SOME PROPERTIES OF LORENTZIAN SASAKIAN MANIFOLDS WITH TANAKA-WEBSTER CONNECTION

(COMMUNICATED BY UDAY CHAND DE)

MEHMET ERDOĞAN, JETA ALO, BERAN PİRİNÇİ AND GÜLSEN YILMAZ

Abstract. In this work we study some curvature properties of Lorentzian Sasakian manifolds with respect to the Tanaka-Webster connection and obtain some results about the slant curves of the 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection.

1. Introduction

If a differentiable manifold has a Lorentzian metric $g$, i.e., a symmetric non-degenerated $(0,2)$ tensor field of index 1, then it is called a Lorentzian manifold. Generally, a differentiable manifold has a Lorentzian metric if and only if it has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. It is very natural and interesting to define both a Sasakian structure and a Lorentzian metric on an odd dimensional manifold. In fact, odd dimensional de Sitter space and Goedell Universe, that are important examples on relativity theory, have Sasakian structure with Lorentzian metric, [6], [8], [15].

In this paper, we will define the Tanaka-Webster connection on a Lorentzian Sasakian manifold and investigate some of its properties like curvature tensor, projective curvature tensor and locally $\phi$-symmetry, see [12], [14]. As is well known, the unit 3-sphere $S^3$ is a typical example of a Sasakian manifold. In 3-dimensional contact metric geometry, Legendre curves play a fundamental role. As a generalisation of Legendre curves, in this paper, we will also study slant curves of a Lorentzian Sasakian 3-manifold $M$ with the Tanaka-Webster connection, [1].

A curve on a manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field $\xi$. It is well known that biharmonic curves in 3-dimensional Sasakian space forms are slant helices, see, [4], [5].

2. Preliminaries

2.1. Sasakian manifolds with Lorentzian metric. Let $M$ be a differentiable manifold of class $C^\infty$ and $\phi, \xi, \eta$ be a tensor field of type $(1,1)$, a vector field, a...
1-form on $M$, respectively, such that

$$\phi^2(X) = -X + \eta(X) \xi, \phi \xi = 0,$$

$$\eta(\phi X) = 0, \eta(\xi) = 1$$  \hspace{1cm} (2.1)

for any vector field $X$ on $M$. Then $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$ and is called an almost contact manifold. The almost contact structure is said to be normal if $N = 2d\eta \otimes \xi = 0$, where

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y], \forall X, Y \in \mathcal{X}(M),$$

is the Nijenhuis tensor field of $\phi$ and $\mathcal{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$. [2].

Since $M$ has a globally defined unique vector field $\xi$ which is also called the Reeb vector field, it is able to have a Lorentzian metric $g$ such that $g(\xi, \xi) = -1$, see [7], [8]. If $M$ has the normal almost contact structure $(\phi, \xi, \eta)$ and the Lorentzian metric $g$ with

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), (\nabla_X \eta)(Y) = g(\phi X, Y), X, Y \in \mathcal{X}(M)$$  \hspace{1cm} (2.2)

where $\nabla$ is the covariant derivative with respect to $g$, then $M$ is called a Sasakian manifold with the Lorentzian metric.

In a Sasakian manifold with Lorentzian metric, we have

$$\eta(X) = -g(\xi, X), \nabla_X \xi = -\phi X, \nabla_X \phi(Y) = -\eta(Y)X - g(X, Y)\xi,$$

$$X, Y \in \mathcal{X}(M).$$  \hspace{1cm} (2.3)

The formulas (2.3) imply that an almost contact manifold is Sasakian if and only if its Reeb vector field $\xi$ is a Killing vector field. A Frenet curve parametrised by arc length $s$ is said to be a slant curve if its contact angle defined by $\cos \theta(s) = g(T(s), \xi)$ is constant, where $T(s)$ is the tangent vector field of the curve. The Riemann curvature tensor $R$ of a Sasakian manifold with Lorentzian metric satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = g(\xi, X)Y - g(\xi, Y)X.$$  \hspace{1cm} (2.4)

If $D$ is the contact distribution in a contact manifold $(M, \phi, \xi, \eta)$, defined by the subspaces $D_x = \{X \in T_xM \mid \eta(X) = 0\}$, then a one-dimensional integral submanifold of $D$ will be called a Legendre curve. These curves are the slant curves of contact angle $\frac{\pi}{2}$. A curve $\gamma : I \rightarrow M$, parametrized by its arc length is a Legendre curve if and only if $\eta(\gamma') = 0$, [1].

