NEW CRITERIA FOR GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM MIXED NORM SPACES INTO BLOCH-TYPE SPACES

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Abstract. New criteria for the boundedness and the compactness of the generalized weighted composition operators from mixed norm spaces into Bloch-type spaces are given in this paper.

1. Introduction

Let \( D \) denote the open unit disk in the complex plane \( \mathbb{C} \), i.e., \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( H(\mathbb{D}) \) be the space of all analytic functions on \( \mathbb{D} \).

Let \( 0 < p, q < \infty, \gamma > -1 \). An \( f \in H(\mathbb{D}) \) is said to belong to the mixed norm space, denoted by \( H_{p,q,\gamma}(\mathbb{D}) \), if

\[
\|f\|_{H_{p,q,\gamma}}^q = \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{q}{p}} (1-r)^\gamma dr < \infty.
\]

Let \( \alpha \in (0, \infty) \). The Bloch-type space (or \( \alpha \)-Bloch space), denoted by \( B^\alpha \), consists of all \( f \in H(\mathbb{D}) \) such that

\[
\beta^\alpha(f) = \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.
\]

Under the natural norm \( \|f\|_{B^\alpha} = |f(0)| + \beta^\alpha(f) \), \( B^\alpha \) is a Banach space. See \([21]\) for more information about Bloch-type space.

Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be an analytic self-map. The composition operator, denoted by \( C_\varphi \), is defined as

\[
C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{D}).
\]

Let \( u \) be a fixed analytic function on \( \mathbb{D} \). The weighted composition operator \( uC_\varphi \), which induced by \( \varphi \) and \( u \), is defined as follows.

\[
(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).
\]

We refer \([2]\) for the theory of the composition operator on function spaces.
The generalized weighted composition operator $D^n_{\varphi,u}$, which induced by Zhu (see [22–24]), is defined as follows.

$$(D^n_{\varphi,u}f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad f \in H(D), \quad z \in D.$$ 

Here $f^{(n)}$ denote the n-th differentiation of $f$. This operator includes many known operators. If $n = 0$, then it is just the weighted composition operator $uC\varphi$. If $n = 0$ and $u(z) \equiv 1$, then we obtain the composition operator $C\varphi$. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = D\varphi$, which was studied in [6, 9, 16, 19]. When $n = 1$ and $u(z) = 1$, then $D^n_{\varphi,u} = C\varphi D$, which was studied in [6, 16, 19]. If we put $n = 1$ and $\varphi(z) = z$, then $D^n_{\varphi,u} = M_uD$. See [17, 18, 22–24] for the study of the generalized weighted composition operator on various function spaces.

Composition operators and weighted composition operators between Bloch-type spaces and some other spaces in one, as well as, in several complex variables were studied, for example, in [1, 4, 5, 7–13, 15–17, 19, 20, 22–25].

In [17], the author studied the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into weighted-type spaces. In [18], the author studied the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into the $m$th weighted-type space. Among others, he obtained the following result.

**Theorem A** Let $u \in H(D)$, $\varphi$ be an analytic self-map of $D$ and $n$ be a non-negative integer. Assume that $0 < p, q < \infty$, $\gamma > -1$ and $0 < \alpha < \infty$. Then the following propositions hold:

(a) The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to B^\alpha$ is bounded if and only if

$$\sup_{z \in D} \frac{(1 - |z|^2)^\alpha |u'(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha}{2} + \frac{1}{p} + n}} < \infty \quad \text{and} \quad \sup_{z \in D} \frac{(1 - |z|^2)^\alpha |\varphi'(z)||u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha}{2} + \frac{1}{p} + n + 1}} < \infty. \quad (1)$$

(b) The operator $D^n_{\varphi,u} : H_{p,q,\gamma} \to B^\alpha$ is compact if and only if

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\alpha |u'(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha}{2} + \frac{1}{p} + n}} = \lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\alpha |\varphi'(z)||u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha}{2} + \frac{1}{p} + n + 1}} = 0. \quad (2)$$

In this paper we give a new criteria for the boundedness and compactness of the generalized weighted composition operators $D^n_{\varphi,u}$ from $H_{p,q,\gamma}$ into $B^\alpha$.

Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the next. The notation $a \asymp b$ means that there is a positive constant $C$ such that $C^{-1}b \leq a \leq Cb$.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need two lemmas as follows.

