ON THE $q$-BESSEL FOURIER TRANSFORM

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Abstract. In this work, we are interested by the $q$-Bessel Fourier transform with a new approach. Many important results of this $q$-integral transform are proved with a new constructive demonstrations and we establish in particular the associated $q$-Fourier-Neumen expansion which involves the $q$-little Jacobi polynomials.

1. Introduction

In the recent mathematical literature one finds many articles which deal with the theory of $q$-Fourier analysis associated with the $q$-Hankel transform. This theory was elaborated first by Koornwinder and R.F. Swarttouw [12] and then by Fitouhi and Al [5, 8].

It should be noticed that in [5] we provided the main results of $q$-Fourier analysis in particular that the $q$-Hankel transform is extended to the $L_{q,2,\nu}$ space like an isometric operator. Often we use the crucial properties namely the positivity of the $q$-Bessel translation operator to prove some results but these last property is not ensured for any $q$ in the interval $]0,1[$. Thus, we will prove some main results of $q$-Fourier analysis without the positivity argument especially the following results:

- Inversion Formula in the $L_{q,p,\nu}$ spaces with $p \geq 1$.
- Plancherel Formula in the $L_{q,p,\nu} \cap L_{q,1,\nu}$ spaces with $p > 2$.
- Plancherel Formula in the $L_{q,2,\nu}$ spaces.

Note that in the paper [7] we have proved that the positivity of the $q$-Bessel translation operator is ensured in all points of the interval $]0,1[\}$ when $\nu \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the $q$-Bessel translation operator plays a role in $q$-Bessel Fourier analysis. In particular, when we try to prove a $q$-version of the Young’s inequality for the associated convolution.

Many interesting result about the uncertainty principle for the $q$-Bessel transform was proved in the last years. We cite for examples [2, 3, 4, 9]. There are some differences of the results cited above and our result:
In this paper the Heisenberg uncertainty inequality is established for functions in $L_{q,2,\nu}$ space. The Hardy’s inequality discussed here is a quantitative uncertainty principle which gives an information about how a function and its $q$-Bessel Fourier transform are linked.

In the end of this paper we use the remarkable work in [1] to establish a new result about the $q$-Fourier-Neumann expansion involving the $q$-little Jacobi polynomials.

2. The $q$-Bessel transform

The reader can see the references [10, 11, 16] about $q$-series theory. The references [5, 8, 12] are devoted to the $q$-Bessel Fourier analysis. Throughout this paper, we consider $0 < q < 1$ and $\nu > -1$. We denote by $\mathbb{R}_q^+ = \{ q^n, \ n \in \mathbb{Z} \}$.

The $q$-Bessel operator is defined as follows [5]

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - (1 + q^{2\nu}) f(x) + q^{2\nu} f(qx) \right].$$

The eigenfunction of $\Delta_{q,\nu}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_{\nu}(\lambda x, q^2)$, where $j_{\nu}(., q^2)$ is the normalized $q$-Bessel function defined by [5, 8, 10, 14, 16]

$$j_{\nu}(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\nu+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$  

The $q$-Jackson integral of a function $f$ defined on $\mathbb{R}_q^+$ is

$$\int_0^{\infty} f(t) dq_t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

We denote by $L_{q,p,\nu}$ the space of functions $f$ defined on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,\nu} = \left( \int_0^{\infty} |f(x)|^p x^{2\nu+1} dq_x \right)^{1/p}$$

exist.

We denote by $C_{q,0}$ the space of functions defined on $\mathbb{R}_q^+$ tending to 0 as $x \to \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $C_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)|.$$

The normalized $q$-Bessel function $j_{\nu}(., q^2)$ satisfies the orthogonality relation

$$c_{q,\nu}^2 \int_0^{\infty} j_{\nu}(xt, q^2) j_{\nu}(yt, q^2) t^{2\nu+1} dq_t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q^+ \quad (1)$$

where

$$\delta_q(x, y) = \left\{ \begin{array}{ll}
0 & \text{if } x \neq y \\
\frac{1}{1-q|x-y|} & \text{if } x = y
\end{array} \right.$$  

and

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$
Let \( f \) be a function defined on \( \mathbb{R}_+^2 \) then
\[
\int_0^\infty f(y)\delta_q(x, y)y^{2\nu+1} dy = f(x).
\]
The normalized \( q \)-Bessel function \( j_\nu(., q^2) \) satisfies
\[
|j_\nu(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty^2(-q^{2\nu+2}; q^2)_\infty}{(q^{2\nu+2}; q^2)_\infty^2} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{\nu^2-(2\nu+1)n} & \text{if } n < 0 \end{cases}.
\]
The \( q \)-Bessel Fourier transform \( \mathcal{F}_{q,\nu} \) is defined by \([5, 8, 12]\)
\[
\mathcal{F}_{q,\nu} f(x) = c_{q,\nu} \int_0^\infty f(t)j_\nu(xt, q^2)t^{2\nu+1} dt,
\]
\( \forall x \in \mathbb{R}_+^2 \).

**Proposition 1.** Let \( f \in \mathcal{L}_{q,1,\nu} \) then \( \mathcal{F}_{q,\nu} f \in \mathcal{C}_{q,0} \) and we have
\[
\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty} \leq B_{q,\nu} \|f\|_{q,1,\nu}
\]
where
\[
B_{q,\nu} = \frac{1}{1 - q} \frac{(-q^2; q^2)_\infty^2(-q^{2\nu+2}; q^2)_\infty}{(q^{2\nu+2}; q^2)_\infty^2}.
\]

**Theorem 1.** Let \( f \) be a function in the \( \mathcal{L}_{q,p,\nu} \) space where \( p \geq 1 \) then
\[
\mathcal{F}_{q,\nu}^2 f = f. \quad (2)
\]

