CLOSE-TO-CONVEXITY AND STARLIKENESS OF CERTAIN ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract. The main object of the present paper is to derive some results for multivalent analytic functions defined by a linear operator $s$. As a special case of these results, we obtain several sufficient conditions for close-to-convexity and starlikeness of certain analytic functions.

1. Introduction

Let $A(p, n)$ denote the class of functions $f$ in the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\})$$

which are analytic and $p$-valent in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We write $A(p, 1) = A(p), A(1, n) = A_n$ and $A_1 = A$. A function $f \in A(p, n)$ is said to be $p$-valent starlike of order $\alpha$ ($0 \leq \alpha < p$) in $\Delta$ if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \Delta,$$

and we denote by $S^*_p(\alpha)$ the class of all such functions. A function $f \in A(p, n)$ is said to be $p$-valent convex of order $\alpha$ ($0 \leq \alpha < p$) in $\Delta$ if

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \Delta,$$

and we denote by $K^*_p(\alpha)$ the class of all such functions. Further a function $f \in A$ is said to be close-to-convex if there exists a (not necessarily normalized) convex function $g$ such that

$$\Re \left( \frac{f'(z)}{g'(z)} \right) > 0, \quad z \in \Delta.$$

We shall denote by $C$ the class of close-to-convex functions in $\Delta$.
For two functions $f$ given by (1) and $g$ given by
\[ g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (p, n \in \mathbb{N}), \]
their Hadamard product (or convolution) is defined by
\[ (f * g)(z) = z^p + \sum_{k=n+p}^{\infty} b_k a_k z^k. \]

Define the function $\phi_p(a, c; z)$ by
\[ \phi_p(a, c; z) := z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad (c \neq 0, -1, -2, \ldots, z \in \Delta), \]
where $(a)_n$ is the Pochhammer symbol defined by
\[ (a)_n := \begin{cases} 1, \quad (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), \quad (n \in \mathbb{N} := \{1, 2, 3\ldots\}) \end{cases}. \]

Corresponding to the function $\phi_p(a, c; z)$, Saitoh [7] introduced a linear operator $L_p(a, c)$ which is defined by means of the following Hadamard product:
\[ L_p(a, c)f(z) = \phi_p(a, c) * f(z) \quad (f \in \mathcal{A}(p, n)), \]
or, equivalently, by
\[ L_p(a, c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} a_k z^{k+p}, \quad z \in \Delta. \]

It follows from (2) that
\[ z(L_p(a, c)f(z))' = aL_p(a+1, c)f(z) - (a-p)L_p(a, c)f(z) \quad (3) \]
Note that $L_p(a, a)f(z) = f(z)$, $L_p(p+1, p)f(z) = \frac{zf(z)}{p}$, $L_1(3, 1)f(z) = zf'(z) + \frac{1}{2}z^2f''(z)$ and $L_p(\delta+1, 1)f(z) = D^{\delta+p}f(z)$, where $D^{\delta+p}f$ is the Ruscheweyh derivative of order $\delta + p$.

Many properties of analytic functions defined by the linear operator $L_p(a, c)f(z)$ were studied by (among others), Aghalary and Ebadian [1], Owa and Srivastava [6], Cho et al. [3], and Carlson and Shaffer [2].

In the present paper we aim to find simple sufficient conditions for close-to-convexity and starlikeness of multivalent analytic functions. The following lemma will be required in our present investigations.