**Lemma 1.** [18] Assume that $0 < p, q < \infty$ and $\gamma > -1$. Let $f \in H_{p,q,\gamma}$. Then there is a positive constant $C$ independent of $f$ such that

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H_{p,q,\gamma}}}{(1 - |z|^2)^{-\frac{\alpha}{2} + \frac{1}{p} + n}}.$$

The following criterion follows from standard arguments similar, for example, to those outlined in Proposition 3.11 of [2].

$$\text{Lemma 1.} \quad \text{Assume that } 0 < p, q < \infty \text{ and } \gamma > -1. \quad \text{Let } f \in H_{p,q,\gamma}. \quad \text{Then there is a positive constant } C \text{ independent of } f \text{ such that}$$

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H_{p,q,\gamma}}}{(1 - |z|^2)^{-\frac{\alpha}{2} + \frac{1}{p} + n}}.$$
Lemma 2. Let \( u \in H(\mathbb{D}) \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a nonnegative integer. Assume that \( 0 < p, q < \infty \), \( \gamma > -1 \) and \( 0 < \alpha < \infty \). The operator \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \) is compact if and only if \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( H_{p,q,\gamma} \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \), we have \( \|D_{\varphi,u}^nf_k\|_{\mathcal{B}^\alpha} \to 0 \) as \( k \to \infty \).

Fix \( 0 < p, q < \infty \), \( \gamma > -1 \). For \( a \in \mathbb{D} \) and \( b > \frac{\gamma+1}{q} \), set

\[
  f_a(z) = \frac{(1 - |a|^2)^{\frac{\gamma+1}{q} + 1}}{(1 - az)^{\frac{\gamma+1}{q} + b}}, \quad \text{and} \quad g_a(z) = \frac{(1 - |a|^2)}{1 - az} f_a(z).
\]

(3)

We use these two families of functions to characterize the generalized weighted composition operators \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \).

Theorem 1. Let \( u \in H(\mathbb{D}) \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a nonnegative integer. Assume that \( 0 < p, q < \infty \), \( \gamma > -1 \) and \( 0 < \alpha < \infty \). Then the following conditions are equivalent:

(a) The operator \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \) is bounded;

(b) \( w\varphi \in \mathcal{B}^\alpha \), \( u \in \mathcal{B}^\alpha \),

\[
  A := \sup_{w \in \mathbb{D}} \|D_{\varphi,u}^nf_{\varphi(w)}\|_{\mathcal{B}^\alpha} < \infty \quad \text{and} \quad B := \sup_{w \in \mathbb{D}} \|D_{\varphi,u}^ng_{\varphi(w)}\|_{\mathcal{B}^\alpha} < \infty.
\]

Proof. (a) \( \Rightarrow \) (b). Assume \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \) is bounded. Taking the functions \( z^n \) and \( z^{n+1} \) and using the boundedness of \( D_{\varphi,u}^n \) we see that \( w\varphi \in \mathcal{B}^\alpha \) and \( u \in \mathcal{B}^\alpha \).

For each \( a \in \mathbb{D} \) and \( b > \frac{\gamma+1}{q} \), from (18) we know that \( f_a, g_a \in H_{p,q,\gamma} \). Moreover \( \|f_a\|_{H_{p,q,\gamma}} \) and \( \|g_a\|_{H_{p,q,\gamma}} \) are bounded by constants independent of \( a \). By the boundedness of \( D_{\varphi,u}^n : H_{p,q,\gamma} \to \mathcal{B}^\alpha \), we get

\[
  \sup_{a \in \mathbb{D}} \|D_{\varphi,u}^nf_{\varphi(a)}\|_{\mathcal{B}^\alpha} \leq \|D_{\varphi,u}^n\| \sup_{a \in \mathbb{D}} \|f_{\varphi(a)}\|_{H_{p,q,\gamma}} \leq C \|D_{\varphi,u}^n\| < \infty
\]

and

\[
  \sup_{a \in \mathbb{D}} \|D_{\varphi,u}^ng_{\varphi(a)}\|_{\mathcal{B}^\alpha} \leq \|D_{\varphi,u}^n\| \sup_{a \in \mathbb{D}} \|g_{\varphi(a)}\|_{H_{p,q,\gamma}} \leq C \|D_{\varphi,u}^n\| < \infty,
\]

as desired.