**Proof.** If \( f \in \mathcal{L}_{q,p,\nu} \) then \( \mathcal{F}_{q,\nu} f \) exist, and we have
\[
\begin{align*}
\mathcal{F}_{q,\nu}^2 f(x) &= c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(t)j_\nu(xt, q^2)t^{2\nu+1} dt \\
&= \int_0^\infty f(y) \left[ c_{q,\nu} \int_0^\infty j_\nu(xt, q^2)j_\nu(yt, q^2)t^{2\nu+1} dt \right] y^{2\nu+1} dy \\
&= \int_0^\infty f(y)\delta_q(x, y)y^{2\nu+1} dy \\
&= f(x).
\end{align*}
\]
The computations are justified by the Fubuni’s theorem: If \( p > 1 \) then we use the Hölder’s inequality
\[
\int_0^\infty |f(y)| \left[ \int_0^\infty |j_\nu(xt, q^2)j_\nu(yt, q^2)|t^{2\nu+1} dt \right] y^{2\nu+1} dy \\
\leq \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} dy \right]^{1/p} \times \left[ \int_0^\infty \sigma(y)^\overline{p} y^{2\nu+1} dy \right]^{1/\overline{p}}.
\]
The numbers \( p \) and \( \overline{p} \) above are conjugates and
\[
\sigma(y) = \int_0^\infty |j_\nu(xt, q^2)j_\nu(yt, q^2)|t^{2\nu+1} dt,
\]
then
\[
\begin{align*}
\int_0^\infty \sigma(y)^\overline{p} y^{2\nu+1} dy \\
= \int_0^1 \sigma(y)^\overline{p} y^{2\nu+1} dy + \int_1^\infty \sigma(y)^\overline{p} y^{2\nu+1} dy.
\end{align*}
\]
Note that
\[
\int_0^1 \sigma(y) \overline{F_{q,\nu}} y^{2\nu+1} dy 
\]
\[
\leq \|j_\nu(.,q^2)\|_{q,\infty} \int_0^1 \left[ \int_0^\infty |j_\nu(xt, q^2)| t^{2\nu+1} dt \right] \overline{F_{q,\nu}} y^{2\nu+1} dy 
\]
\[
\leq \|j_\nu(.,q^2)\|_{q,\infty} \|j_\nu(.,q^2)\|_{q,1,\nu}^{-2(\nu+1)} \left[ \int_0^1 y^{2\nu+1} dy \right] < \infty,
\]
and
\[
\int_1^\infty \sigma(y) \overline{F_{q,\nu}} y^{2\nu+1} dy 
\]
\[
\leq \|j_\nu(.,q^2)\|_{q,\infty} \|j_\nu(.,q^2)\|_{q,1,\nu} \int_1^\infty y^{2\nu+1} \overline{F_{q,\nu}} y^{2\nu+1} dy 
\]
\[
\leq \|j_\nu(.,q^2)\|_{q,\infty} \|j_\nu(.,q^2)\|_{q,1,\nu} \int_1^\infty \frac{1}{y^{2(\nu+1)-1}} dy < \infty.
\]

If \( p = 1 \) then
\[
\int_0^\infty \|f(y)\| \left[ \int_0^\infty |j_\nu(xt, q^2) j_\nu(yt, q^2)| t^{2\nu+1} dt \right] y^{2\nu+1} dy 
\]
\[
\leq \|f\|_{q,1,\nu} \|j_\nu(.,q^2)\|_{q,\infty} \|j_\nu(.,q^2)\|_{q,1,\nu} \times \frac{1}{x^{2(\nu+1)}}.
\]

\[ \square \]

**Theorem 2.** Let \( f \) be a function in the \( L_{q,1,\nu} \cap L_{q,p,\nu} \) space, where \( p > 2 \) then
\[
\|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}.
\]

**Proof.** Let \( f \in L_{q,1,\nu} \cap L_{q,p,\nu} \) then by Theorem 1 we see that
\[
\mathcal{F}_{q,\nu}^2 f = f.
\]
This implies
\[
\int_0^\infty \mathcal{F}_{q,\nu} f(x)^2 x^{2\nu+1} dx = \int_0^\infty \mathcal{F}_{q,\nu} f(x) \left[ c_{q,\nu} \int_0^\infty f(t) j_\nu(xt, q^2) t^{2\nu+1} dt \right] x^{2\nu+1} dx 
\]
\[
= \int_0^\infty f(t) \left[ c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(x) j_\nu(xt, q^2) x^{2\nu+1} dx \right] t^{2\nu+1} dt 
\]
\[
= \int_0^\infty f(t)^2 t^{2\nu+1} dt.
\]
The computations are justified by the Fubuni’s theorem
\[
\int_0^\infty |f(t)| \left[ c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| j_\nu(xt, q^2) x^{2\nu+1} dx \right] t^{2\nu+1} dt 
\]
\[
\leq \left[ \int_0^\infty |f(t)| t^{2\nu+1} dt \right]^{1/p} \times \left[ \int_0^\infty |\phi(t)| \overline{F_{q,\nu}} t^{2\nu+1} dt \right]^{1/\overline{F_{q,\nu}}},
\]
where
\[
\phi(t) = c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| j_\nu(xt, q^2) x^{2\nu+1} dx,
\]
then
\[
\|F_{q,\nu}f(x)\| \leq c_{q,\nu} \int_0^\infty |f(y)| |j_{\nu}(xy, q^2)| y^{2\nu+1} \, dy \\
\leq c_{q,\nu} \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} \, dy \right]^{1/p} \times \left[ \int_0^\infty |j_{\nu}(xy, q^2)| y^{2\nu+1} \, dy \right]^{1/p} \\
\leq c_{q,\nu} \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} \, dy \right]^{1/p} \times \left[ \int_0^\infty |j_{\nu}(y, q^2)| y^{2\nu+1} \, dy \right]^{1/p} x^{-2(\nu+1)/p} \\
\leq c_{q,\nu} \|f\|_{q,p,\nu} \|j_{\nu}(\cdot, q^2)\|_{q,\nu} x^{-2(\nu+1)/p}.
\]

This gives
\[
\phi(t) \leq c_{q,\nu}^2 \|f\|_{q,p,\nu} \|j_{\nu}(\cdot, q^2)\|_{q,\nu} \int_0^\infty |j_{\nu}(xt, q^2)| x^{2(\nu+1)-2(\nu+1)/p} \, dx \\
\leq c_{q,\nu}^2 \|f\|_{q,p,\nu} \|j_{\nu}(\cdot, q^2)\|_{q,\nu} \left[ \int_0^\infty |j_{\nu}(x, q^2)| x^{2(\nu+1)/p} \, dx \right] t^{-2(\nu+1)/p} \\
\leq C_1 t^{-2(\nu+1)/p},
\]

and
\[
\phi(t) = c_{q,\nu} \int_0^\infty |F_{q,\nu}f(x)||j_{\nu}(xt, q^2)| x^{2\nu+1} \, dx \\
= \left[ c_{q,\nu} \int_0^\infty |F_{q,\nu}f(x/t)||j_{\nu}(x, q^2)| x^{2\nu+1} \, dx \right] t^{-2(\nu+1)} \\
\leq c_{q,\nu} \|F_{q,\nu}f\|_{q,\nu} \times \|j_{\nu}(\cdot, q^2)\|_{q,\nu} x^{-2(\nu+1)/p} \\
\leq C_2 t^{-2(\nu+1)/p}.
\]

Note that
\[
\begin{cases}
-1 < -2(\nu+1) \frac{p}{p} + 2\nu + 1 \\
-2(\nu+1) \frac{p}{p} + 2\nu + 1 < -1
\end{cases} \iff \begin{cases}
0 < -2(\nu+1)(\frac{p}{p} - 2) \\
-2(\nu+1)(\frac{p}{p} - 1) < 0
\end{cases} \iff 1 < p < 2 \iff p > 2.
\]

Hence
\[
\int_0^\infty |\phi(t)|^{\frac{2}{\nu}+1} d_q t = \int_1^1 |\phi(t)|^{\frac{2}{\nu}+1} d_q t + \int_1^\infty |\phi(t)|^{\frac{2}{\nu}+1} d_q t \\
\leq C_1 \int_1^1 t^{-2(\nu+1)/p} \pi^{\frac{2}{\nu}+1} d_q t + C_2 \int_1^\infty t^{-2(\nu+1)} \pi^{\frac{2}{\nu}+1} d_q t < \infty,
\]

which prove the result.

\[ \square \]

