DOUBLE SINE SERIES AND HIGHER ORDER LIPSCHITZ CLASSES OF FUNCTIONS

(COMMUNICATED BY HÜSEYN BOR)

DANDAN HAN, GUOCHENG LI, AND DANSHENG YU

Abstract. In the present paper, we generalize the double Lipschitz classes and double Zygmund classes of functions in two variables to the so-called double higher order Lipschitz classes, and give the necessary and sufficient conditions for double sine series belonging to the generalized higher order Lipschitz classes.

1. Introduction

Given a double sequence \( \{a_{jk}; j, k = 1, 2, \ldots \} \) of nonnegative numbers satisfying
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} < \infty, \tag{1}
\]
then the following double sine series
\[
f(x, y) := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky,
\]
is continuous, due to uniform convergence.

Let \( \omega(h, k) \) be a modulus of continuity, that is, \( \omega(h, k) \) is a continuous function on the square \([0, 2\pi] \times [0, 2\pi]\), nondecreasing in each variable, and possessing the following properties:
\[
\omega(0, 0) = 0, \\
\omega(t_1 + t_2, t_3) \leq \omega(t_1, t_3) + \omega(t_2, t_3), \\
\omega(t_1, t_2 + t_3) \leq \omega(t_1, t_2) + \omega(t_1, t_3).
\]

Yu (3) introduced the following classes of functions:
\[
HH^\omega := \{ f(x, y) : \| f(x, y) - f(x + h, y) - f(x, y + k) + f(x + h, y + k) \| = O(\omega(h, k)), h, k > 0 \}.
\]
When \( \omega(u, v) = u^\alpha v^\beta \), \( 0 < \alpha, \beta \leq 1 \), then \( HH^\omega \) becomes the well known double Lipschitz class \( Lip(\alpha, \beta) \). Yu [3] investigated the necessary and sufficient conditions for the double trigonometric series belonging to \( HH^\omega \). In fact, some of his results can be read as follows:

**Theorem 1.** If
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i j a_{ij} = O \left( m n \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
\[
\sum_{i=1}^{m} \sum_{j=n}^{\infty} i a_{ij} = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
\[
\sum_{i=m}^{\infty} \sum_{j=1}^{n} j a_{ij} = O \left( n \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
\[
\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} a_{ij} = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
for \( m, n = 1, 2, \ldots \), then \( f(x, y) \in HH^\omega \).

**Theorem 2.** If \( f(x, y) \in HH^\omega \), then
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i j a_{ij} = O \left( m n \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right).
\]

**Theorem 3.** If \( \{ \omega \left( \frac{1}{m}, \frac{1}{n} \right) \} \) satisfies the following conditions
\[
\sum_{i=m}^{\infty} i^{-1} \omega \left( \frac{1}{i}, \frac{1}{n} \right) = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
\[
\sum_{j=n}^{\infty} j^{-1} \omega \left( \frac{1}{m}, \frac{1}{j} \right) = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
\[
\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} i^{-1} j^{-1} \omega \left( \frac{1}{i}, \frac{1}{j} \right) = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]
for all \( m, n = 1, 2, \ldots \), then \( f(x, y) \in HH^\omega \) if and only if
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i j a_{ij} = O \left( m n \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right).
\]

For any \( f(x,y) \in C(T^2) \), and \( r, s = 1, 2, \ldots \), the \( (r, s) \)-th difference of \( f(x,y) \) at \( x \) with stepsize \( h \) and at \( y \) with stepsize \( k \), is defined by
\[
\Delta^{r,s}(f; x, y; h, k) := \sum_{\mu=0}^{r} \sum_{\gamma=0}^{s} (-1)^{r+s-\mu-\gamma} \binom{r}{\mu} \binom{s}{\gamma} f(x + \mu h, y + \gamma k).
\]
Define the double higher order Lipschitz classes \( \Lambda^\omega_{r,s} \) and the double higher order lipschitz classes \( \lambda^\omega_{r,s} \) as follows:
\[
\Lambda^\omega_{r,s} := \{ f(x,y) \in C(T^2) : \| \Delta^{r,s}(f; x, y; h, k) \| = O(\omega(h, k)), \ h > 0, k > 0 \},
\]
\[
\lambda^\omega_{r,s} := \{ f(x,y) \in C(T^2) : \| \Delta^{r,s}(f; x, y; h, k) \| = o(\omega(h, k)), \ h > 0, k > 0 \}.
\]
Clearly, if $r = s = 1$, $\Lambda_{r,s}^\omega$ reduces to the class $HH^\omega$, and if $r = s = 2$, $\Lambda_{r,s}^\omega$ reduces to the double Zygmund class $ZZ^\omega$, while if $r = 1, s = 2$, $\Lambda_{r,s}^\omega$ is the Lipschitz-Zygmund class $HZ^\omega$ (see [3] for the definitions of $ZZ^\omega$ and $HZ^\omega$, they are generalizations of the multiplicative Zygmund class $Zy(\alpha, \beta)$ and the multiplicative Lipschitz-Zygmund class, respectively).

