beta}{n} \right) \mu_{n,m+1}(x) = x(1 + x) \left[ \mu'_{n,m}(x) + m \mu_{n,m-1}(x) \right] + \left[ (m + nx) + \left( \frac{n + \beta}{n} \right) \left( \frac{\alpha}{n + \beta} - x \right) (n - 2m) \right] \mu_{n,m}(x) - \left( \frac{\alpha}{n + \beta} - x \right) \left[ \left( \frac{\alpha}{n + \beta} - x \right) \left( \frac{n + \beta}{n} \right) - 1 \right] m \mu_{n,m-1}(x).
\]

From the recurrence relation, it easily verified that for all \( x \in [0, \infty) \), we have

\[
\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).
\]

**Lemma 2.5.** There exist the polynomials \( q_{i,j,s}(x) \) on \([0, \infty)\), independent of \( n \) and \( k \) such that

\[
x^s(1 + x)^s \frac{d^n}{dx^n} p_{n,k}(x) = \sum_{i,j \geq s \atop \nu \geq 0} n^i(k - nx)^j q_{i,j,s}(x)p_{n,k}(x).
\]

### 3. Main result

In this section, we give an estimate of the degree of approximation by \( D_{n,a,b}^{(s)}(f(t), x) \) for smooth functions.

**Theorem 3.1.** Let \( f \in C_\nu[0, \infty) \) for some \( \nu > 0 \), \( m > 0 \) and \( s \leq q \leq s + 2 \). If \( f^{(q)} \) exists and is continuous on \((a - \eta, b + \eta) \subset (0, \infty)\), \( \eta > 0 \), then for sufficiently large \( n \)

\[
||D_{n,a,b}^{(s)}(f(x)) - f^{(s)}(x)||_{C[a,b]} \leq C_1 n^{-1} \sum_{i=q}^{q} ||f^{(i)}||_{C[a,b]} + C_2 n^{1/2} \omega(f^{(q)}, n^{1/2}) + O(n^{-m}),
\]

where \( C_1, C_2 \) are constants independent of \( f \) and \( n \), \( \omega(f, \delta) \) is the modulus of continuity of \( f \) on \((a - \eta, b + \eta) \) and \( ||.||_{C[a,b]} \) denotes the sup-norm on \([a,b]\).

**Proof.** By the Taylor's expansion, we have

\[
f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(x) - f^{(q)}(\xi)}{q!}(t-x)^q \chi(t) + h(t,x)(1 - \chi(t))
\]

where \( \xi \) lies between \( t \) and \( x \), and \( \chi(t) \) is the characteristic function on interval \((a - \eta, b + \eta)\).  

For \( t \in (a - \eta, b + \eta) \) and \( x \in [a, b] \), we have

\[
f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(x) - f^{(q)}(\xi)}{q!}(t-\xi)^q.
\]

For \( t \in [0, \infty) \setminus (a - \eta, b + \eta) \) and \( x \in [a, b] \), we define

\[
h(t,x) = f(t) - \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i.
\]
Now,

\[ D_{n,\alpha,\beta}^{(s)}(f, x) - f^{(s)}(x) = \left\{ \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} D_{n,\alpha,\beta}^{(s)}((t-x)^i, x) - f^{(s)}(x) \right\} + D_{n,\alpha,\beta}^{(s)} \left( \frac{f^{(q)}(x) - f^{(q)}(\xi)}{q!} (t-x)^q \chi(t), x \right) + D_{n,\alpha,\beta}^{(s)}(h(t, x)(1-\chi(t)), x) = F_1 + F_2 + F_3. \]

Using Lemma 22, we get

\[ F_1 = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^n}{dx^n} \left[ x^j \frac{n^j}{(n+\beta)^j} \frac{(n+j-1)!(n-j)!}{n!(n-1)!} \right] + \frac{x^{j-1}}{n!} \left( \frac{n+2}{(n+\beta)^{j-1}} \frac{(n+j-2)!(n-j)!}{n!(n-1)!} + j \frac{n^{j-1}}{(n+\beta)^j} \frac{(n+j-2)!(n-j)!}{n!(n-1)!} \right) + \frac{x^{j-2}}{2} \frac{n^{j-2}}{(n+\beta)^j} \frac{(n+j-3)!(n-j+2)!}{n!(n-1)!} + O(n^{-m}) - f^{(s)}(x). \]

Hence

\[ ||F_1||_{C[a, b]} \leq C_1 n^{-1} \sum_{i=0}^{q} ||f^{i}||_{C[a, b]} + O(n^{-m}), \text{ uniformly on } [a, b]. \]