**Theorem 3.** Let \( f \) be a function in the \( L_{q,\nu} \) space then
\[
\|F_{q,\nu}f\|_{q,\nu} = \|f\|_{q,\nu}.
\]

**Proof.** We introduce the function \( \psi_x \), as follows
\[
\psi_x(t) = c_{q,\nu} j_{\nu}(tx, q^2).
\]

The inner product \( \langle \cdot, \cdot \rangle \) in the Hilbert space \( L_{q,\nu} \) is defined by
\[
f, g \in L_{q,\nu} \Rightarrow \langle f, g \rangle = \int_0^\infty f(t)g(t)t^{2\nu+1} \, dt.
\]

Using (1) we write
\[
x \neq y \Rightarrow \langle \psi_x, \psi_y \rangle = 0.
\]
\[ \|\psi_x\|_{q,2,\nu}^2 = \frac{1}{1-q} x^{-2(\nu+1)}. \]

We have
\[ \mathcal{F}_{q,\nu} f(x) = \langle f, \psi_x \rangle, \]
and by Theorem 1
\[ f \in \mathcal{L}_{q,2,\nu} \Rightarrow \mathcal{F}_{q,\nu} f = f, \]
then
\[ \langle f, \psi_x \rangle = 0, \forall x \in \mathbb{R}^+_q \Rightarrow \mathcal{F}_{q,\nu} f(x) = 0, \forall x \in \mathbb{R}^+_q \Rightarrow f = 0. \]

Hence, \( \{\psi_x, x \in \mathbb{R}^+_q\} \) form an orthogonal basis of the Hilbert space \( \mathcal{L}_{q,2,\nu} \) and we have
\[ \{\psi_x, x \in \mathbb{R}^+_q\} = \mathcal{L}_{q,2,\nu}. \]

Now
\[ f \in \mathcal{L}_{q,2,\nu} \Rightarrow \mathcal{F}_{q,\nu} f = \sum_{x \in \mathbb{R}^+_q} \frac{1}{\|\psi_x\|_{q,2,\nu}^2} \langle f, \psi_x \rangle \psi_x, \]
and then
\[ \|f\|_{q,2,\nu}^2 = \sum_{x \in \mathbb{R}^+_q} \|\psi_x\|_{q,2,\nu}^2 \|\langle f, \psi_x \rangle\|^2 = (1-q) \sum_{x \in \mathbb{R}^+_q} x^{2(\nu+1)} \mathcal{F}_{q,\nu} f(x)^2 = \|\mathcal{F}_{q,\nu} f\|_{q,2,\nu}^2, \]
which achieve the proof. \( \square \)

**Proposition 2.** Let \( f \in \mathcal{L}_{q,p,\nu} \) where \( p \geq 1 \) then \( \mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,p,\nu} \). If \( 1 \leq p \leq 2 \) then
\[ \|\mathcal{F}_{q,\nu} f\|_{q,p,\nu} \leq B_{q,\nu}^{-1} \|f\|_{q,p,\nu}. \]

**Proof.** This is an immediate consequence of Proposition 1, Theorem 3, the Riesz-Thorin theorem and the inversion formula (2). \( \square \)

The \( q \)-translation operator is given as follow
\[ T_{q,x}^\nu f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(t) j_\nu(yt, q^2) j_\nu(xt, q^2) t^{2\nu+1} dt. \]

Let us now introduce
\[ Q_\nu = \{ q \in ]0,1[, \ T_{q,x}^\nu \text{ is positive for all } x \in \mathbb{R}^+_q \} \]
the set of the positivity of \( T_{q,x}^\nu \). We recall that \( T_{q,x}^\nu \) is called positive if \( T_{q,x}^\nu f \geq 0 \) for \( f \geq 0 \). In a recent paper [6] it was proved that if \( -1 < \nu < \nu' \) then \( Q_\nu \subset Q_{\nu'} \).

As a consequence :
\begin{itemize}
  \item [-]: If \( 0 \leq \nu \) then \( Q_\nu = ]0,1]. \)
  \item [-]: If \( -\frac{1}{2} \leq \nu < 0 \) then \( ]0,q_0[ \subset Q_{-\frac{1}{2}} \subset Q_\nu \subset ]0,1[, \quad q_0 \simeq 0.43. \)
  \item [-]: If \( -1 < \nu \leq -\frac{1}{2} \) then \( Q_\nu \subset Q_{-\frac{1}{2}}. \)
\end{itemize}

**Theorem 4.** Let \( f \in \mathcal{L}_{q,p,\nu} \) then \( T_{q,x}^\nu f \) exists and we have
\[ \int_0^\infty T_{q,x}^\nu f(y) y^{2\nu+1} dy = \int_0^\infty f(y) y^{2\nu+1} dy. \]

and
\[ T_{q,x}^\nu f(y) = \int_0^\infty f(z) D_\nu(x, y, z) z^{2\nu+1} dz, \]

\[ \|\psi_x\|_{q,2,\nu}^2 = \frac{1}{1-q} x^{-2(\nu+1)}. \]
where

\[ D_\nu(x, y, z) = c_{q, \nu}^2 \int_0^\infty j_\nu(xs, q^2 j_\nu(ys, q^2 j_\nu(zs, q^2))s^{2\nu + 1} dq. \]

If we suppose that \( T_{q, x}^\nu \) is a positive operator then for all \( p \geq 1 \) we have

\[ \|T_{q, x}^\nu f\|_q, \nu \leq \|f\|_q, \nu. \]  

(5)

Proof. We write the operator \( T_{q, x}^\nu \) in the following form

\[ T_{q, x}^\nu f(y) = c_{q, \nu} \int_0^\infty F_{q, \nu} f(z) j_\nu(xz, q^2) j_\nu(yz, q^2)z^{2\nu + 1} dq \]

So we have

\[ \int_0^\infty T_{q, x}^\nu f(y)y^{2\nu + 1} dq = \int_0^\infty F_{q, \nu} \left[ F_{q, \nu} f(z) j_\nu(xz, q^2) \right] (y)y^{2\nu + 1} dq \]

\[ = \frac{1}{c_{q, \nu}} \int_0^\infty F_{q, \nu} f(z) j_\nu(xz, q^2) (y) j_\nu(0, q^2)y^{2\nu + 1} dq \]

\[ = \frac{1}{c_{q, \nu}} F_{q, \nu} f(0) \]

\[ = \int_0^\infty f(y)y^{2\nu + 1} dq. \]

On the other hand

\[ T_{q, x}^\nu f(y) = c_{q, \nu} \int_0^\infty j_\nu(xz, q^2) j_\nu(yz, q^2)z^{2\nu + 1} dq \]

\[ = c_{q, \nu} \int_0^\infty \int_0^\infty j_\nu(xz, q^2) j_\nu(yz, q^2)z^{2\nu + 1} dq \]

\[ = \int_0^\infty D_{q, \nu}(x, y, t)f(t)t^{2\nu + 1} dt. \]