**Lemma 1.1.** (see[4]) Let the (nonconstant) function $\omega$ be analytic in $\Delta$ with $\omega(z) = \omega_n z^n + \cdots$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \Delta$, then
\[ z_0 \omega'(z_0) = c \omega(z_0), \]
where $c$ is a real number and $c \geq n$. 
2. Main Results

**Theorem 2.1.** Let \( a \in \mathbb{C}, |a| > 0, \beta \geq 0, \gamma \geq 0, \) and \( 0 \leq \alpha < p. \) If the function \( f \in \mathcal{A}(p, n) \) satisfies

\[
\frac{\mathcal{L}_p(a, c)f(z)}{z^p} - 1 > \frac{1}{|a|^\beta} \left(1 - \frac{\alpha}{p}\right)^{\gamma + \beta} n^\beta, \quad z \in \Delta,
\]

then

\[
Re\left(\frac{\mathcal{L}_p(a, c)f(z)}{z^p}\right) > \frac{\alpha}{p}, \quad z \in \Delta.
\]

**Proof.** Define the function \( \omega \) by

\[
\frac{\mathcal{L}_p(a, c)f(z)}{z^p} = 1 + \left(1 - \frac{2\alpha}{p}\right)\omega(z), \quad (\omega(z) \neq -1, \; z \in \Delta).
\]

Then, clearly, \( \omega(z) = \omega_n z^n + \cdots \) is analytic in \( \Delta. \) By a simple computation and by making use of the familiar identity (3), we find from (6) that

\[
\frac{\mathcal{L}_p(a, c)f(z)}{z^p} - 1 \right| \mathcal{L}_p(a + 1, c)f(z) - \mathcal{L}_p(a, c)f(z) \right|^{\beta} \]

\[
= \frac{1}{|a|^\beta} \left(1 - \frac{\alpha}{p}\right)^{\gamma + \beta} |zw'(z)|^\beta.
\]

Suppose now that there exists a point \( z_0 \in \Delta \) such that

\[
|\omega(z_0)| = 1 \quad \text{and} \quad |\omega(z)| < 1, \quad \text{when} \quad |z| < |z_0|.
\]

Then by using Lemma 1.1, we have \( \omega(z_0) = e^{i\theta}, 0 < \theta \leq 2\pi \) and \( z_0 \omega'(z_0) = \xi \omega(z_0), \xi \geq n. \) Therefore

\[
\frac{\mathcal{L}_p(a, c)f(z_0)}{z_0^p} - 1 \right| \mathcal{L}_p(a + 1, c)f(z_0) - \mathcal{L}_p(a, c)f(z_0) \right|^{\beta} \]

\[
= \frac{1}{|a|^\beta} \left(1 - \frac{\alpha}{p}\right)^{\gamma + \beta} |\xi|^\beta
\]

\[
> \frac{1}{|a|^\beta} \left(1 - \frac{\alpha}{p}\right)^{\gamma + \beta} n^\beta,
\]

which contradicts our hypothesis (4). Thus, we have

\[
|\omega(z)| < 1, \quad z \in \Delta,
\]

and the proof is complete. \( \square \)

By letting \( a = c = 1 \) and \( p = n = 1 \) in Theorem 2.1 we obtain Theorem 3 of [5] that is:

**Corollary 2.2.** Let \( \gamma \geq 0, \beta \geq 0 \) and \( 0 \leq \alpha < 1. \) If the function \( f \in \mathcal{A} \) satisfies

\[
|f'(z) - 1|^{\gamma} |zf''(z)|^{\beta} < 2^{\beta}(1 - \alpha)^{\gamma + \beta}, \quad z \in \Delta,
\]

then

\[
Re f'(z) > \alpha, \quad z \in \Delta,
\]

i.e. \( f \) is close-to-convex function.
Theorem 2.3. Let $a \in \mathbb{C}$ with $\Re a > 0$, let $\beta \geq 0, \gamma > 0$ and $0 \leq \alpha < p$. If $f \in \mathcal{A}(p)$ satisfies the inequality
\[
\left| \frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1 \right|^\gamma \left| \frac{\mathcal{L}_p(a+1,c)f(z)}{z^p} - 1 \right|^\beta \leq \frac{(1-\frac{\alpha}{p})^{\gamma+\beta}}{|\alpha|^\beta} \left( \Re a + \frac{n}{2} \right)^\beta, \; z \in \Delta, \tag{7}
\]
then
\[
\Re \left( \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \right) > \frac{\alpha}{p}, \; z \in \Delta. \tag{8}
\]