Next, we estimate \( F_2 \) as

\[ |F_2| \leq \sum_{k=1}^{\infty} |p_{n,k}^{(s)}(x)| \int_0^\infty b_{n,k}(t) \left[ \left| \frac{f^{(q)}(x) - f^{(q)}(\xi)}{q!} \right| \frac{nt+\alpha}{n+\beta} - x \right]^q \chi(t) dt + \frac{(n+s-1)!}{(n-1)!} (1+x)^{-n-s} \left( \frac{\alpha}{n+\beta} - x \right)^q \chi(t) \]

\[ \leq \frac{\omega(f^{(s)}, \delta)}{q!} \left[ \sum_{k=1}^{\infty} |p_{n,k}^{(s)}(x)| \int_0^\infty b_{n,k}(t) \left( 1 + \left| \frac{nt+\alpha}{n+\beta} - x \right| \right) \frac{nt+\alpha}{n+\beta} - x \right]^q dt + \frac{(n+s-1)!}{(n-1)!} (1+x)^{-n-s} \left( 1 + \frac{\alpha}{n+\beta} - x \right) \left( \frac{\alpha}{n+\beta} - x \right)^q \]

\[ \leq \frac{\omega(f^{(s)}, \delta)}{q!} \left[ \sum_{k=1}^{\infty} |p_{n,k}^{(s)}(x)| \int_0^\infty b_{n,k}(t) \left( \frac{nt+\alpha}{n+\beta} - x \right)^q + \delta^{-1} \frac{nt+\alpha}{n+\beta} - x \right] dt + \frac{(n+s-1)!}{(n-1)!} (1+x)^{-n-s} \left( \frac{\alpha}{n+\beta} - x \right)^q + \delta^{-1} \frac{\alpha}{n+\beta} - x \right] \left( \frac{\alpha}{n+\beta} - x \right)^{q+1} \]
Now, on application of Schwarz inequality for integration and then for summation, we get
\[
\sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_{0}^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^q \leq \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \left( \int_{0}^{\infty} b_{n,k}(t) dt \right)^{\frac{q}{j}} \left( \int_{0}^{\infty} b_{n,k}(t) \left( \frac{nt+\alpha}{n+\beta} - x \right)^{2q} dt \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2q} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} p_{n,k}(x) \right) \left( \int_{0}^{\infty} b_{n,k}(t) \left( \frac{nt+\alpha}{n+\beta} - x \right)^{2s} dt \right)^{\frac{1}{2}}.
\]

Using Lemma 2.3 we get
\[
\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j} = n^{2j} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x)(k/n-x)^{2j} - (1+x)^{-n}(-x)^{2j} \right\} = n^{2j} \left\{ O(n^{-j}) + O(n^{-r}) \right\} \text{ (for any } r > 0). = O(n^{4j}).
\]

Similarly, using Lemma 2.4 we get
\[
\sum_{k=1}^{\infty} p_{n,k}(x)\int_{0}^{\infty} b_{n,k}(t) \left( \frac{nt+\alpha}{n+\beta} - x \right)^{2q} dt = O(n^{q}) - (1+x)^{-n}(-x)^{2q} = O(n^{-q}) + O(n^{-r}) \text{ (for any } r > 0). = O(n^{-s}).
\]

Hence
\[
\sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_{0}^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^q = O(n^{j/2})O(n^{-q/2}) = O(n^{(j-q)/2}), \text{ uniformly on } [a, b]. \tag{9}
\]

Therefore, by Lemma 2.5 and (9), we get
\[
\sum_{k=1}^{\infty} |p_{n,k}^{(s)}(x)| \int_{0}^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^q \leq \sum_{k=1}^{\infty} \sum_{i,j \leq s \atop i+j \geq 0} n^{i} p_{n,k}(x)|k-nx|^j \int_{0}^{\infty} b_{n,k}(t) \left( \frac{nt+\alpha}{n+\beta} - x \right)^{q} dt \leq K \sum_{i,j \leq s \atop i+j \geq 0} n^{i} \left( \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_{0}^{\infty} b_{n,k}(t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^q dt \right) = K \sum_{i,j \leq s \atop i+j \geq 0} n^{i} O(n^{(j-q)/2}) = O(n^{(s-q)/2}), \text{ uniformly on } [a, b]. \tag{10}
\]
where $K = \sup_{2i+1 \leq s, 0 \leq j \leq b} \frac{n^s q_i}{x(1+x)^s}$. Choosing $\delta = n^{-1/2}$ and making use of \cite{10}, we get for any $m > 0$,

$$
\|F_2\|_{C[a, b]} \leq \frac{\omega(f^{(q)}, n^{-1/2})}{q!} \left( O(n^{(s-q)/2}) + n^{1/2} O(n^{(s-q-1)/2}) + O(n^{-m}) \right) \leq C_2 \left( n^{-(q-s)/2} \right) \omega(f^{(q)}, n^{-1/2}).
$$

For $t \in (0, \infty) \setminus (a - \eta, b + \eta)$, we can choose $\delta$ such that $|t - x| \geq \delta$ for all $x \in [a, b]$. Thus by Lemma \ref{2.5} we get

$$
|F_2| \leq \sum_{2i+1 \leq s, 0 \leq j} n^s q_{i,j,s}(x) x^n \sum_{k=1}^{\infty} p_{n,k}(x) |k - nx|^2 \int_{|t-x| \geq \delta} b_{n,k}(t)|h(t, x)|dt
+ \frac{(n+s-1)!}{(n-1)!} (1+x)^{-n-s} \left| b \left( \frac{\alpha n + \beta}{n + \beta}, x \right) \right|.
$$

We can find a constant $M_1$ such that

$$
|h(t, x)| \leq M_1 \left| \frac{nt + \alpha}{n + \beta} - x \right|^\beta \text{ for } |t - x| \geq \delta,
$$

where $\beta \geq (\nu, q)$. Hence applying Schwarz inequality, \ref{7} and \ref{5}, it is easy to see that $F_3 = O(n^{-r})$ for any $r > 0$ uniformly on $[a, b]$. Combining the estimates of $F_1, F_2$ and $F_3$, the required result follows.

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