Our main purpose is to generalize Theorem 1-Theorem 3 to the double higher Lipschitz classes $\Lambda_{r,s}^\omega$ (see Theorem A-Theorem C below). Our results also generalize some well known results considering single trigonometric series from Lipscitz class and Zygmund class to the higher order Lipschitz classes.

2. Main results

In what follows, we always assume that $\{a_{jk}\}$ is a double sequence of nonnegative numbers satisfying (1). We first give a sufficient condition for $f \in \Lambda_{r,s}^\omega$.

**Theorem A.** If

$$\sum_{j=1}^{m} \sum_{k=1}^{n} j^r k^s a_{jk} = O \left(m^n \omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

$$\sum_{j=1}^{m} \sum_{k=n+1}^{\infty} j^r a_{jk} = O \left(m^n \omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

$$\sum_{j=m+1}^{\infty} \sum_{k=1}^{n} k^s a_{jk} = O \left(n^s \omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

$$\sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} a_{jk} = O \left(\omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

then $f \in \Lambda_{r,s}^\omega$.

We have the following necessary conditions for $f \in \Lambda_{r,s}^\omega$ :

**Theorem B.** If $f \in \Lambda_{r,s}^\omega$ then

$$\sum_{j=1}^{m} \sum_{k=1}^{n} j^{r^*} k^{s^*} a_{jk} = O \left(m^n \omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

where $r^* := \begin{cases} r+1 & \text{if } r \text{ is even}, \\ r & \text{if } r \text{ is odd} \end{cases}$, $s^* := \begin{cases} s+1 & \text{if } s \text{ is even}, \\ s & \text{if } s \text{ is odd} \end{cases}$.

If $\omega(\delta, \eta)$ satisfies some further conditions, we can obtain the necessary and sufficient conditions for $f \in \Lambda_{r,s}^\omega$. In fact, we have the following:

**Theorem C.** (i). If $r$ and $s$ are all odd, $\omega \left(\frac{1}{m}, \frac{1}{n}\right)$ satisfies

$$\sum_{j=m}^{\infty} \frac{1}{j} \omega \left(\frac{1}{j}, \frac{1}{n}\right) = O \left(\omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

$$\sum_{k=n}^{\infty} \frac{1}{k} \omega \left(\frac{1}{m}, \frac{1}{k}\right) = O \left(\omega \left(\frac{1}{m}, \frac{1}{n}\right)\right),$$
for all $m, n = 1, 2, \ldots$, then $f \in \Lambda_{r,s}^\omega$ if and only if (2.1) holds.

(ii). If $r$ is even and $s$ is odd, $\omega \left( \frac{1}{m}, \frac{1}{n} \right)$ satisfies (2.5),(2.6) and
\[
\sum_{j=1}^{m} j^{r-1} \omega \left( \frac{1}{j}, 1 \right) n = O \left( m^{r-\alpha} n^{s-\beta} \right),
\]
then $f \in \Lambda_{r,s}^\omega$ if and only if (2.3) holds.

(iii). If $r$ is odd and $s$ is even, and $\omega \left( \frac{1}{m}, \frac{1}{n} \right)$ satisfies (2.5),(2.6) and
\[
\sum_{k=1}^{n} k^{s-1} \omega \left( 1, \frac{1}{k} \right) m = O \left( n^{s-\alpha} m^{r-\beta} \right),
\]
then $f \in \Lambda_{r,s}^\omega$ if and only if (2.2) holds.

