ON AN INTEGRAL-TYPE OPERATOR FROM MIXED NORM SPACES TO ZYGMUND-TYPE SPACES

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Abstract. This paper studies the boundedness and compactness of an integral-type operator from mixed norm spaces to Zygmund-type spaces and little Zygmund-type spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let $\beta > 0$. An $f \in H(\mathbb{D})$ is said to belong to the Zygmund-type space, denoted by $Z_\beta$, if $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f''(z)| < \infty$. It is easy to see that $Z_\beta$ is a Banach space with the norm $\|f\|_{Z_\beta} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f''(z)|$.

Let $Z_{\beta,0}$ denote the subspace of $Z_\beta$ consisting of those $f \in Z_\beta$ such that $\lim_{|z| \to 1} (1 - |z|^2)^\beta |f''(z)| = 0$. We call $Z_{\beta,0}$ the little Zygmund-type space. When $\beta = 1$, the induced spaces $Z_1$ and $Z_{1,0}$ becomes the classical Zygmund space and the little Zygmund space respectively (see [2]).

If $0 < r < 1$ and $f \in H(\mathbb{D})$, we set $M_r^q(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^q dt, 0 < q < \infty$.

Let $\nu$ be a normal function on $[0, 1)$ (see [7]). For $0 < p, q < \infty$, the mixed norm space $H(p, q, \nu) = H(p, q, \nu)(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that $\|f\|_{H(p, q, \nu)} = \left( \int_0^1 M_r^p(f, r) \frac{\nu^p(r)}{1 - r} dr \right)^{1/p} < \infty$.

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Let \( g \in H(\mathbb{D}) \), \( n \) be a nonnegative integer and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). In [11], Zhu defined a new integral-type operator as follows.

\[
(C_n^{\varphi, g} f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
\]

When \( n = 1 \), then \( C_1^{\varphi, g} \) is the generalized composition operator \( C_1^{\varphi} \), which firstly defined and studied in [4]. See some related results about the generalized composition operator \( C_1^{\varphi} \) and the operator \( C_n^{\varphi, g} \) in [3, 4, 5, 6, 8, 9, 10]. The purpose of this paper is to study the operator \( C_n^{\varphi, g} \). The boundedness and compactness of the operator \( C_n^{\varphi, g} \) from \( H(p, q, \nu) \) to Zygmund-type spaces and little Zygmund-type spaces are completely characterized, which generalized the results of [8].

Throughout this paper, constants are denoted by \( C \), they are positive and may differ from one occurrence to the other. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( B/C \leq A \leq CB \).

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g., Proposition 3.11 in [1]).

**Lemma 2.1.** Assume that \( 0 < p, q, \beta < \infty \), \( \nu \) is a normal function, \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( n \) is a nonnegative integer. Then \( C_n^{\varphi, g} : H(p, q, \nu) \to Z_\beta \) is compact if and only if \( C_n^{\varphi, g} : H(p, q, \nu) \to Z_\beta \) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( H(p, q, \nu) \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \), we have \( \|C_n^{\varphi, g}f_k\|_{Z_\beta} \to 0 \) as \( k \to \infty \).

**Lemma 2.2.** ([8]) A closed set \( G \) in \( Z_{\beta, 0} \) is compact if and only if it is bounded and satisfies

\[
\lim_{|z| \to 1} \sup_{f \in G} (1 - |z|^2)^\beta |f''(z)| = 0.
\]

**Lemma 2.3.** ([8]) Assume that \( 0 < p, q < \infty \) and \( \nu \) is a normal function. If \( f \in H(p, q, \nu) \), then there is a positive constant \( C \) independent of \( f \) such that

\[
|f^{(n)}(z)| \leq C \frac{\|f\|_{H(p, q, \nu)}}{\nu(z)(1 - |z|^2)^{\frac{1}{q} + n}}, \quad z \in \mathbb{D}.
\]

Now we are in a position to state and prove the main results of this paper.