The computations are justified by the Fubuni’s theorem

\[ \int_0^\infty \left[ \int_0^\infty \left| f(t) \right| j_\nu(tz, q^2) \left| t^{2\nu + 1} dq \right| \right] \left| j_\nu(xz, q^2) \right| \left| j_\nu(ys, q^2) \right| z^{2\nu + 1} dz \]

\[ \leq \|f\|_{q, \nu} \int_0^\infty \left[ \int_0^\infty \left| j_\nu(tz, q^2) \right| \left| t^{2\nu + 1} dq \right| \right] \left| j_\nu(xz, q^2) \right| \left| j_\nu(ys, q^2) \right| z^{2\nu + 1} dz \]

\[ \leq \|f\|_{q, \nu} \|j_\nu(., q^2)\|_{q, \nu} \int_0^\infty \left| j_\nu(xz, q^2) \right| \left| j_\nu(ys, q^2) \right| z^{2(\nu + 1)}(1 - 1)z^{-1} dz. \]
Now suppose that $T_{q,x}^\nu$ is positive. Given a function $f \in \mathcal{C}_{q,0}$ we obtain
\[
|T_{q,x}^\nu f(y)| = \left| \int_0^\infty D_{q,y}(x,y,t) f(t) t^{2\nu+1} dt \right|
\leq \int_0^\infty |D_{q,y}(x,y,t)| |f(t)| t^{2\nu+1} dt
\leq \left[ \int_0^\infty D_{q,y}(x,y,t) t^{2\nu+1} dt \right] \|f\|_{q,\infty} = \|f\|_{q,\infty}
\]
which implies
\[
\|T_{q,x}^\nu f\|_{q,\infty} \leq \|f\|_{q,\infty}.
\]
If the function $f \in \mathcal{L}_{q,1,\nu}$ then we obtain
\[
\|T_{q,x}^\nu f\|_{q,1,\nu} = \int_0^\infty |T_{q,x}^\nu f(y)| y^{2\nu+1} dy
\leq \int_0^\infty \left[ \int_0^\infty |D_{q,y}(x,y,t)| |f(t)| t^{2\nu+1} dt \right] y^{2\nu+1} dy
\leq \int_0^\infty \left[ \int_0^\infty D_{q,y}(x,y,t) y^{2\nu+1} dy \right] |f(t)| t^{2\nu+1} dt
\leq \int_0^\infty |f(t)| t^{2\nu+1} dt = \|f\|_{q,1,\nu}.
\]
The result is a consequence of the Riesz-Thorin theorem.

Notice that the kernel $D_{q,y}(x,y,t)$ can be written as follows
\[
D_{q,y}(x,y,t) = c_{q,y}^2 \int_0^\infty j_\nu(xz,q^2) j_\nu(yz,q^2) t^{2\nu+1} dz
= c_{q,y} \mathcal{F}_{q,y} \left[ j_\nu(xz,q^2) j_\nu(yz,q^2) \right](t),
\]
which implies
\[
\int_0^\infty D_{q,y}(x,y,t) t^{2\nu+1} dt = c_{q,y} \int_0^\infty \mathcal{F}_{q,y} \left[ j_\nu(xz,q^2) j_\nu(yz,q^2) \right](t) t^{2\nu+1} dt
= \mathcal{F}_{q,y}^2 \left[ j_\nu(xz,q^2) j_\nu(yz,q^2) \right](0) = 1.
\]

The $q$-convolution product is defined by
\[
f \ast_q g = \mathcal{F}_{q,y} \left[ \mathcal{F}_{q,y} f \times \mathcal{F}_{q,y} g \right].
\]

**Theorem 5.** Let $1 \leq p, r, s$ such that
\[
\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}
\]
Given two functions $f \in \mathcal{L}_{q,p,\nu}$ and $g \in \mathcal{L}_{q,r,\nu}$ then $f \ast_q g$ exists and we have
\[
f \ast_q g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(y) g(y) y^{2\nu+1} dy.
\]
and
\[
f \ast_q g \in \mathcal{L}_{q,s,\nu}.
\]
\[
\mathcal{F}_{q,y}(f \ast_q g) = \mathcal{F}_{q,y}(f) \times \mathcal{F}_{q,y}(g).
\]
If \( s \geq 2 \) then
\[
||f * g||_{q,s,\nu} \leq B_{q,\nu} ||f||_{q,p,\nu} ||g||_{q,r,\nu}.
\] (6)

If we suppose that \( T_{q,x}^{\nu} \) is a positive operator then
\[
||f * g||_{q,s,\nu} \leq c_{q,\nu} ||f||_{q,p,\nu} ||g||_{q,r,\nu}.
\] (7)

Proof. We have
\[
f * g(x) = \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g](x)
\]
\[
= c_{q,\nu} \int_{0}^{\infty} \mathcal{F}_{q,\nu} f(y) \times \mathcal{F}_{q,\nu} g(y) j_{\nu}(xy, q^{2}) y^{2\nu+1} d_q y
\]
\[
= c_{q,\nu} \int_{0}^{\infty} \mathcal{F}_{q,\nu} f(y) \times \left[ c_{q,\nu} \int_{0}^{\infty} g(z) j_{\nu}(zy, q^{2}) z^{2\nu+1} d_q z \right] j_{\nu}(xy, q^{2}) y^{2\nu+1} d_q y
\]
\[
= c_{q,\nu} \int_{0}^{\infty} \left[ c_{q,\nu} \int_{0}^{\infty} \mathcal{F}_{q,\nu} f(y) j_{\nu}(zy, q^{2}) j_{\nu}(xy, q^{2}) y^{2\nu+1} d_q y \right] g(z) z^{2\nu+1} d_q z
\]
\[
= c_{q,\nu} \int_{0}^{\infty} T_{q,x}^{\nu} f(z) g(z) z^{2\nu+1} d_q z.
\]

The computations are justified by the Fubunin’s theorem
\[
\int_{0}^{\infty} |F_{q,\nu} f(y)| \times \left[ \int_{0}^{\infty} |g(z)| \times |j_{\nu}(zy, q^{2})| z^{2\nu+1} d_q z \right] j_{\nu}(xy, q^{2}) |y^{2\nu+1} d_q y
\]
\[
\leq ||g||_{q,r,\nu} \int_{0}^{\infty} |F_{q,\nu} f(y)| \times \left[ \int_{0}^{\infty} |j_{\nu}(zy, q^{2})| z^{2\nu+1} d_q z \right] j_{\nu}(xy, q^{2}) |y^{2\nu+1} d_q y
\]
\[
\leq ||g||_{q,r,\nu} ||j_{\nu}(., q^{2})||_{q,\pi,\nu} \int_{0}^{\infty} |F_{q,\nu} f(y)| \times \left[ |j_{\nu}(xy, q^{2})| y^{-\frac{2\nu+1}{p}} \right] y^{2\nu+1} d_q y
\]
\[
\leq ||g||_{q,r,\nu} ||j_{\nu}(., q^{2})||_{q,\pi,\nu} ||F_{q,\nu} f||_{q,\pi,\nu} \left( \int_{0}^{\infty} |j_{\nu}(xy, q^{2})| y^{-\frac{2\nu+1}{p}} \right)^{\frac{1}{p}} y^{2(\nu+1)} d_q y \right)^{\frac{1}{p}}
\]
\[
\leq ||g||_{q,r,\nu} ||j_{\nu}(., q^{2})||_{q,\pi,\nu} ||F_{q,\nu} f||_{q,\pi,\nu} \left( \int_{0}^{\infty} |j_{\nu}(xy, q^{2})| y^{2(\nu+1)(1-\frac{1}{p})-1} d_q y \right)^{\frac{1}{p}}.
\]