(iv). If $r$ and $s$ are all even , $\omega \left( \frac{1}{m}, \frac{1}{n} \right)$ satisfies (2.5)-(2.8), then $f \in \Lambda_{r,s}^\omega$ if and only if (2.4) holds.

Now, we give some useful corollaries of Theorem C.

**Corollary A.** Assume that there are $\mu_1, \nu_1$ ($\mu_1, \nu_1 > 0$) such that $\{m^{\mu_1} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ and $\{n^{\nu_1} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ are almost decreasing on $m$ and $n$ respectively, then

(i). If $r$ and $s$ are all odd, $f \in \Lambda_{r,s}^\omega$ if and only if (2.1) holds.

(ii). If $r$ is even and $s$ is odd, and there is a $\mu_2$ ($0 < \mu_2 < r$) such that $\{m^{\mu_2} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ is almost increasing on $m$ , then $f \in \Lambda_{r,s}^\omega$ if and only if (2.3) holds.

(iii). If $r$ is odd and $s$ is even, and there is a $\nu_2$ ($0 < \nu_2 < s$) such that $\{n^{\nu_2} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ is almost increasing on $n$, then $f \in \Lambda_{r,s}^\omega$ if and only if (2.2) holds.

(iv). If $r$ and $s$ are all even , and there are $\mu_3, \nu_3$ ($0 < \mu_3 < r, 0 < \nu_3 < s$) such that $\{m^{\mu_3} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ and $\{n^{\nu_3} \omega \left( \frac{1}{m}, \frac{1}{n} \right)\}$ are almost increasing on $m$ and $n$ respectively, then $f \in \Lambda_{r,s}^\omega$ if and only if (2.4) holds.

**Corollary B.** (i). If $r$ and $s$ are all odd, and $\omega(\delta, \eta) = \delta^\alpha \eta^\beta$ ($0 < \alpha \leq r, 0 < \beta \leq s$), then $f \in \Lambda_{r,s}^\omega$ if and only if
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} j^r k^s a_{jk} = O \left( m^{r-\alpha} n^{s-\beta} \right).
\]
(ii). If $r$ is even and $s$ is odd, and $\omega(\delta, \eta) = \delta^\alpha \eta^\beta$ ($0 < \alpha < r, 0 < \beta \leq s$), then $f \in \Lambda_{r,s}^\omega$ if and only if
\[
\sum_{j=m}^{\infty} \sum_{k=1}^{n} k^s a_{jk} = O \left( m^{-\alpha} n^{s-\beta} \right).
\]
(iii). If $r$ is odd and $s$ is even, and $\omega(\delta, \eta) = \delta^\alpha \eta^\beta$ ($0 < \alpha \leq r, 0 < \beta < s$), then $f \in \Lambda_{r,s}^\omega$ if and only if
\[
\sum_{j=1}^{m} \sum_{k=n}^{\infty} j^r a_{jk} = O \left( m^{r-\alpha} n^{-\beta} \right).
\]
If $r$ and $s$ are all even, and $\omega(\delta, \eta) = \delta^\alpha \eta^\beta (0 < \alpha < r, 0 < \beta < s)$, then $f \in \Lambda^\omega_{r,s}$ if and only if
\[
\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} a_{jk} = O \left( m^{-\alpha} n^{-\beta} \right).
\]

**Remark.** When ‘$O$’ is replaced by ‘$o$’, and $\Lambda^\omega_{r,s}$ is replaced by $\lambda^\omega_{r,s}$, the corresponding results still hold. When $r = s = 2$, our results also generalize the corresponding results in [4].