A plane section in $T_pM$ is called a $\phi$-section if there exists a vector $X \in T_pM$ orthogonal to $\xi$ such that $\{X, \phi X\}$ span the section. The sectional curvature, $K(X, \phi X)$, is called $\phi$-sectional curvature. A Sasakian manifold of constant $\phi$-sectional curvature with Lorentzian metric $g$ will be called a Lorentzian Sasakian space form and denoted by $M(c)$. The curvature tensor of a Sasakian space form with Lorentzian metric is given by

$$4R(X, Y)Z = (c - 3)\{g(Y, Z)X - g(X, Z)Y\} + (c + 1)\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + (c + 1)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$  \hspace{1cm} (2.5)

where $c$ is a constant, [8].
2.2. Tanaka-Webster connection on a Sasakian manifold. Now, we review the Tanaka-Webster connection, on a \((2n + 1)\)-dimensional Sasakian manifold \(M\) with Lorentzian metric \(g\), see \([8]\) and \([17]\). We denote by \(\nabla\) the Lorentzian connection defined by \(g\). Let \(r\) be arbitrary fixed real number, and let \(A\) be a tensor fields of type (1,2) defined by

\[
A(X)Y = g(\phi X, Y)\xi + r\eta(X)\phi(Y) + \eta(Y)\phi X
\]

for all vector fields \(X, Y\) on \(M\). Then we can define a linear connection \(D\) \((D\)-connection, for short\) as

\[
D_X Y = \nabla_X Y + A(X)Y. \tag{2.7}
\]

The tensor fields \(\xi, \eta, g\) and \(A\) are parallel with respect to the \(D\)-connection, for the proof, see \([8]\). If we choose \(r = 1\) in (2.6) we get the special form of \(D\)-connection which is called the Tanaka-Webster connection and denoted by \(\hat{\nabla}\), that is we will define

\[
\hat{\nabla}_X Y = \nabla_X Y + g(\phi X, Y)\xi + \eta(X)\phi(Y) + \eta(Y)\phi X. \tag{2.8}
\]

We see that the Tanaka-Webster connection \(\hat{\nabla}\) for Sasakian manifold \(M\) with Lorentzian metric \(g\) has the torsion

\[
\hat{T}(X,Y) = -2g(X,\phi Y)\xi. \tag{2.9}
\]

Lemma 2.1. ([8]) The tensor \(A\) satisfies followings

\[
A(A(Z)X)Y = g(X,\phi Z)\phi Y - g(X, Y)\eta(Z)\xi - g(Y, Z)\eta(X)\xi
- \eta(Y)\eta(Z)X - \eta(X)\eta(Y)Z,
\]

\[
A(Z)A(X)Y - A(X)A(Z)Y = \eta(X)g(Z, Y)\xi - \eta(Z)g(X, Y)\xi + \eta(Y)\eta(X)Z
+ g(\phi X, Y)\phi Z - g(\phi Z, Y)\phi X - \eta(Z)\eta(Y)X.
\]

3. Curvature tensors of Tanaka-Webster connection

Since the curvature tensor \(\hat{R}\) of the Tanaka-Webster connection and the curvature tensor \(R\) of the Lorentzian connection satisfies

\[
\hat{R}(X,Y)Z = R(X,Y)Z + A(A(Y)X)Z - A(A(X)Y)Z + A(X)A(Y)Z - A(Y)A(X)Z,
\]

from Lemma 2.1, we have the following:

Proposition 3.1. ([8]) Curvature tensors \(\hat{R}\) and \(R\) satisfies following equation

\[
\hat{R}(X,Y)Z = R(X,Y)Z + 2g(\phi X, Y)\phi Z + g(Z, Y)\eta(X)\xi - g(X, Z)\eta(Y)\xi
+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X. \tag{3.1}
\]

As the Reeb vector field \(\xi\) is a parallel vector field with respect to the Tanaka-Webster connection, we obtain following

Theorem 3.2. ([8]) Let \(M\) be a Sasakian manifold with Lorentzian metric. Then the sectional curvature \(\hat{K}(X,\xi)\) of the Tanaka-Webster connection with respect to a section spanned by \(\xi\) and \(X\) is identically zero.

