FIXED POINT THEOREMS FOR SET-VALUED GENERALIZED
ASYMPTOTIC CONTRACTIONS

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Abstract. The purpose of this paper is to obtain some coincidence and fixed
point theorems for a generalized hybrid pair of single-valued and set-valued
non continuous maps. Our results generalize some recent results.

1. Introduction

Kirk [14] introduced a new class of maps known as asymptotic contractions on a
metric space and obtained a fixed point theorem (see Definition 1.1 and Theorem
1.2) below.

Definition 1.1. Let \((X, d)\) be a metric space. A self-map \(T\) of \(X\) is an asymptotic
contraction on \(X\) if

\[
d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \quad \text{for} \quad x, y \in X,
\]

where \(\varphi\) is a continuous function, from \([0, \infty)\) into itself, \(\varphi(t) < t\) for all \(t > 0\)
and \(\{\varphi_n\}\) is a sequence of functions from \([0, \infty)\) into itself such that \(\{\varphi_n\} \to \{\varphi\}\)
uniformly on the range of \(d\).

Theorem 1.2. Let \((X, d)\) be a complete metric space and \(T\) an asymptotic contrac-
tion on \(X\) with \(\{\varphi_n\}\) and \(\varphi\) as in Definition 1.1. Assume that there exists \(x \in X\)
such that the orbit \(\{T^n x : n \in \mathbb{N}\}\) of \(x\) is bounded, and that \(\varphi_n\) is continuous for
\(n \in \mathbb{N}\). Then there exists a unique fixed point \(z \in X\). Moreover \(\lim_{n \to \infty} T^n x = z\) for all
\(x \in X\).

Remark 1.3. We remark that:

(1) Theorem 1.2 is an asymptotic version of Boyd and Wong contraction [4]
(see [12]).
(2) Jachymski and Jóźwiec [12] showed that the continuity of the map \(T\) is
essential for the conclusion of Theorem 1.2 to hold.
(3) In respect of Definition 1.1 it has been observed (cf. [1, 12, 21, 22]) that
\(\varphi(0) = 0\).

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(4) For the equivalent formulation of Theorem 1.2 in topological spaces with the so called TCS-convergence, we refer to Tasković [24, 25].

Subsequently many extensions and generalizations of Theorem 1.2 appeared (see, for instance, [1–5, 13, 16, 18–23, 26, 27]). Underlying the power and importance of this new class of maps, Briseid [5, 7] has observed that a continuous self-map of a compact metric space satisfying any one of the first 50 contractive conditions listed by Rhoades [17] is an asymptotic contraction.

Recently, Fakhar [10] and Wlodarczyk et al. [26, 27] extended Kirk’s asymptotic contraction to set-valued maps and obtained some endpoint theorems for such contractions. In [26, 27] some applications of the theory of asymptotic contractions to the analysis of set-valued dynamical systems are also discussed. On the other hand, a generalization of the well known Banach contraction principle due to Meir-Keeler [15] has been of continuing interest in fixed point theory. Recently Suzuki [21] combined the ideas of Meir-Keeler contraction and Kirk’s asymptotic contraction and introduced the following notion of asymptotic contraction of Meir-Keeler type.

**Definition 1.4.** Let $(X, d)$ be a metric space. A self-map $T$ of $X$ is called an asymptotic contraction of Meir-Keeler type if there exists a sequence $\varphi_n$ of functions from $[0, \infty)$ into itself satisfying the following conditions:

\begin{align*}
(S1): & \quad \lim \sup \varphi_n(\varepsilon) \leq \varepsilon \text{ for all } \varepsilon \geq 0; \\
(S2): & \quad \text{for each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ and } \nu \in \mathbb{N} \text{ such that } \varphi_\nu(t) \leq \varepsilon \text{ for all } t \in [\varepsilon, \varepsilon + \delta]; \\
(S3): & \quad d(T^n x, T^n y) < \varphi_n(d(x, y)), \text{ for all } n \in \mathbb{N} \text{ and } x, y \in X \text{ with } x \neq y.
\end{align*}

In this paper first we introduce the notion of set-valued generalized asymptotic contraction of Meir-Keeler type, which includes the known notions of asymptotic contractions due to Kirk [14], Suzuki [21] and Fakhar [10] (see Example 2.7 for illustration). Subsequently, this notion is utilized to obtain some coincidence and fixed point theorems for such contractions which generalize, and unify several known results including [10], [26] and others.