**Theorem 2.1.** Assume that \( 0 < p, q, \beta < \infty \), \( \nu \) is a normal function, \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( n \) is a nonnegative integer. Then \( C_n^{\varphi, g} : H(p, q, \nu) \to Z_\beta \) is bounded if and only if

\[
M_1 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q} + n}} < \infty \quad (2.2)
\]

and

\[
M_2 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q} + n + 1}} < \infty. \quad (2.3)
\]
Proof. Suppose that (2.2) and (2.3) hold. First it is easy to see that \((C_n^{\varphi,g}f)(0) = 0\) and \((C_n^{\varphi,g}f)'(z) = f^{(n)}(\varphi(z))g(z)\) for every \(f \in H(\mathbb{D})\). For any \(z \in \mathbb{D}\) and \(f \in H(p,q,\nu)\), by Lemma 2.3 we have
\[
(1 - |z|^2)^\beta |(C_n^{\varphi,g}f)'(z)| \leq (1 - |z|^2)^\beta |(f^{(n)} \circ \varphi \cdot g)'(z)|
\]
\[
\leq (1 - |z|^2)^\beta |f^{(n+1)}(\varphi(z))g(z)\varphi'(z)| + (1 - |z|^2)^\beta |f^{(n)}(\varphi(z))g'(z)|
\]
\[
\leq C \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)||f||_{H(p,q,\nu)}}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q}} + n + 1} + C \frac{(1 - |z|^2)^\beta |g'(z)||f||_{H(p,q,\nu)}}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{q}} + n + 1} \quad (2.4)
\]
Moreover \(|(C_n^{\varphi,g}f)'(0)| \leq \frac{C|\varphi(0)||f||_{H(p,q,\nu)}}{\nu(\varphi(0))(1 - |\varphi(0)|^2)^{\frac{1}{q}} + n} \). Taking the supremum in (2.4) for \(z \in \mathbb{D}\), then employing (2.2) and (2.3) we see that \(C_n^{\varphi,g} : H(p,q,\nu) \to Z_\beta\) is bounded.

Conversely, suppose that \(C_n^{\varphi,g} : H(p,q,\nu) \to Z_\beta\) is bounded, i.e., there exists a constant \(C\) such that \(\|C_n^{\varphi,g}f\|_{Z_\beta} \leq C\|f\|_{H(p,q,\nu)}\) for all \(f \in H(p,q,\nu)\). Taking the functions \(f(z) \equiv z^n\) and \(f(z) \equiv z^{n+1}\), which belongs to \(H(p,q,\nu)\), we get
\[
M_3 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| < \infty, \quad M_4 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g(z)||\varphi'(z)| < \infty. \quad (2.5)
\]

For \(a \in \mathbb{D}\), set \(f_a(z) = \frac{(1 - |a|^2)^2 + 1}{\nu(a)(1 - |a|^2)^{\frac{1}{q}} + n + 1}\). From [8] we see that \(f_a \in H(p,q,\nu)\).

Moreover there is a positive constant \(C\) such that \(\sup_{a \in \mathbb{D}} \|f_a\|_{H(p,q,\nu)} \leq C\),
\[
|f_a^{(n)}(a)| = \frac{\prod_{k=1}^n (\frac{1}{q} + t + k)a^n \nu(a)(1 - |a|^2)^{\frac{1}{q} + n + 1}}{\nu(a)(1 - |a|^2)^{\frac{1}{q}} + n + 1}, \quad |f_a^{(n+1)}(a)| = \frac{\prod_{k=1}^{n+1} (\frac{1}{q} + t + k)a^{n+1} \nu(a)(1 - |a|^2)^{\frac{1}{q} + n + 1}}{\nu(a)(1 - |a|^2)^{\frac{1}{q} + n + 1}}.
\]

Hence,
\[
\infty > \|C_n^{\varphi,g}f_\varphi(\lambda)\|_{Z_\beta} \geq \frac{\prod_{k=1}^n (\frac{1}{q} + t + k)(1 - |\lambda|^2)^\beta |g'(\lambda)||\varphi(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{q}} + n + 1} \frac{\prod_{k=1}^{n+1} (\frac{1}{q} + t + k)(1 - |\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{q}} + n + 1} \quad (2.6)
\]
for each \(\lambda \in \mathbb{D}\).