From Proposition\(^2\) we deduce that
\[
\mathcal{F}_{q,\nu} f \in \mathcal{L}_{q,\pi,\nu} \text{ and } \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\pi,\nu}.
\]
Then, using the Hölder inequality and the fact that
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{s}
\]
to conclude that
\[
\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g \in \mathcal{L}_{q,\pi,\nu}.
\]
Which implies that
\[
f * g = \mathcal{F}_{q,\nu} [\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g] \in \mathcal{L}_{q,s,\nu}
\]
and by the inversion formula\(^2\) we obtain
\[
\mathcal{F}_{q,\nu} (f * g) = \mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g.
\]
Suppose that \( s \geq 2 \), so \( 1 \leq s \leq 2 \) and we can write
\[
\| f * g \|_{q,s,\nu} = \| \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu} f \times \mathcal{F}_{q,\nu} g] \|_{q,s,\nu}
\]
\[
\leq B_{q,\nu}^{s-1} \| \mathcal{F}_{q,\nu} f \|_{q,\nu} \| \mathcal{F}_{q,\nu} g \|_{q,\nu}
\]
\[
\leq B_{q,\nu}^{s-1} B_{q,\nu}^{\frac{2}{\nu}} B_{q,\nu}^{\frac{2}{\nu} - 1} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}
\]
\[
\leq B_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}.
\]
Now suppose that \( T_{q,x}^\nu \) is a positive operator.
We introduce the operator \( K_f \) as follows
\[
K_f g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(z) g(z) z^{2\nu+1} d_q z.
\]
By the Hölder inequality and (5) we get
\[
\| K_f g \|_{q,\infty} \leq c_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}.
\]
The Minkowski inequality leads to
\[
\| K_f g \|_{q,p,\nu} \leq c_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,1,\nu}.
\]
Hence we have
\[
K_f : \mathcal{L}_{q,\nu} \rightarrow \mathcal{L}_{q,0}, \quad K_f : \mathcal{L}_{q,1,\nu} \rightarrow \mathcal{L}_{q,p,\nu}.
\]
Then the operator \( K_f \) satisfies
\[
K_f : \mathcal{L}_{q,r,\nu} \rightarrow \mathcal{L}_{q,s,\nu}
\]
and
\[
\| f * g \|_{q,s,\nu} = \| K_f g \|_{q,s,\nu} \leq c_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}.
\]

\[\square\]

**Remark 1.** We discuss here the sharp results for the Hausdorff-Young inequality provided above. An inequality already sharper than (6) is given in formula (7). In fact we have \( c_{q,\nu} < B_{q,\nu} \).
To obtained (7) without the positivity argument, we can do by using which is a \( q \)-Riemann-Liouville fractional integral generalizing the \( q \)-Mehler integral representation for the \( q \)-Bessel function \( j_\nu(\cdot, q^2) \) which can be proved in a straightforward way \[8\]
\[
j_\nu(\lambda, q^2) = [2\nu]_q \int_0^\infty (q^2 t^2, q^2) \int_0^\infty (q^2 t^2, q^2) j_0(\lambda t, q^2) t d_q t
\]
together with the inequalities for the \( q \)-Bessel function which is given as formula (24) in the paper \[4\]
\[
|j_0(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.
\]
Combine this formulas we arrive at
\[
|j_\nu(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+, \quad \nu \geq 0.
\]
Then the inequalities \[4\] can be written as follows
\[
\| \mathcal{F}_{q,\nu} f \|_{q,\nu} \leq c_{q,\nu}^{\frac{2}{\nu} - 1} \| f \|_{q,p,\nu}.
\]
This should give the sharpest version of (6) in the cases \( \nu \geq 0 \). Unfortunately the positivity of the operator \( T_{q,x}^\nu \) is satisfied in this case.
In fact we can prove that if we are in the positivity cases then
\[ \|j_\nu(., q^2)\|_{q, \infty} \leq 1. \]

To prove this recalling that
\[ T_{q,x}^{\nu} j_\nu(y, q^2) = j_\nu(x, q^2) j_\nu(y, q^2). \]

So we have
\[ \int_0^\infty D_{q,\nu}(x, y, t) j_\nu(t, q^2) t^{2\nu+1} d_q t = j_\nu(x, q^2) j_\nu(y, q^2). \]

We obtain for all \( x, y \in \mathbb{R}_q^+ \)
\[ |j_\nu(x, q^2)| \times |j_\nu(y, q^2)| \leq \int_0^\infty D_{q,\nu}(x, y, t) |j_\nu(t, q^2)| t^{2\nu+1} d_q t \]
\[ \leq \left[ \int_0^\infty D_{q,\nu}(x, y, t) t^{2\nu+1} d_q t \right] \|j_\nu(., q^2)\|_{q, \infty}. \]

The fact that
\[ \int_0^\infty D_{q,\nu}(x, y, t) t^{2\nu+1} d_q t = 1 \]
implies
\[ \|j_\nu(., q^2)\|_{q, \infty} \leq \|j_\nu(., q^2)\|_{q, \infty} \]
which gives the result.

3. Uncertainty Principle

We introduce two \( q \)-difference operators
\[ \partial_q f(x) = \frac{f(q^{-1}x) - f(x)}{x} \]
and
\[ \partial_q^* f(x) = \frac{f(x) - q^{2\nu+1} f(qx)}{x}. \]

Then we have
\[ \partial_q \partial_q^* f(x) = \partial_q^* \partial_q f(x) = \Delta_{q,\nu} f(x). \]

**Proposition 3.** If \( \langle \partial_q f, g \rangle \) exist and \( \lim_{a \to \infty} |a^{2\nu+1} f(q^{-1}a) g(a)| = 0 \) then
\[ \langle \partial_q f, g \rangle = - \langle f, \partial_q^* g \rangle. \]
Proof. The following computation
\[
\int_0^a \partial_q f(x)g(x)x^{2\nu+1}d_qx \\
= \int_0^a f(q^{-1}x) - f(x)g(x)x^{2\nu+1}d_qx \\
= \int_0^a f(q^{-1}x)g(x)x^{2\nu+1}d_qx - \int_0^a f(x)g(x)x^{2\nu+1}d_qx \\
= q^{2\nu+1}\int_0^a f(q^{-1}x)g(qx)x^{2\nu+1}d_qx - \int_0^a f(x)g(x)x^{2\nu+1}d_qx \\
= q^{2\nu+1}\int_0^a f(x)g(qx)x^{2\nu+1}d_qx - \int_0^a f(x)g(x)x^{2\nu+1}d_qx + a^{2\nu+1}f(q^{-1}a)g(a) \\
= -\int_0^a f(x)\partial_q^* g(x)x^{2\nu+1}d_qx + a^{2\nu+1}f(q^{-1}a)g(a)
\]
leads to the result.
\[\square\]

Corollary 1. If \(f \in L_{q,2,\nu}\) such that \(xF_{q,\nu}f \in L_{q,2,\nu}\) then
\[
\|\partial_q f\|_2 = \|xF_{q,\nu}f\|_2.
\]