### 3. Auxiliary Results

**Lemma 1.** When $r = 2m, m = 1, 2, \ldots$, we have
\[
\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sin k(x + jh) = 2^m (\cos kh - 1)^m \sin k(x + mh).
\] (10)

When $r = 2m - 1, m = 1, 2, \ldots$, we have
\[
\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sin k(x + jh) = 2^m (\cos kh - 1)^{m-1} \cos k(x + mh - \frac{h}{2}) \sin \frac{kh}{2}.
\] (11)

**Proof.** First, we have (Móricz ([2])) for $m = 1, 2, \ldots, t \in \mathbb{R}$, that
\[
S_{2m-1} := \sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} e^{i(m-j)t} = 2^{m-1} (\cos t - 1)^{m-1} (e^{it} - 1),
\] (12)
and
\[
S_{2m} := \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} e^{i(m-j)t} = 2^m (\cos t - 1)^m.
\] (13)

When $r = 2m$, by (3.4), we have
\[
\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sin k(x + jh) = \Im \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} e^{ik(x+jh)}
\]
\[
= \Im \left( \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} e^{i(m-j)(-kh)} e^{i(mkh+lx)} \right)
\]
\[
= 2^m (\cos kh - 1)^m \Im \left( e^{i(mkh+lx)} \right)
\]
\[
= 2^m (\cos kh - 1)^m \sin k(x + mh),
\]
which prove (3.1).
Lemma 3. If implies \( \sum \) transformation, we conclude that Lemma 2. If \( \square \) which prove (3.2).

Proof. \( \text{implies (3.5)-(3.7).} \)

Let \( M < M, 1 \leq m < M, 1 \leq n < N \), by Abel’s transformation, we have

\[
\sum_{j=0}^{r} (-1)^{r-j} {r \choose j} \sin k(x + jh) = Im \sum_{j=0}^{r} (-1)^{r-j} {r \choose j} e^{ik(x+jh)}
\]

\[
= Im \left( 2^{m-1} (1-j) {r \choose j} e^{i(-kh)} e^{i(kx+mh)} \right)
\]

\[
= 2^{m-1} (\cos kh - 1)^{m-1} Im \left( e^{i(kx+mh)} - e^{i(kx+mh)} \right)
\]

\[
= 2^{m-1} (\cos kh - 1)^{m-1} \cos k(x + mh - \frac{h}{2}) \sin \frac{kh}{2},
\]

which prove (3.2). \( \square \)

Lemma 2. If \( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \) satisfies \( \square \), \( \square \), then for any \( \delta \geq r, \eta \geq s \),

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} i^{j} j^{n} a_{jk} = O \left( m^{\delta}, n^{\eta} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right) \quad (14)
\]

implies

\[
\sum_{j=1}^{m} \sum_{k=n+1}^{\infty} j^{\delta} a_{jk} = O \left( m^{\delta}, \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right), \quad (15)
\]

\[
\sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} k^{\eta} a_{jk} = O \left( n^{\delta}, \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right), \quad (16)
\]

\[
\sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} a_{jk} = O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right). \quad (17)
\]

Lemma 2 can be proved in a way similar to that of Lemma 3.1 in \( \square \).

Lemma 3. If \( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \) satisfies \( \square \) and \( \square \), then, for any \( \delta \geq r, \eta \geq s \), (3.8)

implies (3.5)-(3.7).

Proof. Let \( M \) and \( N \) be integers for which \( 1 \leq m < M, 1 \leq n < N \), by Abel’s transformation, we conclude that

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^{\delta} j^{n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} (j_{1}^{\eta} - (j_{1} - 1)^{\eta}) \sum_{j=j_{1}}^{N} a_{ij} - n^{\eta} \sum_{j=n+1}^{N} a_{ij}
\]

\[
\leq \sum_{j_{1}=1}^{n} \eta j_{1}^{\eta-1} \sum_{j=j_{1}}^{N} a_{ij}
\]

\[
= \sum_{j_{1}=1}^{n} \eta j_{1}^{\eta-1} \sum_{j=j_{1}}^{N} \left( \sum_{i_{1}=1}^{m} (i_{1}^{\delta} - (i_{1} - 1)^{\delta}) \sum_{i=i_{1}}^{M} a_{ij} - m^{\delta} \sum_{i=m+1}^{M} a_{ij} \right)
\]

\[
\leq \sum_{j_{1}=1}^{n} \eta j_{1}^{\eta-1} \sum_{i_{1}=1}^{m} \delta i_{1}^{\delta-1} \sum_{j=j_{1}}^{N} \sum_{i=i_{1}}^{M} a_{ij}.
\]
Letting $M$ and $N$ tend to $\infty$, by (3.8), we have