**2. Generalized asymptotic contractions**

Throughout this section, $Y$ denotes an arbitrary nonempty set, $(X, d)$ a metric space, $CB(X)$ the collection of all nonempty closed bounded subsets of $X$, $\varphi_n$ as in Definition 1.4 and $H$ the Hausdorff metric induced by $d$, i.e.,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A) \right\},$$

for all $A, B \subseteq CB(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

We denote by $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Further, let

$$m(x, y) : = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)] \right\};$$

$$M(x, y) : = \max \left\{ d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] \right\}.$$

Now, we introduce the notion of set-valued generalized asymptotic contraction of Meir-Keeler type as follows.
**Definition 2.1.** Let \((X,d)\) be a metric space \(f : Y \to X\) and \(T : Y \to CB(X)\). The map \(T\) will be called a \textit{generalized asymptotic contraction of Meir-Keeler type} with respect to \(f\) if the following hold:

1. \((G1): \limsup_n \varphi_n(\varepsilon) \leq \varepsilon\) for all \(\varepsilon \geq 0\);
2. \((G2):\) for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\varphi_k(t) < \varepsilon\) for all \(t \in [\varepsilon, \varepsilon + \delta]\) and \(k \in \mathbb{N}\);
3. \((G3): H(T^n x, T^n y) < \varphi_n(M(x, y))\) for all \(n \in \mathbb{N}\) and \(x, y \in Y\) with \(M(x, y) > 0\).

As a special case of the above definition, we have the following:

**Definition 2.2.** Let \((X,d)\) be a metric space and \(T : X \to CB(X)\). The map \(T\) will be called a \textit{generalized asymptotic contraction of Meir-Keeler type} if the following hold:

- \(\limsup_n \varphi_n(\varepsilon) \leq \varepsilon\) for all \(\varepsilon \geq 0\);
- for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\varphi_k(t) < \varepsilon\) for all \(t \in [\varepsilon, \varepsilon + \delta]\) and \(k \in \mathbb{N}\);
- \(H(T^n x, T^n y) < \varphi_n(m(x, y))\) for all \(n \in \mathbb{N}\) and \(x, y \in X\) with \(m(x, y) > 0\).

The following theorem is our main result.

**Theorem 2.3.** Let \((X,d)\) be a metric space, \(f : Y \to X\) and \(T : Y \to CB(X)\) such that \(TY \subseteq fY\). Let \(T\) be a generalized asymptotic contraction of Meir-Keeler type with respect to \(f\).

If \(T(Y)\) or \(f(Y)\) is a complete subspace of \(X\) then \(T\) and \(f\) have a coincidence point.

Further, if \(Y = X\), then \(T\) and \(f\) have a common fixed point provided that \(ffu = fu\) and \(T\) and \(f\) commute at a coincidence point.

**Proof.** Pick \(x_0 \in Y\). We construct a sequence \(\{x_n\}\) in the following manner. Since \(TY \subseteq fY\), we may choose a point \(x_1 \in Y\) such that \(fx_1 \in Tx_0\). If \(Tx_0 = Tx_1\) then \(x_1 = z\) is a coincidence point of \(T\) and \(f\) and we are done. So assume that \(Tx_0 \neq Tx_1\) and choose \(x_2 \in Y\) such that \(fx_2 \in Tx_1\) and

\[
d(fx_1, fx_2) \leq H(Tx_0, Tx_1).
\]

If \(Tx_1 = Tx_2\), i.e., \(x_2\) is a coincidence point of \(T\) and \(f\), we are done. If not continuing in the same manner we have

\[d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).\]

By \((G3)\),

\[d(fx_n, fx_{n+1}) \leq H(Tx_{n-1}, Tx_n) < \varphi_n(M(x_0, x_1)).\]