Proof. In fact we have
\[
\|\partial_q f\|_2^2 = \langle \partial_q f, \partial_q f \rangle = -\langle f, \partial_q^* \partial_q f \rangle = -\langle f, \Delta_q f \rangle = -\langle F_{q,\nu}f, F_{q,\nu} \Delta_q f \rangle = \langle F_{q,\nu}f, x^2 F_{q,\nu}f \rangle = \|xF_{q,\nu}f\|_2^2,
\]
which prove the result.
\[\square\]

Theorem 6. Assume that \(f\) belongs to the space \(L_{q,2,\nu}\). Then the \(q\)-Bessel transform satisfies the following uncertainty principal
\[
\|f\|_2^2 \leq k_{q,\nu} \|xf\|_2 \|xF_{q,\nu}f\|_2
\]
where
\[
k_{q,\nu} = \left[1 + \sqrt{q} \times q^{\nu+1}\right]_{1 - q^{2(\nu+1)}}.
\]

Proof. In fact
\[
\partial_q^* xf = f(x) - q^{2\nu+2}f(qx) \\
x\partial_q f = f(q^{-1}x) - f(x).
\]
We introduce the following operator
\[
\Lambda_q f(x) = f(qx),
\]
then
\[
\langle \Lambda_q f, g \rangle = q^{-2(\nu+1)} \langle f, \Lambda_q^{-1} g \rangle.
\]
So
\[
\frac{1}{1 - q^{2(\nu+1)}} \left[ \partial_x^q x f(x) - q^{2\nu+2} \Lambda_q x \partial_q f(x) \right] = f(x)
\]
Assume that \(xf\) and \(x F_{q,\nu} f\) belongs to the space \(L_{q,2,\nu}\). Then we have
\[
\langle f, f \rangle = -\frac{1}{1 - q^{2(\nu+1)}} \langle xf, \partial_q f \rangle - \frac{1}{1 - q^{2(\nu+1)}} \langle \partial_q f, x \Lambda_q^{-1} f \rangle.
\]
By Cauchy-Schwartz inequality we get
\[
\langle f, f \rangle \leq \frac{1}{1 - q^{2(\nu+1)}} \|xf\|_2 \|\partial_q f\|_2 + \frac{1}{1 - q^{2(\nu+1)}} \|\partial_q f\|_2 \|x \Lambda_q^{-1} f\|_2.
\]
On the other hand
\[
\|x \Lambda_q^{-1} f\|_2 = \sqrt{q \times q^{\nu+1}} \|xf\|_2,
\]
Corollary [1] leads to the result. □

4. HARDY’S THEOREM

The following Lemma from complex analysis is crucial for the proof of our main theorem.

**Lemma 1.** For every \(p \in \mathbb{N}\), there exist \(\sigma_p > 0\) for which
\[
|z|^{2p} j_\nu(z, q^2) < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}.
\]

**Proof.** In fact
\[
|z|^{2p} j_\nu(z, q^2) \leq \frac{1}{(q^2, q^2)_{\infty}(q^{2\nu+2}, q^2)_{\infty}} \sum_{n=0}^{\infty} q^{n(n-1)} |z|^{2n+2p} \leq \frac{q^{p(p+1)}}{(q^2, q^2)_{\infty}(q^{2\nu+2}, q^2)_{\infty}} \sum_{n=p}^{\infty} q^{n(n-2p-1)} |z|^{2n}.
\]
Now using the Stirling’s formula
\[
n! \sim \sqrt{2\pi n} \frac{n^n}{e^n},
\]
we see that there exist an entire \(n_0 \geq p\) such that
\[
q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0,
\]
which implies
\[
\sum_{n=n_0}^{\infty} q^{n(n-2p-1)} |z|^{2n} \leq \sum_{n=n_0}^{\infty} \frac{1}{(2n)!} |z|^{2n} < e^{|z|}.
\]
Finally there exist \(\sigma_p > 0\) such that
\[
\frac{|z|^{2p} j_\nu(z, q^2)}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}
\]
This complete the proof. □
Lemma 2. Let $h$ be an entire function on $\mathbb{C}$ such that

$$|h(z)| \leq C e^{a|z|^2}, \quad z \in \mathbb{C},$$

$$|h(x)| \leq C e^{-ax^2}, \quad x \in \mathbb{R},$$

for some positive constants $a$ and $C$. Then there exist $C^* \in \mathbb{R}$ such that

$$h(x) = C^* e^{-ax^2}.$$

The reader can see the reference [17] for the proof.

Now we are in a position to state and prove the $q$-analogue of the Hardy’s theorem.

Theorem 7. Suppose $f \in L_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}^+,$$  \hspace{1cm} (8)

$$|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where $C$ is a positive constant. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu} \left( e^{-\frac{1}{2}z^2} \right)(z), \quad \forall z \in \mathbb{C}.$$

Proof. We claim that $\mathcal{F}_{q,\nu}f$ is an analytic function and there exist $C' > 0$ such that

$$|\mathcal{F}_{q,\nu}f(z)| \leq C'e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.$$

We have

$$|\mathcal{F}_{q,\nu}f(z)| \leq c_{q,\nu} \int_0^\infty |f(x)||j_{\nu}(zx,q^2)|x^{2\nu+1}d_qx.$$  \hspace{1cm} (9)

From the Lemma[1] if $|z| > 1$ then there exist $\sigma_1 > 0$ such that

$$x^{2\nu+1}|j_{\nu}(zx,q^2)| = \frac{1}{|z|^{2\nu+1}} \left( \frac{|z|}{x} \right)^{2\nu+1} \frac{1}{1+\left( \frac{|z|}{x} \right)^2} \frac{1}{e^{\frac{1}{2}x^2}} < \frac{\sigma_1}{1+|z|^2|x|^2} e^{\frac{1}{2}|z|^2}, \quad \forall x \in \mathbb{R}^+.$$  \hspace{1cm} (10)

Then we obtain

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_1 c_{q,\nu} \left[ \int_0^\infty \frac{e^{-\frac{1}{2}(x-|z|)^2}}{1+|z|^2x^2}d_qx \right] e^{\frac{1}{2}|z|^2} < C\sigma_1 c_{q,\nu} \left[ \int_0^\infty \frac{1}{1+x^2}d_qx \right] e^{\frac{1}{2}|z|^2}.$$  \hspace{1cm} (11)

Now, if $|z| \leq 1$ then there exist $\sigma_2 > 0$ such that

$$x^{2\nu+1}|j_{\nu}(zx,q^2)| \leq \sigma_2 e^x, \quad \forall x \in \mathbb{R}^+.$$  \hspace{1cm} (12)

Therefore

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_2 c_{q,\nu} \left[ \int_0^\infty e^{-\frac{1}{2}x^2+z^2}d_qx \right] \leq C\sigma_2 c_{q,\nu} \left[ \int_0^\infty e^{-\frac{1}{2}x^2+x}d_qx \right] e^{\frac{1}{2}|z|^2},$$

which leads to the estimate (11). Using Lemma[2] we obtain

$$\mathcal{F}_{q,\nu}f(z) = \text{const}. e^{-\frac{1}{2}z^2}, \quad \forall z \in \mathbb{C},$$

and by Theorem[1] we conclude that

$$f(z) = \text{const.} \mathcal{F}_{q,\nu} \left( e^{-\frac{1}{2}z^2} \right)(z), \quad \forall z \in \mathbb{C}.$$
Corollary 2. Suppose \( f \in \mathcal{L}_{q,1,\nu} \) satisfying the following estimates
\[
|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,
\]
\[
|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},
\]
where \( C, p, \sigma \) are a positive constant and \( ps = \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}_q^+ \) such that \( a^2 p = \frac{1}{2} \). Then there exist \( A \in \mathbb{R} \) such that
\[
f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\sigma t^2}\right)(z), \quad \forall z \in \mathbb{C}.
\]
Proof. Let \( a \in \mathbb{R}_q^+ \), and put
\[
f_a(x) = f(ax),
\]
then
\[
\mathcal{F}_{q,\nu}f_a(x) = \frac{1}{a^{2\nu+2}}\mathcal{F}_{q,\nu}f(x/a).
\]
In the end, applying Theorem 7 to the function \( f_a \). \( \square \)