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^{\delta} j^{\eta} a_{ij} = O \left( \sum_{i_1=1}^{m} \sum_{j_1=1}^{n} i_1^{\delta-1} j_1^{\eta-1} \omega \left( \frac{1}{i_1}, \frac{1}{j_1} \right) \right)
\]

\[
= O \left( m^{\delta-r} n^{\eta-s} \sum_{i_1=1}^{m} \sum_{j_1=1}^{n} i_1^{r-1} j_1^{s-1} \omega \left( \frac{1}{i_1}, \frac{1}{j_1} \right) \right)
\]

\[
= O \left( m^{\delta} n^{\eta-s} \sum_{j_1=1}^{n} j_1^{s-1} \omega \left( \frac{1}{i_1}, \frac{1}{j_1} \right) \right)
\]

\[
= O \left( m^{\delta} n^{\eta} \omega \left( 1, \frac{1}{j_1} \right) \right),
\]

which proves (3.5).

By Abel’s transformation again, we have

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^{\delta} a_{ij} \leq \sum_{j=n}^{\infty} \left( \sum_{i_1=1}^{m} (i_1^\delta - (i_1 - 1)^\delta) \sum_{i=i_1}^{M} a_{ij} - m^{\delta} \sum_{i=m+1}^{M} a_{ij} \right)
\]

\[
\leq \sum_{i_1=1}^{m} \delta i_1^{\delta-1} \sum_{i=i_1}^{M} a_{ij}.
\]

Letting $M$ tend to $\infty$, by (2.7), we have

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} i^{\delta} a_{ij} = O \left( m^{\delta-r} \sum_{i_1=1}^{m} i_1^{r-1} \omega \left( \frac{1}{i_1}, \frac{1}{n} \right) \right)
\]

\[
= O \left( m^{\delta} \omega \left( 1, \frac{1}{n} \right) \right).
\]

which proves (3.6).

In a similar way to the proof of (3.6), we have (3.7).

Remark 2. When ‘$O$’ is replaced by ‘$o$’, the corresponding results of Lemma 2-Lemma 5 still hold.

Analogue to Lemma 3, we have the following lemmas.

**Lemma 4.** If $\omega \left( \frac{1}{n}, \frac{1}{n} \right)$ satisfies (6) and (9), then, for any $\delta \geq r, \eta \geq s$ (15) implies (14), (16) and (17).

**Lemma 5.** If $\omega \left( \frac{1}{m}, \frac{1}{n} \right)$ satisfies (7), (8), then, for any $\delta \geq r, \eta \geq s$, (16) implies (14), (15) and (17).

Remark 2. When ‘$O$’ is replaced by ‘$o$’, the corresponding results of Lemma 2-Lemma 5 still hold.
4. Proof of the Theorem

Proof of Theorem A. Write \( m := \left[ \frac{1}{\delta} \right] \), \( n := \left[ \frac{1}{\eta} \right] \) for given \( \delta > 0, \eta > 0 \). Direct calculations yield that

\[
|\Delta^{r,s}(f; x, y; \delta, \eta)| = \left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sum_{\mu=0}^{r} (-1)^{r-\mu} \binom{r}{\mu} \sin j(x + \mu \delta) \sum_{s=0}^{s} (-1)^{s-\gamma} \binom{s}{\gamma} \sin k(y + \gamma \eta) \right|
\]

By (2), we have

\[
|\Delta^{r,s}(f; x, y; \delta, \eta)| \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} |e^{ijx} e^{iky} (1 - e^{ij \delta})^r (1 - e^{ik \eta})^s|
\]

By (3) and (4), we have

\[
|\Delta^{r,s}(f; x, y; \delta, \eta)| \leq 2^{r+s} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \left| \sin \frac{j \delta}{2} \right| \left| \sin \frac{k \eta}{2} \right|^{s}
\]

\[
=: S_1 + S_2 + S_3 + S_4.
\]

By (2), we have

\[
S_1 \leq \delta^r \eta^s \sum_{j=1}^{m} \sum_{k=1}^{n} j^r k^s a_{jk} = O(\omega(\delta, \eta)).
\]

By (3) and (4), we have

\[
S_2 \leq 2^s \delta^r \sum_{j=1}^{m} \sum_{k=n+1}^{\infty} j^s a_{jk} = O(\omega(\delta, \eta)),
\]

and

\[
S_3 \leq 2^r \eta^s \sum_{j=m+1}^{\infty} \sum_{k=1}^{n} k^s a_{jk} = O(\omega(\delta, \eta)),
\]

respectively. Finally, by (4), we have

\[
S_4 \leq 2^{r+s} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} a_{jk} = O(\omega(\delta, \eta)).
\]

Combining all the above estimates, the proof of Theorem A is complete.