From Proposition 3.1, we have the following equations about the Ricci tensors and the scalar curvatures.
Proposition 3.3. ([8]) The Ricci tensor $\hat{Ric}$ of the Tanaka-Webster connection and the Ricci tensor $Ric$ of the Lorentzian connection satisfies

$$\hat{Ric}(X,Y) = Ric(X,Y) - 2g(X,Y) - 2(n+1)\eta(X)\eta(Y).$$  \hspace{1cm} (3.2)

The scalar curvature $\hat{\rho}$ of the Tanaka-Webster connection and the scalar curvature of the Lorentzian connection satisfies $\hat{\rho} = \rho - 2n$. Now, we will prove the following theorem:

Theorem 3.4. Let $M$ be a Sasakian manifold with Lorentzian metric. If $M$ is of constant curvature $c$ with respect to the Tanaka-Webster connection, then $c = 0$.

Proof. From Proposition 3.1, it follows that

$$R(X,Y)Z = cg(Y,Z)X - cg(X,Z)Y - 2g(\phi X,Y)\phi Z - g(Z,Y)\eta(X)\xi + g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(\phi X,Z)\phi Y + g(\phi Y,Z)\phi X$$

by virtue of the assumption. Hence, using (2.4), we have

$$R(X,\xi)Z = (1-c)\eta(Z)X + (1-c)g(X,Z)\xi = \eta(Z)X + g(X,Z)\xi,$$

so that \(c\{\eta(Z)X + g(X,Z)\xi\} = 0\), for any vectors $X$ and $Z$. Putting $Z = \xi$ and $\eta(X) = 0$ in this equation, we obtain $c = 0$.

4. Projective curvature tensor on Lorentzian Sasakian manifolds with Tanaka-Webster connection

Let $M$ be an $(2n+1)$-dimensional Lorentzian Sasakian manifold equipped with a Tanaka-Webster connection. Since the Ricci tensor $\hat{Ric}$ of the Tanaka-Webster connection is symmetric, the projective curvature tensor of the Sasakian manifold with respect to the Tanaka-Webster connection can be defined by

$$\hat{P}(X,Y)Z = \hat{R}(X,Y)Z - \frac{1}{2n}\{\hat{Ric}(Y,Z)X - \hat{Ric}(X,Z)Y\}. \hspace{1cm} (4.1)$$

Using (3.1) and (3.2), (4.1) reduces to

$$\hat{P}(X,Y)Z = R(X,Y)Z + 2g(\phi X,Y)\phi Z + g(Z,Y)\eta(X)\eta(Y)\xi + \eta(Y)\eta(Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$$

$$- \frac{1}{2n}\{Ric(Y,Z)X - 2g(Y,Z)X - 2(n+1)\eta(Y)\eta(Z)X - Ric(X,Z)Y + 2g(X,Z)Y + 2(n+1)\eta(X)\eta(Z)Y\}$$

or

$$\hat{P}(X,Y)Z = P(X,Y)Z + 2ng(\phi X,Y)\phi Z + g(Z,Y)\eta(X)\eta(Y)\xi - \eta(Y)\eta(Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X$$

$$+ g(Y,Z)X - g(X,Z)Y,$$
where $P$ is the projective curvature tensor with respect to the Lorentzian metric on the manifold, [12]. Putting $Z = \xi$ in the last equation and using (2.3), we have that

$$\hat{P}(X, Y)\xi = 0. \quad (4.2)$$

**Definition.** A Sasakian manifold is called $\xi$-projectively flat if the condition $P(X, Y)\xi = 0$ is satisfied on the manifold.