First we show that

\[
\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0. \tag{1}
\]

It initially holds if \(x_1 = x_2\). In the other case of \(x_1 \neq x_2\), we assume that

\[\alpha := \limsup_n d(fx_{n+1}, fx_{n+2}) > 0.\]

From the condition \((G2)\), we can choose \(k \in \mathbb{N}\) satisfying \(\varphi_k(d(fx_1, fx_2)) < d(fx_1, fx_2)\). By \((G3)\) and \((G1)\),

\[d(fx_{k+1}, fx_{k+2}) \leq H(Tx_k, Tx_{k+1}) < \varphi_k(M(x_0, x_1)) < M(x_1, x_2). \tag{2}\]
Now, we have
\[
\alpha : = \lim_{n \to \infty} \sup d(f_{x_{k+n+1}}, f_{x_{k+n+2}}) \leq \lim_{n \to \infty} \sup H(T_{x_{k+n}}, T_{x_{k+n+1}})
\]
\[
\leq \lim_{n \to \infty} \sup \varphi_n(M(x_k, x_{k+1})) \leq M(x_k, x_{k+1})
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_k}, T_{x_k}), d(f_{x_{k+1}}, T_{x_{k+1}}),
\frac{1}{2}[d(f_{x_k}, T_{x_{k+1}}) + d(f_{x_{k+1}}, T_{x_k})]
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_k}, f_{x_{k+2}}), d(f_{x_{k+1}}, f_{x_{k+2}}),
\frac{1}{2}[d(f_{x_k}, f_{x_{k+1}}) + d(f_{x_{k+1}}, f_{x_{k+2}})]\}
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_{k+1}}, f_{x_{k+2}})\}.
\]

If
\[
\max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_{k+1}}, f_{x_{k+2}})\} = d(f_{x_k}, f_{x_{k+2}})
\]
then
\[
d(f_{x_{k+1}}, f_{x_{k+2}}) \leq H(T_{x_k}, T_{x_{k+1}})
\]
\[
< \varphi_1(M(x_k, x_{k+1})) < M(x_k, x_{k+1})
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_k}, T_{x_k}), d(f_{x_{k+1}}, T_{x_{k+1}}),
\frac{1}{2}[d(f_{x_k}, T_{x_{k+1}}) + d(f_{x_{k+1}}, T_{x_k})]\}
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_k}, f_{x_{k+2}}), d(f_{x_{k+1}}, f_{x_{k+2}}),
\frac{1}{2}[d(f_{x_k}, f_{x_{k+1}}) + d(f_{x_{k+1}}, f_{x_{k+2}})]\}
\]
\[
= \max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_{k+1}}, f_{x_{k+2}})\} = d(f_{x_k}, f_{x_{k+1}}),
\]

a contradiction. Therefore
\[
\max\{d(f_{x_k}, f_{x_{k+1}}), d(f_{x_{k+1}}, f_{x_{k+2}})\} = d(f_{x_k}, f_{x_{k+1}})
\]
and we conclude that \( M(x_k, x_{k+1}) = d(f_{x_k}, f_{x_{k+1}}) \).

By (2),
\[
d(f_{x_{k+2}}, f_{x_{k+3}}) \leq H(T_{x_{k+1}}, T_{x_{k+2}})
\]
\[
< \varphi_k(M(x_1, x_2)) < M(x_1, x_2)
\]
\[
= \max\{d(f_{x_1}, f_{x_2}), d(f_{x_1}, T_{x_2}), d(f_{x_1}, T_{x_2}),
\frac{1}{2}[d(f_{x_1}, T_{x_2}) + d(f_{x_1}, T_{x_2})]\}
\]
\[
= d(f_{x_1}, f_{x_2}).
\]

So \( \alpha < d(f_{x_1}, f_{x_2}) \). By a similar argument, we obtain \( \alpha < d(f_{x_{k+1}}, f_{x_{k+2}}) \) for all \( k \in \mathbb{N} \). Hence \( \{d(f_{x_n}, f_{x_{n+1}})\} \) converges to \( \alpha \).