(b) \( \Rightarrow \) (a). Suppose that \( w\varphi \in \mathcal{B}^\alpha \), \( u \in \mathcal{B}^\alpha \), \( A \) and \( B \) are finite. Now we need to show that the inequalities in (11) hold. A calculation shows that

\[
  f_{\varphi(a)}^{(n)}(w) = \prod_{j=0}^{n-1} \left( \frac{1}{p} + b + j \right) \frac{\prod_{j=0}^{n-1}}{\left( 1 - |\varphi(w)|^2 \right)^{\frac{\gamma+1}{q} + 1 + b + n}}, \quad (4)
\]

and

\[
  g_{\varphi(a)}^{(n)}(w) = \prod_{j=1}^{n} \left( \frac{1}{p} + b + j \right) \frac{\prod_{j=1}^{n}}{\left( 1 - |\varphi(w)|^2 \right)^{\frac{\gamma+1}{q} + 1 + b + n}}, \quad (5)
\]

From (4), for \( w \in \mathbb{D} \), we have

\[
  (D_{\varphi,u}^nf_{\varphi(w)})'(w) = \prod_{j=0}^{n-1} \left( \frac{1}{p} + b + j \right) \frac{u'(w)\varphi(w)^n}{\left( 1 - |\varphi(w)|^2 \right)^{\frac{\gamma+1}{q} + 1 + b + n}}
\]

\[
  + \prod_{j=0}^{n} \left( \frac{1}{p} + b + j \right) \frac{u(w)\varphi'(w)\varphi(w)^{n+1}}{\left( 1 - |\varphi(w)|^2 \right)^{1 + \frac{\gamma+1}{q} + 1 + b + n}}. \quad (6)
\]
For simplicity, we denote \( \prod_{j=0}^{n-1} \left( \frac{1}{p} + b + j \right) \) and \( \prod_{j=0}^{n} \left( \frac{1}{p} + b + j \right) \) by \( Q \) and \( R \), respectively. Therefore

\[
\frac{(1 - |w|^2)^{n} |u'(w)| |\varphi(w)|^{n}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q}} + \frac{1}{p} + n}
\leq \frac{1}{Q} \left( 1 - |w|^2 \right)^{n} |u(w)\varphi'(w)| |\varphi(w)|^{n+1}
\leq \frac{A}{Q} + \frac{R}{Q} \left( 1 - |w|^2 \right)^{n} |u(w)\varphi'(w)| |\varphi(w)|^{n+1}
\leq \frac{A}{Q} + \frac{R}{Q} \left( 1 - |\varphi(w)|^2 \right)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}.
\]

In addition,

\[
(D_{\varphi,u}g_{\varphi(w)})'(w) = \prod_{j=1}^{n} \left( \frac{1}{p} + b + j \right) \frac{u'(w)|\varphi(w)|^{n}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q}} + \frac{1}{p} + n}
+ \prod_{j=1}^{n+1} \left( \frac{1}{p} + b + j \right) \frac{u(w)\varphi'(w)|\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}}.
\]

Therefore, subtracting \([6]\) from \([8]\) and taking the modulus, we obtain

\[
\frac{|u(w)\varphi'(w)||\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}}
\leq \frac{1}{R} \left( \frac{1}{p} + b + n \right) \frac{|(D_{\varphi,u}f_{\varphi(w)})'(w)|}{A}
+ \frac{1}{R} \left( \frac{1}{p} + b + n \right) \frac{|(D_{\varphi,u}g_{\varphi(w)})'(w)|}{B}.
\]

which yields

\[
\frac{(1 - |w|^2)^{n} |u(w)\varphi'(w)||\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}} \leq \frac{1}{R} \left( \frac{1}{p} + b + n \right) \frac{A}{Q} + \frac{1}{R} \left( \frac{1}{p} + b + n \right) \frac{B}{R}.
\]

Hence, by \([7]\), we get

\[
\frac{(1 - |w|^2)^{n} |u'(w)||\varphi(w)|^{n}}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}} \leq \frac{1}{Q} \left( \frac{1}{p} + b + 1 + n \right) \frac{A}{Q} + \frac{1}{Q} \left( \frac{1}{p} + b \right) \frac{B}{Q}.
\]