So from (4.2) we have that the following

**Theorem 4.1.** A Lorentzian Sasakian manifold with Tanaka-Webster connection is $\xi$-projectively flat with respect to the Tanaka-Webster connection $\hat{\nabla}$.

Now we express the following definitions which we will need later.

**Definition.** A Sasakian manifold is called $\phi$-projectively flat if the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0 \quad (4.3)$$

is satisfied on the manifold, [13].

**Definition.** A Sasakian manifold is called $\eta$-Einstein manifold if it satisfies the condition $\text{Ric}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$ for any real numbers $\alpha$ and $\beta$, [18].

Let us assume that $M$ is a $\phi$-projectively flat Lorentzian Sasakian manifold with a Tanaka-Webster connection on it. Then, it can be verified that

$$g(\hat{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \quad (4.4)$$

so from (4.1) we have

$$g(\hat{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}\{\hat{\text{Ric}}(\phi Y, \phi Z)g(\phi X, \phi W) - \hat{\text{Ric}}(\phi X, \phi Z)g(\phi Y, \phi W)\}. \quad (4.5)$$

for $X, Y, Z, W \in T(M)$.

Let $\{e_1, e_2, ..., e_{2n}, \xi\}$ be an orthonormal basis of the vector fields in $M$. Putting $X = W = e_i$ in the last equation and summing up over $i$, we have

$$\sum_{i=1}^{2n} g(\hat{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2n} \sum_{i=1}^{2n} \{\hat{\text{Ric}}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \hat{\text{Ric}}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \quad (4.5)$$

Using (2.1)-(2.3) and (3.2), it can be easily verified that

$$\sum_{i=1}^{2n} g(\hat{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + (2n + 1)g(Y, Z) + \eta(Y)\eta(Z)$$

$$= \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + 2ng(Y, Z) + g(\phi Y, \phi Z)$$

$$= \text{Ric}(Y, Z) + R(\xi, Y, Z, \xi) + 2ng(Y, Z) + g(\phi Y, \phi Z)$$

$$= \text{Ric}(Y, Z) + \eta(Z)Y - \eta(Y)Z + (2n + 1)g(Y, Z) + \eta(Y)\eta(Z)$$

$$= \text{Ric}(Y, Z) + (4n + 1)g(Y, Z) - (4n + 1)\eta(Y)\eta(Z).$$

On the other hand taking into account that

$$\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \sum_{i=1}^{2n} \hat{\text{Ric}}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = \hat{\text{Ric}}(\phi Y, \phi Z),$$
from (4.5) we obtain that
\[
\frac{2n-1}{2n} \hat{\text{Ric}}(\phi Y, \phi Z) = \frac{2n-1}{2n} \{ \hat{\text{Ric}}(Y, Z) + \eta(Y)\eta(Z) \}
\]
Using (3.2) and \( \text{Ric}(X, \xi) = 2n\eta(X) \), from (4.6) we get
\[
\hat{\text{Ric}}(Y, Z) = \alpha g(Y, Z) + \beta \eta(Y)\eta(Z),
\]
where \( \alpha = -2n(4n+1) \) and \( \beta = 4n(2n+1) - 1 \).

Hence we can state the following

**Theorem 4.2.** If a Lorentzian Sasakian manifold is \( \phi \)-projectively flat with respect to the Tanaka-Webster connection then the manifold is an \( \eta \)-Einstein manifold with respect to the Tanaka-Webster connection.

5. **Locally \( \phi \)-symmetric Lorentzian Sasakian Manifolds with Tanaka-Webster Connection**

**Definition.** A Sasakian manifold \( M \) is called to be locally \( \phi \)-symmetric if the condition \( \phi^2(\nabla_W R)(X,Y)Z = 0 \) for all vector fields \( X, Y, Z, W \in T(M) \) orthogonal to \( \xi \). This notion was introduced by Takahashi [14].

Now, we will consider a Lorentzian Sasakian manifold \( M \) with Tanaka-Webster connection and define locally \( \phi \)-symmetry on it by
\[
\phi^2(\nabla_W \hat{R})(X,Y)Z = 0
\]
for all vector fields \( X, Y, Z, W \in T(M) \) orthogonal to \( \xi \).