Proof. Let define the function $\omega$ by
\[
\frac{\mathcal{L}_p(a,c)f(z)}{z^p} = \frac{1 + \left( 1 - \frac{2\alpha}{p} \right) \omega(z)}{1 - \omega(z)}, \quad (\omega(z) \neq -1, \; z \in \Delta).
\]
Then $\omega$ is analytic in $\Delta$, $\omega(z) = \omega_n z^n + \cdots$. By making use of the identity (3), we obtain
\[
\left| \frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1 \right|^\gamma \left| \frac{\mathcal{L}_p(a+1,c)f(z)}{z^p} - 1 \right|^\beta
\]
\[
= \left| \frac{2(1 - \frac{\alpha}{p})\omega(z)}{1 - \omega(z)} \right|^\gamma \left| \frac{2(1 - \frac{\alpha}{p})\omega'(z)}{a(1 - \omega(z))^2} + \frac{2(1 - \frac{\alpha}{p})\omega(z)}{1 - \omega(z)} \right|^\beta
\]
\[
= \frac{2^{\gamma+\beta}(1 - \frac{\alpha}{p})^{\gamma+\alpha}}{|a|^\beta} \left| \frac{\omega(z) - a + \frac{\omega'(z)}{(1 - \omega(z))\omega(z)}}{1 - \omega(z)} \right|^{\gamma+\beta}.
\]
Suppose that there exists a point $z_0 \in \Delta$ such that $\max |\omega(z)| = |\omega(z_0)| = 1 (|z| \leq |z_0|)$. Then by using Lemma 1.1, we have $\omega(z_0) = e^{i\theta}, \; 0 < \theta \leq 2\pi$ and $z_0 \omega'(z_0) = \xi \omega(z_0), \; \xi \geq n$. Therefore
\[
\left| \frac{\mathcal{L}_p(a,c)f(z_0)}{z_0^p} - 1 \right|^\gamma \left| \frac{\mathcal{L}_p(a+1,c)f(z_0)}{z_0^p} - 1 \right|^\beta
\]
\[
= \frac{2^{\gamma+\beta}(1 - \frac{\alpha}{p})^{\gamma+\beta}}{|a|^\beta} \left| \frac{1}{1 - \omega(z_0)} \right|^{\gamma+\beta} \left| a + \frac{\xi}{(1 - e^{i\theta})} \right|^\beta
\]
\[
\geq \frac{(1 - \frac{\alpha}{p})^{\gamma+\beta}}{|a|^\beta} \left( \Re a + \frac{n}{2} \right)^\beta.
\]
Which contradicts obviously our hypothesis (7). Thus, we have $|\omega(z)| < 1$ for all $z \in \Delta$, and hence (8) holds true. $\square$

By letting $c = a - 1 = 1, \; \gamma = \beta = \frac{1}{2}$ and $p = n = 1$ in Theorem 2.1, we obtain the following Corollary:

Corollary 2.4. If the function $f \in \mathcal{A}$ satisfies the inequality
\[
|f'(z) - 1|^\frac{1}{2} \left| f'(z) + \frac{1}{2} z f''(z) - 1 \right|^\frac{1}{2} \leq \frac{(1 - \alpha)}{\sqrt{2}}(2 + \frac{1}{2})^\frac{1}{2}, \; z \in \Delta,
\]
then
\[
\Re f'(z) > \alpha, \; z \in \Delta,
\]
i.e. $f$ is close-to-convex function.