Fix \( r \in (0, 1) \). If \(|\varphi(w)| > r\), then from \([10]\) we obtain

\[
\frac{(1 - |w|^2)^{n} |u(w)\varphi'(w)|}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}} \leq \frac{1}{R} \left( \frac{1}{p} + b + n \right) \frac{A}{R} + \frac{1}{R} \left( \frac{1}{p} + b \right) \frac{B}{R}.
\]

On the other hand, if \(|\varphi(w)| \leq r\), by the fact that

\[
(1 - |w|^2)^{n} |u(w)\varphi'(w)| \leq ||u\varphi||_{B^q} + ||u||_{B^q},
\]

we get

\[
\frac{(1 - |w|^2)^{n} |u(w)\varphi'(w)|}{(1 - |\varphi(w)|^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}} \leq \frac{1}{(1 - r^2)^{\frac{2n}{q} + \frac{1}{q} + \frac{1}{p} + n}} \left( ||u\varphi||_{B^q} + ||u||_{B^q} \right).
\]

From \([12]\) and \([13]\) we see that the second inequality in \([1]\) holds. Using similar arguments and \([11]\) we can obtain that the first inequality in \([1]\) holds as well. The proof of this theorem is finished. \(\square\)
\textbf{Theorem 2.} Let \( u \in H(D) \), \( \varphi \) be an analytic self-map of \( D \) and \( n \) be a nonnegative integer. Assume that \( p, q > 0 \), \( \gamma > -1 \) and \( 0 < \alpha < \infty \). Suppose that the operator \( D^n_{\varphi,u} : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha \) is bounded, then the following conditions are equivalent:

(a) The operator \( D^n_{\varphi,u} : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha \) is compact;

(b) \[
\lim_{|\varphi(w)| \to 1} \| D^n_{\varphi,u} f_{\varphi(w)} \|_{\mathcal{B}^\alpha} = 0 \quad \text{and} \quad \lim_{|\varphi(w)| \to 1} \| D^n_{\varphi,u} g_{\varphi(w)} \|_{\mathcal{B}^\alpha} = 0.
\]

\textbf{Proof.} (a) \( \Rightarrow \) (b). Assume that \( D^n_{\varphi,u} : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha \) is compact. Let \( \{w_k\}_{k \in \mathbb{N}} \) be a sequence in \( D \) such that \( \lim |\varphi(w_k)| = 1 \). Since the sequences \( \{f_{\varphi(w_k)}\} \) and \( \{g_{\varphi(w_k)}\} \) are bounded in \( H_{p,q,\gamma} \) and converge to 0 uniformly on compact subsets of \( D \), by Lemma 2, we get

\[
\| D^n_{\varphi,u} f_{\varphi(w_k)} \|_{\mathcal{B}^\alpha} \rightarrow 0 \quad \text{and} \quad \| D^n_{\varphi,u} g_{\varphi(w_k)} \|_{\mathcal{B}^\alpha} \rightarrow 0
\]

as \( k \to \infty \), which means that (b) holds.

(b) \( \Rightarrow \) (a). Suppose that the limits in (b) are 0. Using the inequality (9), we get

\[
\frac{(1 - |w|^2)^\alpha |u(w)| \varphi'(w)}{(1 - |\varphi(w)|^2)^{\frac{n+1}{2} + \frac{1}{q} + n}} \leq \frac{1}{R |\varphi(w)|^{n+1}} \left[ \| D^n_{\varphi,u} f_{\varphi(w)} \|_{\mathcal{B}^\alpha} + \frac{1}{Q} \| D^n_{\varphi,u} g_{\varphi(w)} \|_{\mathcal{B}^\alpha} \right] \rightarrow 0
\]

as \( |\varphi(w)| \to 1 \). Moreover, using (7), we deduce

\[
\frac{(1 - |w|^2)^\alpha |u'(w)|}{(1 - |\varphi(w)|^2)^{\frac{n+1}{2} + \frac{1}{q} + n}} \leq \frac{1}{Q |\varphi(w)|^n} \left[ \| D^n_{\varphi,u} f_{\varphi(w)} \|_{\mathcal{B}^\alpha} + \frac{R}{Q} \left( 1 - |w|^2 \right)^\alpha |u(w)| \varphi'(w) |\varphi(w)| \right] \rightarrow 0,
\]

as \( |\varphi(w)| \to 1 \). By Theorem A we see that \( D^n_{\varphi,u} : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha \) is compact. The proof of this theorem is complete. \( \square \)

\textbf{References}


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