From (2.8) we have that
\[
(\nabla_W \hat{R})(X,Y)Z = (\nabla_W \hat{R})(X,Y)Z + g((\phi \hat{R})(X,Y)Z, W)\xi
+ \eta(W)(\phi \hat{R})(X,Y)Z + \eta(\hat{R}(X,Y)Z)\phi(W)
\]
and from (3.1), we may rewrite the curvature tensor of Tanaka-Webster connection as follows
\[
\hat{R}(X,Y)Z = R(X,Y)Z + 2d\eta(Y,Z)\phi Z + d\eta(X,Z)\phi(Y) - d\eta(Y,Z)\phi(X)
+ g(Z,Y)\eta(X)\xi - g(Z,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X.
\]
Differentiating this equation in the direction of \( W \) we get
\[
(\nabla_W \hat{R})(X,Y)Z = (\nabla_W R)(X,Y)Z + 2d\eta(X,Y)\phi Z + d\eta(X,Z)\phi(Y) - d\eta(Y,Z)\phi(X)
- d\eta(Y,Z)(\nabla_W \phi)(X) + \{(\nabla_W \eta)(X)g(Y,Z) - (\nabla_W \eta)(Y)g(X,Z)\}\xi
+ (\nabla_W \xi)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} + (\nabla_W \eta)(X)\eta(Z)Y + (\nabla_W \eta)(Z)\eta(X)Y
- (\nabla_W \eta)(Y)\eta(Z)X - (\nabla_W \eta)(Z)\eta(Y)X.
\]
Then, using (2.2), (2.3) in this equation we have
\[
(\nabla_W \hat{R})(X,Y)Z = (\nabla_W R)(X,Y)Z - 2d\eta(X,Y)\{ g(Z,W)\xi + \eta(Z)W \}
- d\eta(X,Z)\{ g(Y,W)\xi + \eta(Y)W \} + d\eta(Y,Z)\{ g(X,W)\xi + \eta(X)W \}
+ \{ g(X,\phi W)g(Y,Z) - g(Y,\phi W)g(X,Z)\} \xi - \phi W\{ \eta(X)g(Y,Z) - \eta(Y)g(X,Z) \}
+ g(X,\phi W)\eta(Z)Y + g(Z,\phi W)\eta(X)Y - g(Y,\phi W)\eta(Z)X - g(Z,\phi W)\eta(Y)X.
\]
Now, applying $\phi^2$ both sides of (5.2) we have
\[
\phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z = \phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z + \eta(W)\phi^2(\phi \hat{R})(X, Y)Z \\
- \phi(W)\eta(\hat{R}(X, Y)Z)
\]
and using (2.1) and this equation in (5.2) we get
\[
\phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z = \eta(W)\phi^2(\phi \hat{R})(X, Y)Z - \phi(W)\eta(\hat{R}(X, Y)Z) + \phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z \\
- 2d\eta(X, Y)\{\eta(Z)\eta(W) + \eta(Z)W\} - d\eta(X, Z)\{\eta(Y)\eta(W) + \eta(Y)W\} \\
+ d\eta(Y, Z)\{\eta(X)\eta(W) + \eta(X)W\} + \phi W\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\} \\
- \{g(X, \phi W)\eta(Y)\eta(Z) + g(Y, \phi W)\eta(X)\eta(Z)\}\xi - g(X, \phi W)\eta(Z)Y - g(Z, \phi W)\eta(X)Y \\
+ g(Y, \phi W)\eta(Z)X - g(Z, \phi W)\eta(Y)X.
\]
If we take $X, Y, Z, W$ orthogonal to $\xi$, from the last equation we find that
\[
\phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z = \phi^2(\hat{\nabla}_{W} \hat{R})(X, Y)Z.
\]
Hence we can state the following:

**Theorem 5.1.** For a Lorentzian Sasakian manifold the Lorentzian connection $\nabla$ is locally $\phi$-symmetric if and only if the Tanaka-Webster connection $\hat{\nabla}$ is locally $\phi$-symmetric.