Since \( 0 < \alpha < d(f_{x_1}, f_{x_2}) < \infty \), there exists \( \delta_2 > 0 \) and \( l \in \mathbb{N} \) such that
\[
\varphi(t) \leq \alpha \text{ for all } t \in [\alpha, \alpha + \delta_2].
\]
We choose $p \in \mathbb{N}$ with $d(f_{x_{p+1}}, f_{x_{p+2}}) < \alpha + \delta_2$. Then we have
\[ d(f_{x_{t+p+1}}, f_{x_{t+p+2}}) \leq H(T f_{x_{t+p}}, T f_{x_{t+p+1}}) < \varphi d(f_{x_t}, f_{x_{t+1}}) \leq \alpha, \]
a contradiction. This proves that $\lim d(f_{x_n}, f_{x_{n+1}}) = 0$. Now following the proof of Theorem 3.1 [20], it can be easily shown that $\{f x_n\}$ is a Cauchy sequence.

Suppose $f(Y)$ is complete. Then $\{fx_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it $z$. Let $u \in f^{-1} z$. Then $fu = z$. Using (G2),
\[ d(fu, Tu) \leq H(Tx_n, Tu) < \varphi_1(M(u, x_n)) = \varphi_1(\max\{d(fu, fx_n), d(fu, Tu), d(fu, Tx_n), \frac{1}{2}d(fu, Tx_n) + d(fx_n, Tu)\}). \]
Making $n \to \infty$, $d(fu, Tu) \leq \varphi_1(d(fu, Tu)) < d(fu, Tu)$. This yields $fu \in Tu$.

Further, if $Y = X$, $ffu = fu$, and the maps $f$ and $T$ commute at their coincidence point $u$ then $fu \in fTu \subseteq Tfu$ and $fu$ is a common fixed point of $f$ and $T$.

In case $TY$ is a complete subspace of $X$, the condition $TY \subseteq fY$ implies that the sequence $\{fx_n\}$ converges in $fY$ and the previous argument works. □

**Remark 2.4.** We remark that a set-valued asymptotic contraction of Meir-Keeler type is the set-valued generalized contraction of Meir-Keeler type when $m(z, y) = d(x, y)$. Further it includes the set-valued asymptotic contraction given in [10] and [20].

Now in the view of Definition 2.3 and the above remark we have the following corollaries.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ a generalized asymptotic contraction of Meir-Keeler type. Then $T$ has a fixed point in $X$.

**Corollary 2.6.** Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ an asymptotic contraction of Meir-Keeler type. Then $T$ has a fixed point in $X$.

The following example shows the generality of Theorem 2.3 over [26, Th. 2.1] and [10, Th. 2.3].

**Example 2.7.** Let $Y = (-\infty, \infty)$ and $X = [0, \infty)$ endowed with the usual metric $d$. Let $f : Y \to X$ and $T : Y \to CB(X)$ be defined by
\[ fx = \begin{cases} -2x & \text{if } x < 0, \\ 2x & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad Tx = \begin{cases} \{-x\} & \text{if } x < 0, \\ [0, x] & \text{if } 0 \leq x \leq 1, \\ \{x\} & \text{if } x > 1 \end{cases} \]
for all $x \in Y$. Let $\varphi_n(t) = \frac{3}{4}t$ for $t > 0$.

Then for $x > 1$ and $y > 1$,
\[ H(T^n x, T^n y) = |x - y| > \frac{3}{4} |x - y| = \varphi_n(d(x, y)), \]
and the contractive condition of Theorem 2.3 [10] is not satisfied.

Further, $\delta(T^n([0, 1])) = \delta([0, 1])$ and condition (d) of Theorem 2.1 [26] is not satisfied. It can be verified that the maps $f$ and $T$ satisfy all the hypotheses of Theorem
Notice that $TY \subseteq fY$ and $f$ and $T$ commute at 0. Hence $f0 \in T0$ is a common fixed point of $f$ and $T$.

REFERENCES

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