Proof of Theorem B. We prove the result by considering the following many cases.

Case 1. \( r \) and \( s \) are both odd, say \( r = 2m_0 - 1 \) for some \( m_0 = 1, 2, ..., s = 2n_0 - 1 \) for some \( n_0 = 1, 2, ..., \). Since \( f \in \Lambda^{\omega}_{r,s} \), by (3.2), there exists a constant \( C \) such that

\[
|\Delta^{r,s}(f; x, y; \delta, \eta)| = 2^{m_0+n_0} \left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} (1 - \cos j \delta)^{m_0 - 1} (1 - \cos k \eta)^{n_0 - 1} \right| \times
\]

\[
\left| \sin \left( \frac{j \delta}{2} \right) \cos j \left( x + \left( m_0 - \frac{1}{2} \right) \delta \right) \sin \left( \frac{k \eta}{2} \right) \cos k \left( y + \left( n_0 - \frac{1}{2} \right) \eta \right) \right| \leq C \omega(\delta, \eta), \quad \delta > 0, \eta > 0.
\]
Noting that $f$ is uniformly convergent (due to (11)), we can integrate both sides of the above inequality with respects to $x$ on $(-m_0 \delta, -(m_0 - \frac{1}{2}) \delta)$ and $y$ on $(-n_0 \eta, -(n_0 - \frac{1}{2}) \eta)$ to obtain that

$$2^{2(m_0+n_0)} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \sin^2 m_0 \frac{j \delta}{2} \sin^2 n_0 \frac{k \eta}{2} \delta, \eta)dx \leq \left| \int_{-m_0 \delta}^{-(m_0 - \frac{1}{2}) \delta} \int_{-n_0 \eta}^{-(n_0 - \frac{1}{2}) \eta} \Delta^{r,s}(f; x, y; \delta, \eta)dx \right|$$

By using the well known inequality

$$\sin t \geq \frac{2t}{\pi}, \quad 0 \leq t \leq \frac{\pi}{2}, \quad (19)$$

and (18), we obtain

$$2^{2(m_0+n_0)} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{a_{jk}}{jk} \left( \frac{j \delta}{2} \right)^{2m_0} \left( \frac{k \eta}{2} \right)^{2n_0} \leq C \delta \eta \omega(\delta, \eta), \quad \delta > 0, \eta > 0,$$

where $m := \left[ \frac{1}{2} \right], n := \left[ \frac{1}{2} \right]$. Hence,

$$\sum_{j=1}^{m} \sum_{k=1}^{n} j^r k^s a_{jk} = O \left( m^r n^s \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),$$

which proves Theorem B in the case when $r$ and $s$ are all odd.

**Case 2.** $r$ is odd, $s$ is even, say $r = 2m_0 - 1$ for some $m_0 = 1, 2, ..., s = 2n_0$ for some $n_0 = 1, 2, ....$ Since $f \in \Delta_{r,s}^{r,s}$, by Lemma 1, there exists a constant $C$ such that

$$|\Delta^{r,s}(f; x, y; \delta, \eta)| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} (1 - \cos j \delta)^{m_0 - 1} (1 - \cos k \eta)^{n_0} \sin \left( \frac{j \delta}{2} \right) \cos j \left( x + \left( m_0 - \frac{1}{2} \right) \delta \right) \sin k (y + n_0 \eta)$$

$$\leq C \omega(\delta, \eta), \quad \delta > 0, \eta > 0.$$

By integrating both sides of the above inequality with respects to $x$ on $(-m_0 \delta, -(m_0 - \frac{1}{2}) \delta)$ and $y$ on $(-n_0 \eta, -(n_0 - \frac{1}{2}) \eta)$, we have

$$2^{2(m_0+n_0)} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \sin^2 m_0 \frac{j \delta}{2} \sin^2 n_0 \frac{k \eta}{2} + 2 \cdot \left| \int_{-m_0 \delta}^{-(m_0 - \frac{1}{2}) \delta} \int_{-n_0 \eta}^{-(n_0 - \frac{1}{2}) \eta} \Delta^{r,s}(f; x, y; \delta, \eta)dx \right|$$