For \(a \in \mathbb{D}\), set
\[
h_a(z) = \frac{(1 - |a|^2)^2 + 1}{\nu(a)(1 - |a|^2)^{\frac{1}{q}} + n + 1} - \frac{\frac{1}{q} + t + 1}{\nu(a)(1 - |a|^2)^{\frac{1}{q}} + n + 1} (1 - |a|^2)^2 + 1.
\]

Then
\[
\sup_{a \in \mathbb{D}} \|h_a\|_{H(p,q,\nu)} \leq C, \quad |h_a^{(n)}(a)| = 0, \quad |h_a^{(n+1)}(a)| = \frac{\prod_{k=1}^n (\frac{1}{q} + t + k)(1 - |a|^2)^2 |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{q}} + n + 1}.
\]

Hence,
\[
\infty > \|C_n^{\varphi,g}h_\varphi(\lambda)\|_{Z_\beta} \geq \frac{\prod_{k=1}^n (\frac{1}{q} + t + k)(1 - |\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{q}} + n + 1} \quad (2.7)
\]
for each \(\lambda \in \mathbb{D}\). Therefore, we obtain
\[
\sup_{\lambda \in \mathbb{D}} \frac{(1 - |\lambda|^2)^\beta |g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{q}} + n + 1} \leq C \|C_n^{\varphi,g}\|_{H(p,q,\nu) \to Z_\beta} < \infty. \quad (2.8)
\]
From (2.8), we have
\[ \sup_{|\varphi(\lambda)| > \frac{1}{2}} \frac{(1 - |\lambda|^2)^{\beta}||g(\lambda)||\varphi'(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{2} + n + 1}} < \sup_{|\varphi(\lambda)| > \frac{1}{2}} 2^{n+1}(1 - |\lambda|^2)^{\beta}||g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1} < \infty. \]  
(2.9)

Inequality (2.6) and the normality of \( \nu \) give
\[ \sup_{|\varphi(\lambda)| \leq \frac{1}{2}} \frac{(1 - |\lambda|^2)^{\beta}||g(\lambda)||\varphi'(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{2} + n + 1}} \leq \frac{CM_4}{\nu(\frac{1}{2})}\frac{(1 - |\lambda|^2)^{\beta}||g(\lambda)||\varphi'(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{2} + n + 1}} < \infty. \]  
(2.10)

Therefore, (2.8) follows from (2.6) and (2.10). From (2.6) and (2.7), we obtain
\[ \sup_{\lambda \in \mathbb{B}} \frac{(1 - |\lambda|^2)^{\beta}||g(\lambda)||\varphi(\lambda)|}{\nu(\varphi(\lambda))(1 - |\varphi(\lambda)|^2)^{\frac{1}{2} + n}} < \infty. \]  
(2.11)

Using (2.6) and (2.11), similarly to the above proof we obtain that (2.2) holds. The proof is completed.

**Theorem 2.2.** Assume that \( 0 < p, q, \beta < \infty, \nu \) is a normal function, \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( n \) is a nonnegative integer. Then \( C^n_{\varphi, \beta} : H(p, q, \nu) \rightarrow \mathbb{Z}_\beta \) is compact if and only if \( C^n_{\varphi, \beta} : H(p, q, \nu) \rightarrow \mathbb{Z}_\beta \) is bounded and
\[ \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^{\beta}||g'(z)||}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{2} + n}} = \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^{\beta}||g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{2} + n + 1}} = 0. \]  
(2.12)

Proof. Suppose that \( C^n_{\varphi, \beta} : H(p, q, \nu) \rightarrow \mathbb{Z}_\beta \) is bounded and (2.12) hold. Let \( (f_k)_{k \in \mathbb{N}} \) be a sequence in \( H(p, q, \nu) \) such that \( sup_{k \in \mathbb{N}} ||f_k||_{H(p, q, \nu)} \leq C \) and \( f_k \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( k \rightarrow \infty \). By the assumption, for any \( \varepsilon > 0 \), there exists a \( \delta \in (0, 1) \) such that
\[ \frac{(1 - |z|^2)^{\beta}||g'(z)||}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{2} + n}} < \varepsilon, \quad \frac{(1 - |z|^2)^{\beta}||g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{1}{2} + n + 1}} < \varepsilon \]  
(2.13)