### 6. Slant curves in 3-dimensional Lorentzian Sasakian manifolds with Tanaka-Webster connection

Let $M$ be 3-dimensional Lorentzian Sasakian manifold equipped with Tanaka-Webster connection $\hat{\nabla}$ and $\gamma$ be a slant curve parametrised by arc length $s$ in $M$. Then $\cos\theta(s) = g(T(s), \xi)$ is constant and satisfies $\cos\theta(s) = -\eta(T)$, where $g$ is the Lorentzian metric. Since $\hat{\nabla}$ is a metrical connection, i.e., $\hat{\nabla}g = 0$, there exists an orthonormal frame field $\{\hat{T}, \hat{N}, \hat{B}\}$ along $\gamma$ such that $\hat{T} = \gamma$ and satisfies the following Frenet-Serret equation with respect to the Tanaka-Webster connection:
\[
\nabla_{\hat{T}} \hat{T} = \hat{\kappa} \hat{N} \\
\nabla_{\hat{T}} \hat{N} = -\hat{\kappa} \hat{T} + \hat{\tau} \hat{B} \\
\nabla_{\hat{T}} \hat{B} = -\hat{\kappa} \hat{N}.
\]
Here $\hat{\kappa}$ and $\hat{\tau}$ are the curvature and the torsion of $\gamma$, respectively. For a unit speed curve $\gamma(s)$ in 3-dimensional Lorentzian Sasakian manifold, by virtue of (2.3) and (2.8) we get
\[
\nabla_{\hat{T}} \hat{T} = \nabla_{\hat{T}} \hat{T} + 2\eta(\phi)\phi \hat{T} = \nabla_{\hat{T}} \hat{T} + 2\cos\theta(s)\phi \hat{T},
\]
where $\nabla$ is the Lorentzian connection of $M$, [4],[5]. Now, if $\gamma(s)$ is a Legendre curve in $M$ and $\{T, N, B\}$ the Frenet frame along $\gamma(s)$, then the tangent vector field $T$ can be defined by $T(s) = \gamma$ and the curvature $\kappa(s)$ of $\gamma(s)$ is given by $\nabla_{\gamma} T = \kappa N$. Since the formula (6.1) implies that every Legendre curve $\gamma(s)$ in $M$ satisfies
\[
\nabla_{\gamma} \gamma = \nabla_{\gamma} \gamma,
\]
the mean curvature vector field $\nabla_{\gamma} \gamma$ coincides with the $\nabla_{\gamma} \gamma$ so that we have $\hat{N} = N = \phi \hat{T}$ and $\hat{\kappa} = \kappa$. Thus every Legendre curve has zero torsion with respect to the Tanaka-Webster connection and so we have that every Legendre curve in $M$ is
\( \nabla \)-geodesic if and only if it is a \( \nabla \)-geodesic, [1]. Now we choose an adapted local orthonormal frame field \( X, \phi X, \xi \) of \( M \) such that \( \eta(X) = 0 \).

Let \( \gamma(s) \) be a non-geodesic Frenet curve in 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection.

Differentiating the equation \( \cos \theta(s) = g(\hat{T}(s), \xi) \) along \( \gamma(s) \), then it follows that

\[
-\theta' \cdot \sin \theta = g(\nabla_{\hat{T}} \hat{T}, \xi) + g(\hat{T}, \nabla_{\hat{T}} \xi) = g(\hat{\kappa} \hat{N}, \xi) + g(\hat{T}, \nabla_{\hat{T}} \xi + \phi \hat{T} + g(\phi \hat{T}, \xi) \xi).
\]

If \( \gamma(s) \) is a slant curve of \( M \), then from (6.3) it follows that

\[
g(\hat{\kappa} \hat{N}, \xi) + g(\hat{T}, \nabla_{\hat{T}} \xi) + g(\hat{T}, \phi \hat{T}) + g(\phi \hat{T}, \xi) = 0.
\]

This equation and from (2.1) we have \( \eta(\hat{N}) = 0 \). Hence we proved the following result, see [4],[9].

**Proposition 6.1.** A non-geodesic curve \( \gamma(s) \) in a 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection is a slant curve if and only if it satisfies \( \eta(\hat{N}) = 0 \).