By letting $c = a = 1, \; \gamma = \beta = \frac{1}{2}$ and $p = 1$ in Theorem 2.3, we conclude the following result:
Corollary 2.5. If the function \( f \in A \) satisfies the inequality
\[
|f(z) - 1|^p |f'(z) - 1|^q < \frac{3}{2} (1 - \alpha), \quad z \in \Delta,
\]
then
\[
\text{Re} \left( \frac{f(z)}{z} \right) > \alpha, \quad z \in \Delta.
\]

Finally we prove:

Theorem 2.6. Suppose that \( a \in \mathbb{C}, \text{Re} \, a \geq 0, \beta \geq 0, \gamma \geq 0 \) and \( 0 \leq \alpha < p \). If the function \( f \in \mathcal{A}(p, n) \) satisfies the inequality
\[
\left| \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)} - 1 \right|^{\gamma} \left| \frac{L_p(a + 2, c) f(z)}{L_p(a + 1, c) f(z)} - 1 \right|^{\beta} < N(\alpha, p, \gamma, \beta), \quad z \in \Delta,
\]
where
\[
N(\alpha, p, \gamma, \beta) = \left\{ \begin{array}{ll}
(1 - \frac{\beta}{\gamma})^\gamma (\text{Re} a)(1 - \frac{\beta}{\gamma} + \frac{2}{\gamma})^\beta, & 0 \leq \alpha \leq \frac{p}{2}, \\
(1 - \frac{\beta}{\gamma})^{\gamma + \beta} (\text{Re} a + \frac{\beta}{\gamma})^\beta, & \frac{p}{2} \leq \alpha < p.
\end{array} \right.
\]
Then
\[
\text{Re} \left( \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)} \right) > \frac{\alpha}{p}, \quad z \in \Delta.
\]
Proof. Define the function \( M \) by
\[
M(z) = \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)}.
\]

Then by a simple computation and by making use of the identity (3), we have
\[
\left| \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)} - 1 \right|^{\gamma} \left| \frac{L_p(a + 2, c) f(z)}{L_p(a + 1, c) f(z)} - 1 \right|^{\beta} = \left| M(z) - 1 \right|^{\gamma} \left| \frac{1}{a + 1} \left( \frac{z M'(z)}{M(z)} + a(M(z) - 1) \right) \right|^{\beta}.
\]

Now we distinguish two cases,
Case(i). If \( 0 \leq \alpha \leq \frac{p}{2} \), define a function \( \omega \)
\[
M(z) = \frac{1 + \left( 1 - \frac{2\alpha}{p} \right) \omega(z)}{1 - \omega(z)}, \quad z \in \Delta.
\]

Then \( \omega \) is analytic in \( \Delta, \omega(z) = \omega_n z^n + \cdots \) and \( \omega(z) \neq 1 \) in \( \Delta \). We find from (11) that
\[
\left| \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)} - 1 \right|^{\gamma} \left| \frac{L_p(a + 2, c) f(z)}{L_p(a + 1, c) f(z)} - 1 \right|^{\beta} = \frac{2^{\gamma + \beta} \left( 1 - \frac{\beta}{\gamma} \right)^{\gamma + \beta}}{|a + 1|^{\beta}} \left| \frac{\omega(z)}{1 - \omega(z)} \right|^{\gamma + \beta} \left| a + \frac{z \omega'(z)}{1 + \left( 1 - \frac{2\alpha}{p} \right) \omega(z)} \right|^\beta.
\]
Suppose now that there exists a point \( z_0 \in \Delta \) such that \( \max |\omega(z)| = |\omega(z_0)| = 1 \) (\(|z| \leq |z_0|\)). Then by using Lemma 1.1, we have \( \omega(z_0) = e^{i\theta} \), \( 0 < \theta < 2\pi \) and \( z_0\omega'(z_0) = m\omega(z_0), \ m \geq 1 \). Therefore from (12), we obtain

\[
\left| \frac{L_p(a + 1, c) f(z_0)}{L_p(a, c) f(z_0)} - 1 \right|^\gamma \left| \frac{L_p(a + 2, c) f(z_0)}{L_p(a + 1, c) f(z_0)} - 1 \right|^\beta 
\geq \left( \frac{1 - \frac{\alpha}{p}}{|a + 1|} \right)^{\gamma + \beta} \left( \frac{\omega(z_0)}{\omega(z)} \right)^\gamma \left| a + \frac{z \omega'(z)}{\omega(z)} \right|^\beta.
\]

which contradicts (10) for \( 0 \leq \alpha < \frac{p}{2} \). Hence, we must have \(|\omega(z)| < 1\) for all \( z \in \Delta \), and the first part of theorem complete.