Corollary 3. Suppose \( f \in \mathcal{L}_{q,1,\nu} \) satisfying the following estimates
\[
|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,
\]
\[
|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},
\]
where \( C, p, \sigma \) are a positive constant and \( ps > \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}_q^+ \) such that \( a^2 p = \frac{1}{2} \). Then \( f \equiv 0 \).

Proof. In fact there exists \( \sigma' < \sigma \) such that \( ps' = \frac{1}{4} \). Then the function \( f \) satisfying the estimates of Corollary 2 if we replacing \( \sigma \) by \( \sigma' \). Which implies
\[
\mathcal{F}_{q,\nu}f(x) = \text{const.} e^{-\sigma' x^2}, \quad \forall x \in \mathbb{R}.
\]
On the other hand, \( f \) satisfying the estimates (9), then
\[
\left|\text{const.} e^{-\sigma' x^2}\right| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}.
\]
This implies \( \mathcal{F}_{q,\nu}f \equiv 0 \), and by Theorem 1 we conclude that \( f \equiv 0 \). \( \square \)

5. The \( q \)-Fourier-Neumann Expansions

The little \( q \)-Jacobi polynomials are defined for \( \nu, \beta > -1 \) by [15]
\[
p_n(x; q^\nu, q^\beta; q) = 2\phi_1\left(q^{n+\nu+\beta+1}, q^{-n}; q, qx\right).
\]
We define the functions
\[
P_{q,\nu}(x; q^2) = \sigma_{q,\nu}(n)q^{-n(\nu+1)}\frac{(q^{2+2n}, q^{2\nu+2}; q^2)_\infty}{(q^{2+2n+2\nu}, q^{2\nu+2}; q^2)_\infty}p_n(x^2; q^{2\nu}, 1; q^2)
\]
and
\[
J_{q,\nu}(x; q^2) = \sigma_{q,\nu}(n)\frac{J_{q^{2+2n+1}(q^n x; q^2)}}{x^{\nu+1}},
\]
where
\[
\sigma_{q,\nu}(n) = \sqrt{\frac{1 - q^{2\nu+4n+2}}{1 - q}}.
\]
Consider \( \mathcal{L}^{\nu}_{q,2} \) as an Hilbert space with the inner product
\[
(f|g) = \int_{0}^{1} f(x)g(x)x^{2\nu+1}d_{q}x.
\]
The \( q \)-Paley-Wiener space is defined by
\[
PW^{\nu}_{q} = \left\{ f \in \mathcal{L}_{q,2,\nu} : f(x) = c_{q,\nu} \int_{0}^{1} u(t)j_{\nu}(xt,q)\,t^{2\nu+1}d_{q}t, \quad u \in L^{\nu}_{q,2} \right\}.
\]

**Proposition 4.** \( PW^{\nu}_{q} \) is a closed subspace of \( \mathcal{L}_{q,2,\nu} \) and with the inner product given in (3) is an Hilbert space.

**Proof.** In fact, given \( f \in \mathcal{L}_{q,2,\nu} \) and let \( \{ f_{n} \}_{n \in \mathbb{N}} \) be a sequence of element of \( PW^{\nu}_{q} \) which converge to \( f \) in \( L^{2} \)-norm. For \( n \in \mathbb{N} \), there exist \( u_{n} \in \mathcal{L}^{\nu}_{q,2} \) such that
\[
f_{n}(x) = c_{q,\nu} \int_{0}^{1} u_{n}(t)j_{\nu}(xt,q)\,t^{2\nu+1}d_{q}t.
\]
Moreover
\[
\lim_{n \to \infty} \| f_{n} - f \|_{q,2,\nu} = 0.
\]
This give
\[
\lim_{n \to \infty} \| F_{q,\nu}f_{n} - F_{q,\nu}f \|_{q,2,\nu} = 0,
\]
and then
\[
\lim_{n \to \infty} \left[ \int_{0}^{1} |F_{q,\nu}f_{n}(x) - F_{q,\nu}f(x)|^{2}x^{2\nu+1}d_{q}x + \int_{1}^{\infty} |F_{q,\nu}f(x)|^{2}x^{2\nu+1}d_{q}x \right] = 0,
\]
which implies
\[
\int_{1}^{\infty} |F_{q,\nu}f(x)|^{2}x^{2\nu+1}d_{q}x = 0 \Rightarrow F_{q,\nu}f(x) = 0, \quad \forall x \in \mathbb{R}_{q}^{+} \cap [1, +\infty[.
\]
Then \( f \in PW^{\nu}_{q} \).

**Proposition 5.** We have
\[
F_{q,\nu}(J_{\nu,n})(x) = P_{\nu,n}(x; q^{2})\chi_{[0,1]}(x), \quad \forall x \in \mathbb{R}_{q}^{+}.
\]
As a consequence
\[
\int_{0}^{1} P_{\nu,n}(x; q^{2})P_{\nu,m}(x; q^{2})x^{2\nu+1}d_{q}x = \delta_{n,m},
\]