By using the well known inequality

$$\sin t \geq \frac{2t}{\pi}, \quad 0 \leq t \leq \frac{\pi}{2}, \quad (19)$$

and (18), we obtain

$$2^{2(m_0+n_0)} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{a_{jk}}{jk} \left( \frac{j \delta}{2} \right)^{2m_0} \left( \frac{k \eta}{2} \right)^{2n_0} \leq C \delta \eta \omega(\delta, \eta), \quad \delta > 0, \eta > 0,$$
By (4.2) and (20), we have

$$
2^{2(m_0+n_0)} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{a_{jk}}{jk} \left( \frac{j\delta}{2} \right)^{2m_0} \left( \frac{k\eta}{2} \right)^{2n_0+2} \leq C\delta\eta\omega(\delta, \eta), \quad \delta > 0, \eta > 0,
$$

where \( m := \left\lfloor \frac{1}{\delta} \right\rfloor \), \( n := \left\lfloor \frac{1}{\eta} \right\rfloor \). Hence,

$$
\sum_{j=1}^{m} \sum_{k=1}^{n} j^r k^{s+1} a_{jk} = O \left( m^r n^{s+1} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
$$

which proves Theorem B in the case when \( r \) is odd and \( s \) is even.

**Case 3.** \( r \) is even, \( s \) is odd. By similar discussion to Case 2, we see that Theorem B holds in this case.

**Case 4.** \( r \) and \( s \) are both even, say \( r = 2m_0 \) for some \( m_0 = 1, 2, \ldots \), \( s = 2n_0 \) for some \( n_0 = 1, 2, \ldots \). Since \( f \in \Lambda_{r,s}^\omega \), there exists a constant \( C \) such that

$$
|\Delta^{r,s}(f; x, y; \delta, \eta)| = 2^{2m_0+n_0} \left| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} (1 - \cos j\delta)^{m_0} (1 - \cos k\eta)^{n_0} \sin j(x + m_0\delta) \sin k(y + n_0\eta) \right|
$$

$$
= 2^{2(m_0+n_0)} \left| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \sin^{2m_0} \frac{j\delta}{2} \sin^{2n_0} \frac{k\eta}{2} \sin j(x + m_0\delta) \sin k(y + n_0\eta) \right|
$$

$$
\leq C\omega(\delta, \eta), \quad \delta > 0, \eta > 0.
$$

By integrating both sides of the above inequality with respects to \( x \) on \((-m_0\delta, -m_0\delta + \delta)\) and \( y \) on \((-n_0\eta, -n_0\eta + \eta)\), we have

$$
2^{2(m_0+n_0)} \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \sin^{2m_0+2} \frac{j\delta}{2} \sin^{2n_0+2} \frac{k\eta}{2} = \left| \int_{-m_0\delta}^{-m_0\delta+\delta} \int_{-n_0\eta}^{-n_0\eta+\eta} \Delta^{r,s}(f; x, y; \delta, \eta) dxdy \right|
$$

$$
\leq \int_{-m_0\delta}^{-m_0\delta+\delta} \int_{-n_0\eta}^{-n_0\eta+\eta} |\Delta^{r,s}(f; x, y; \delta, \eta)| dxdy
$$

$$
\leq C \int_{-m_0\delta}^{-m_0\delta+\delta} \int_{-n_0\eta}^{-n_0\eta+\eta} \omega(\delta, \eta) dxdy
$$

$$
\leq C\delta\eta\omega(\delta, \eta), \quad \delta > 0, \eta > 0.
$$

(21)

By (4.2) and (21), we have

$$
2^{2(m_0+n_0)} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{a_{jk}}{jk} \left( \frac{j\delta}{2} \right)^{2m_0+2} \left( \frac{k\eta}{2} \right)^{2n_0+2} \leq C\delta\eta\omega(\delta, \eta), \quad \delta > 0, \eta > 0,
$$

where \( m := \left\lfloor \frac{1}{\delta} \right\rfloor \), \( n := \left\lfloor \frac{1}{\eta} \right\rfloor \). Hence,

$$
\sum_{j=1}^{m} \sum_{k=1}^{n} j^{r+1} k^{s+1} a_{jk} = O \left( m^{r+1} n^{s+1} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
$$

which proves Theorem B in the case when \( r \) and \( s \) are both even.