Case (ii). When \( \frac{p}{2} \leq \alpha < p \), let a function \( \omega \) be defined by

\[
M(z) = \frac{\alpha}{p} \omega(z) - \left( 1 - \frac{\alpha}{p} \right)\omega(z), \ z \in \Delta.
\]

Then \( \omega \) is analytic in \( \Delta \) and \( \omega(z) = \omega_n z^n + \cdots \) proceeding the same as case (i). We find from (12) that

\[
\left| \frac{L_p(a + 1, c) f(z)}{L_p(a, c) f(z)} - 1 \right|^\gamma \left| \frac{L_p(a + 2, c) f(z)}{L_p(a + 1, c) f(z)} - 1 \right|^\beta 
= \left( \frac{1 - \frac{\alpha}{p}}{|a + 1|} \right)^{\gamma + \beta} \left( \frac{\omega(z_0)}{\omega(z)} \right)^\gamma \left| a + \frac{z \omega'(z)}{\omega(z)} \right|^\beta.
\]

Suppose that there exists a point \( z_0 \in \Delta \) such that \( \max |\omega(z)| = |\omega(z_0)| = 1 \) (\(|z| \leq |z_0|\)). Then by using Lemma 1.1, we have obtain \( \omega(z_0) = e^{i\theta} \), \( 0 \leq \theta < 2\pi \) and \( z_0\omega'(z_0) = m\omega(z_0), \ m \geq 1 \). Now from (13) we have

\[
\left| \frac{L_p(a + 1, c) f(z_0)}{L_p(a, c) f(z_0)} - 1 \right|^\gamma \left| \frac{L_p(a + 2, c) f(z_0)}{L_p(a + 1, c) f(z_0)} - 1 \right|^\beta 
= \left( \frac{1 - \frac{\alpha}{p}}{|a + 1|} \right)^{\gamma + \beta} \left( \frac{\omega(z_0)}{\omega(z)} \right)^\gamma \left| a + \frac{z \omega'(z_0)}{\omega(z)} \right|^\beta 
\geq \left( \frac{1 - \frac{\alpha}{p}}{|a + 1|} \right)^{\gamma + \beta} \left( \frac{\omega(z_0)}{\omega(z)} \right)^\gamma \left| a + \frac{z \omega'(z_0)}{\omega(z)} \right|^\beta.
\]

which contradicts (9) for \( \frac{p}{2} \leq \alpha < p \). Therefore, we must have \(|\omega(z)| < 1\) for all \( z \in \Delta \), and the proof is complete.

By letting \( c = a = 1 \) and \( p = 1 \) in the theorem 2.6, we have:

**Corollary 2.7.** If the function \( f \in A_n \) satisfies the inequality

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right|^\gamma \left| \frac{zf''(z)}{f'(z)} \right|^\beta \leq M(\alpha, \beta, \gamma, n), \ z \in \Delta,
\]

Then \( \omega(z) = e^{i\theta} \), \( 0 < \theta < 2\pi \) and \( z_0\omega'(z_0) = m\omega(z_0), \ m \geq 1 \).
where

\[ M(\alpha, \beta, \gamma, n) = \begin{cases} 
(1 - \alpha)^\gamma (1 + \frac{\alpha}{2} - \alpha)^\beta, & 0 \leq \alpha \leq \frac{1}{2}, \\
(1 - \alpha)^\gamma + \beta (1 + n)^\beta, & \frac{1}{2} \leq \alpha < 1.
\end{cases} \]

Then

\[ \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \Delta. \]

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