**Proof.** The following proof is identical to the proof of Lemma 1 in [1]. Using an identity established in [12] [13]
\[
\int_{0}^{\infty} t^{-\lambda}J_{\mu}(q^{m}t; q^{2})J_{\mu}(q^{n}t; q^{2})d_{q}t
\]
\[
= (1 - q)q^{\nu}q^{(\lambda-1)+(m-n)\mu}(1-\frac{\lambda+\theta+\mu}{\lambda+\theta+\mu}; q^{2})_{\infty}
\]
\[
\times 2\phi_{1}\left( \frac{q^{1-\lambda+\mu+\theta}}{q^{2}}; q^{2m-2n+1+\lambda+\theta-\mu} \right)_{\infty},
\]
where \( n, m \in \mathbb{Z} \) and \( \theta, \mu, \lambda \in \mathbb{C} \) such that \( \text{Re}(1 - \lambda + \theta + \mu) > 0 \), \( \theta, \mu \) are not equal to a negative integer and
\[
(\lambda + \theta + 1 - \mu)/2, \quad m - n + (\lambda + \theta + 1 - \mu)/2
\]
are not a non-positive integer $[13]$. To evaluate $F_{n,m}(\mathcal{J}_{\nu,n})(x)$ when $x = q^n \leq 1$, we take in $[10]$ $q^m = x, \mu = \nu, \theta = \nu + 2n + 1, \lambda = 0$
then
$$F_{q,\nu}(\mathcal{J}_{\nu,n})(x) = \sigma_{q,\nu}(n)\frac{x^{-\nu}}{1-q} \int_0^\infty J_{\nu}(xt; q^2)J_{\nu+2n+1}(q^\nu t; q^2) \, dt$$
$$= \sigma_{q,\nu}(n)q^{-n(\nu+1)}\frac{(q^{2+2n}, q^{2\nu+2}; q^2)_{\infty}}{(q^{2+2n+2\nu}, q^2, q^2)_{\infty}} \phi_1\left(\frac{q^{2+2n+2\nu}, q^{-2n}}{q^2, q^2 x^2}\right)$$
$$= P_{\nu,n}(x; q^2).$$
To evaluate $F_{q,\nu}(\mathcal{J}_{\nu,m})(x)$ when $x = q^n > 1$, we consider in $[10]$
$$q^m = x, \mu = \nu + 2m + 1, \theta = \nu, \lambda = 0$$
In this way, $1 + \lambda + \theta - \mu = -2m$. This gives for $m \in \mathbb{N}$ a factor
$$(q^{-2m}; q^2)_{\infty} = 0$$
on the numerator and then
$$F_{q,\nu}(\mathcal{J}_{\nu,m})(x) = 0, \quad x > 1$$
By setting $\lambda = 1, \theta = \nu + 2n + 1, \mu = \nu + 2m + 1$ in , it is clear that, for $n, m = 0, 1, 2, \ldots$,
$$\int_0^\infty J_{\nu+2n+1}(q^nx; q^2)J_{\nu+2m+1}(q^\nu x; q^2) \frac{dq^2}{x} = \frac{1}{\sigma_{q,\nu}(n)^2} \delta_{n,m}$$
and then
$$\int_0^\infty J_{\nu,n}(x; q^2)J_{\nu,m}(x; q^2) x^{2\nu+1} \, dq^2 x = \delta_{n,m}.$$
Now we use the arguments of $q$-Bessel Fourier analysis provided in this paper to show that
$$\langle P_{\nu,n} \chi_{[0,1]}, P_{\nu,m} \chi_{[0,1]} \rangle = \langle F_{q,\nu}(\mathcal{J}_{\nu,n}), F_{q,\nu}(\mathcal{J}_{\nu,m}) \rangle = \langle \mathcal{J}_{\nu,n}, \mathcal{J}_{\nu,m} \rangle = \delta_{n,m}. \quad (11)$$
Another proof of the orthogonality of the little $q$-Jacobi polynomials can be found in $[13]$.

**Proposition 6.** The systems
$$\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty, \quad \{P_{\nu,n}\}_{n=0}^\infty$$
form two orthonormals basis respectively of the Hilbert spaces $PW^v_q$ and $L^v_{q,2}$.

**Proof.** From (11) we derive the orthonormalization. To prove that the system $\{\mathcal{J}_{\nu,n}\}_{n=0}^\infty$ is complet in $PW^v_q$, given a function $f \in PW^v_q$ such that
$$\langle f, \mathcal{J}_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$Then
$$\langle F_{q,\nu}(f), \mathcal{J}_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N},$$which implies
$$\langle F_{q,\nu}(f), P_{\nu,n} \chi_{[0,1]} \rangle = \langle F_{q,\nu}(f) \chi_{[0,1]}, P_{\nu,n} \rangle = \langle F_{q,\nu}(f), P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}.$$From the definition of the polynomial $P_{\nu,n}$ we conclude that
$$\langle F_{q,\nu}(f), t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}.\]
Then
\[ c_{q,\nu} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2\nu+2}, q^2)_n} \langle F_{q,\nu}(f), t^{2n} \rangle x^{2n} = 0, \quad \forall x \in \mathbb{R}_q^+, \]
which can be written as
\[ F_{q,\nu}^2(f)(x) = 0, \quad \forall x \in \mathbb{R}_q^+. \]
By the inversion formula (2) we conclude that \( f = 0 \). From (11) we derive the orthonormality. To prove that the system \( \{P_{\nu,n}\}_{n=0}^{\infty} \) is complete in \( L_{\nu, q}^2 \), given a function \( f \in L_{\nu, q}^2 \) such that
\[ \langle f | P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N} \]
Then
\[ \langle f | t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}. \]
Which leads to the result. \( \square \)

**Proposition 7.** Let \( \lambda \in \mathbb{R}_q^+ \) then
\[ c_{q,\nu} j_{\nu}(\lambda x; q^2) = \sum_{n=0}^{\infty} J_{n,\nu}(\lambda; q^2) P_{\nu,n}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+. \]
As a consequence we have
\[ \sum_{n=0}^{\infty} [P_{\nu,n}(x; q^2)]^2 = \frac{x^{-2(\nu+1)}}{1-q}, \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+ \]
and for all \( \lambda \in \mathbb{R}_q^+ \)
\[ \sum_{n=0}^{\infty} [J_{n,\nu}(\lambda; q^2)]^2 = -\frac{q^{\nu'}}{2(1-q)^{\lambda+2\nu}} \times \left[ \frac{\lambda}{q} J_{\nu+1}(\lambda; q^2) J'_{\nu}(\lambda/q; q^2) - J_{\nu+1}(\lambda; q^2) J_{\nu}(\lambda/q; q^2) - J_{\nu+1}(\lambda; q^2) J_{\nu}(\lambda/q; q^2) \right]. \]

**Proof.** Let \( \lambda \in \mathbb{R} \) and consider the function
\[ \psi_{\lambda}: [0, 1] \cap \mathbb{R}_q^+ \to \mathbb{R}, \quad x \mapsto c_{q,\nu} j_{\nu}(\lambda x; q^2). \]
Then \( \psi_{\lambda} \in \mathcal{L}_{\nu, q}^2 \) and we can write
\[ \psi_{\lambda}(x) = \sum_{n=0}^{\infty} \langle \psi_{\lambda} | P_{\nu,n} \rangle P_{\nu,n}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}_q^+. \]  
(12)
Note that
\[ \langle \psi_{\lambda} | P_{\nu,n} \rangle = \langle \psi_{\lambda}, P_{\nu,n} \chi_{[0,1]} \rangle = \langle \psi_{\lambda}, F_{q,\nu}(J_{n,\nu}) \rangle = F_{q,\nu}^2(J_{n,\nu})(\lambda) = J_{n,\nu}(\lambda; q^2). \]
Then we deduce the result. Using the Parseval’s theorem and (12) we obtain
\[ \sum_{n=0}^{\infty} [P_{\nu,n}(x; q^2)]^2 = \|\psi_{\lambda}\|^2_{q, 2, \nu} = \frac{x^{-2(\nu+1)}}{1-q}. \]
The second identity is deduced also from the Parseval’s theorem
\[ \sum_{n=0}^{\infty} [J_{n,\nu}(\lambda; q^2)]^2 = N_{q, \nu, 2}^2(\psi_{\lambda}), \]
and the following relation proved in [14]
\[
\int_0^1 \left[ J_\nu(aqt; q^2) \right]^2 \, tdq = -\frac{(1-q)q^{\nu-1}}{2a} \times \left[ aJ_{\nu+1}(aq; q^2)J'_{\nu}(a; q^2) - J_{\nu+1}(aq; q^2)J'_{\nu}(a; q^2) \right].
\]

References