Hence the reeb vector field \( \xi \) can be written as follows \( \xi = \cos \theta \hat{T} \mp \sin \theta \hat{B} \).

This means that the reeb vector field is in the plane spanned by \( \hat{T} \) and \( \hat{B} \), namely \( g(\xi, \hat{N}) = 0 \). On the other hand, with respect to an adapted local orthonormal frame fields \( X, \phi X, \xi \) of \( M \) such that \( \eta(X) = 0 \) we have the following equalities of the Frenet vector fields \( \hat{T}, \hat{N} \) and \( \hat{B} \) for some function \( \lambda(s) \),

\[
\hat{T} = \sin \theta \{\cos \lambda X + \sin \lambda \phi X\} + \cos \theta \xi,
\]

\[
\hat{N} = -\sin \lambda X + \cos \lambda \phi X,
\]

\[
\hat{B} = \mp \cos \theta \cos \lambda X \mp \cos \theta \sin \phi \lambda X \pm \cot \theta \xi.
\]

Differentiating the equation \( g(\xi, \hat{N}) = 0 \) along the slant curve \( \gamma(s) \) of \( M \) and using (6.1) Frenet-Serre equations and the following identities

\[
\phi \hat{T} = -\sin \theta \sin \lambda X + \sin \theta \cos \phi \lambda X - g(X, \xi) \sin \theta \sin \lambda \xi
\]

\[
\phi \hat{N} = -\cos \lambda X - \sin \lambda \phi X - g(X, \xi) \cos \lambda \xi,
\]

it follows that

\[
g(\nabla_{\hat{T}} \hat{N}, \xi) + g(\hat{N}, \nabla_{\hat{T}} \xi) = 0,
\]

\[
g(\nabla_{\hat{T}} \hat{N} - g(\hat{T}, \xi) \phi \hat{N} + g(\phi \hat{T}, \hat{N}) \xi, \xi) + g(\hat{N}, g(\phi \hat{T}, \xi), \xi) = 0,
\]

\[
g(-\hat{\kappa} \hat{T} + \hat{\tau} \hat{B} - \cos \theta \phi \hat{N}, \xi) - g(\phi \hat{T}, \hat{N}) + g(\hat{N}, g(\phi \hat{T}, \xi) \xi) = 0,
\]

\[
\hat{\kappa} \cos \theta \pm \hat{\tau} \sin \theta - \cos \theta g(\phi \hat{N}, \cos \hat{T} \pm \sin \theta \hat{B}) - g(\phi \hat{T}, \hat{N}) = 0,
\]

\[
\hat{\kappa} \cos \theta \pm \hat{\tau} \sin \theta - \cos \theta \{\pm \cos \theta \pm \frac{\cos \lambda}{\sin \theta} g(X, \xi)\} - \sin \theta = 0,
\]

from this we have that \( \hat{\kappa} \cos \theta + (\pm \hat{\tau} + 1) \sin \theta = 0 \) or \( \frac{\hat{\kappa}}{\pm \hat{\tau} + 1} = \text{const.} \). Thus we proved that the following result

**Theorem 6.2.** If a non-geodesic curve of a 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection is a slant curve, then \( \frac{\hat{\kappa}}{\pm \hat{\tau} + 1} = \text{const.} \).
References


Mehmet ERDOĞAN
Department of Biomedical Engineering, Faculty of Engineering and Architecture, Yeni Yuzyl University, Topkapi, Istanbul, TURKEY.
E-mail address: mehmet.erdogan@yeniyuzyl.edu.tr

Jeta ALO
Department of Mathematics and Computing, Faculty of Science and Letters, Beykent University, Ayazağa, Istanbul, TURKEY.
E-mail address: jeta@beykent.edu.tr

Beran PIRÎNCÎ
Department of Mathematics, Faculty of Science, Istanbul University, Vezneciler, Istanbul, TURKEY.
E-mail address: beran@istanbul.edu.tr

Gülsen YILMAZ
Department of Architecture, Faculty of Engineering and Architecture, Yeni Yuzyl University, Topkapi, Istanbul, TURKEY.
E-mail address: gulsen.yilmaz@yeniyuzyl.edu.tr