We prove Theorem B by combining the results of Case 1-Case 4.
Proof of Theorem C. (i) The necessity follows from Theorem B, while the sufficiency follows from Theorem A and Lemma 2 (with \( \delta = r, \eta = s \)).

(ii) The necessity follows from Theorem B and Lemma 2 (with \( \delta = r + 1, \eta = s \)), while the sufficiency follows from Theorem A and Lemma 5 (with \( \delta = r, \eta = s + 1 \)).

(iii) The necessity follows from Theorem B and Lemma 2 (with \( \delta = r, \eta = s + 1 \)), while the sufficiency follows from Theorem A and Lemma 4 (with \( \delta = r, \eta = s \)).

(iv) The necessity follows from Theorem B and Lemma 2 (with \( \delta = r + 1, \eta = s + 1 \)), while the sufficiency follows from Theorem A and Lemma 3 (with \( \delta = r, \eta = s \)).

Proof of Corollary A. (i) If there are \( \mu_1, \nu_1 (\mu_1, \nu_1 > 0) \) such that \( \{m^{\mu_1} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \} \) and \( \{n^{\nu_1} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \} \) are almost decreasing on \( m \) and \( n \) respectively, then

\[
\sum_{j=m}^{\infty} j^{-1} \omega \left( \frac{1}{j}, \frac{1}{n} \right) = \sum_{j=m}^{\infty} j^{-1-\mu_1} \left( j^{\mu_2} \omega \left( \frac{1}{j}, \frac{1}{n} \right) \right) = O \left( m^{\mu_1} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \sum_{j=1}^{m} j^{-1-\mu_1} \right)
\]

\[
= O \left( \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right),
\]

which implies (i).

Similarly, we have (ii).

Therefore, the result follows from (i) of Theorem C.

(ii) If there is \( \mu_2 (0 < \mu_2 < r) \) such that \( \{m^{\mu_2} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \} \) is almost increasing on \( m \), then

\[
\sum_{j=1}^{m} j^{r-1} \omega \left( \frac{1}{j}, \frac{1}{n} \right) = \sum_{j=1}^{m} j^{r-1-\mu_2} \left( j^{\mu_2} \omega \left( \frac{1}{j}, \frac{1}{n} \right) \right) = O \left( m^{\mu_2} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \sum_{j=1}^{m} j^{r-1-\mu_2} \right)
\]

\[
= O \left( m^{r} \omega \left( \frac{1}{m}, \frac{1}{n} \right) \right).
\]

Thus, the result follows from (ii) of Theorem C.

Similarly, (iii) and (iv) of Corollary A follow from (iii) and (iv) of Theorem C, respectively.

Proof of Corollary B. Set

\[ \omega (u, v) = u^\alpha v^\beta, \quad \alpha, \beta > 0. \]

Then \( \omega (u, v) \) satisfies the conditions of Theorem C under assumptions of Corollary B on the parameters \( \alpha, \beta \). Therefore, Corollary B follows from Theorem C immediately.
Acknowledgements. Research of the second author is supported by NSF of China (10901044), and Program for Excellent Young Teachers in HZNU.

References


D. Han
DEPARTMENT OF MATHEMATICS,
HANGZHOU NORMAL UNIVERSITY,
HANGZHOU, ZHEJIANG 310036, CHINA.

G. C. Li (Corresponding Author)
HANGZHOU POLYTECHNIC,
FUYang, HANGZHOU, ZHEJIANG PROVINCE, P.R. CHINA 311402.

E-mail address: 79487694@qq.com

D. S. Yu
DEPARTMENT OF MATHEMATICS,
HANGZHOU NORMAL UNIVERSITY,
HANGZHOU, ZHEJIANG 310036, CHINA.