when \( \delta < |\varphi(z)| < 1 \). Since \( C^n_{\varphi, \beta} : H(p, q, \nu) \rightarrow \mathbb{Z}_\beta \) is bounded, then from the proof of Theorem 2.1 we have \( M_3 < \infty \), \( M_4 < \infty \). Let \( K = \{ z \in \mathbb{D} : |\varphi(z)| \leq \delta \} \). Then, by \( M_3 < \infty \), \( M_4 < \infty \) and (2.13) we have
\[ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta}||C^n_{\varphi, \beta}f_k(z)|| \leq \sup_{K} (1 - |z|^2)^{\beta}||g'(z)||f_k(n)_{\varphi(z)} + \sup_{K} (1 - |z|^2)^{\beta}||g(z)||\varphi'(z)||f_k(n+1)_{\varphi(z)}|| + \sup_{D \setminus K} (1 - |z|^2)^{\beta}||g(z)||f_k_{\varphi(z)}||f_k||_{H(p, q, \nu)} + \sup_{D \setminus K} (1 - |z|^2)^{\beta}||g(z)||\varphi'(z)||f_k||_{H(p, q, \nu)}^{n+1} \leq M_3 \sup_{|w| \leq \delta} |f_k(n)_{w}| + M_4 \sup_{|w| \leq \delta} |f_k(n+1)_{w}| + C \varepsilon ||f_k||_{H(p, q, \nu)}, \]
i.e.,
\[ ||C^n_{\varphi, \beta}f_k||_{Z_\beta} = |f_k(n)_{\varphi(0)}g(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta}||C^n_{\varphi, \beta}f_k(z)|| \leq |f_k(n)_{\varphi(0)}g(0)| + M_3 \sup_{|w| \leq \delta} |f_k(n)_{w}| + M_4 \sup_{|w| \leq \delta} |f_k(n+1)_{w}| + C \varepsilon ||f_k||_{H(p, q, \nu)}. \]
From Cauchy’s estimate and the assumption that \( f_k \to 0 \) as \( k \to \infty \) on compact subsets of \( \mathbb{D} \), we see that \( f_k^{(n)} \to 0 \) as \( k \to \infty \) on compact subsets of \( \mathbb{D} \). Letting \( k \to \infty \) in the last inequality and using the fact that \( \varepsilon \) is an arbitrary positive number, we obtain \( \lim_{k \to \infty} \| C_{\varphi,g}^n f_k \|_{Z_\beta} = 0 \). Applying Lemma 2.1, the result follows.

Conversely, suppose that \( C_{\varphi,g}^n : H(p,q,\nu) \to Z_\beta \) is compact. Then it is clear that \( C_{\varphi,g}^n : H(p,q,\nu) \to Z_\beta \) is compact. Let \( (z_k)_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_k)| \to 1 \) as \( k \to \infty \) (if such a sequence does not exist then conditions in (2.12) are vacuously satisfied). Let

\[
h_k(z) = \frac{(1 - |\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1 - \varphi(z_k)z)^{\frac{1}{2}+t+1}} - \frac{\frac{1}{q} + t + 1}{\frac{1}{q} + t + n + 1} \frac{(1 - |\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1 - \varphi(z_k)z)^{\frac{1}{2}+t+1}}.
\]

Then

\[
\sup_{k \in \mathbb{N}} \| h_k \|_{H(p,q,\nu)} < \infty, \quad h_k^{(n+1)}(\varphi(z_k)) = \frac{\prod_{j=1}^n \left( \frac{1}{q} + t + j \right) |\varphi(z_k)|^{n+1}}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n+1}},
\]

and \( h_k \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Since \( C_{\varphi,g}^n : H(p,q,\nu) \to Z_\beta \) is compact, by Lemma 2.1 we have \( \lim_{k \to \infty} \| C_{\varphi,g}^n h_k \|_{Z_\beta} = 0 \). On the other hand, we have

\[
\| C_{\varphi,g}^n h_k \|_{Z_\beta} \geq \prod_{j=1}^n \left( \frac{1}{q} + t + j \right) |g(z_k)| |\varphi'(z_k)| |\varphi(z_k)|^{n+1} \frac{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n+1}}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+t+1}}.
\]

which together with \( \lim_{|\varphi(z_k)| \to 1} \frac{1 - |z_k|^2|^2 |\varphi'(z_k)| |\varphi(z_k)|^n}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n+1}} = 0 \) implies that

\[
\lim_{|\varphi(z_k)| \to 1} \frac{(1 - |z_k|^2 |g(z_k)| |\varphi'(z_k)| |\varphi(z_k)|^n}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n+1}} = 0. \tag{2.14}
\]

Then the second equality of (2.12) follows. Let

\[
f_k(z) = \frac{(1 - |\varphi(z_k)|^2)^{t+1}}{\nu(\varphi(z_k))(1 - \varphi(z_k)z)^{\frac{1}{2}+t+1}}.
\]

Then \( f_k \in H(p,q,\nu) \) and \( f_k \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Since \( C_{\varphi,g}^n : H(p,q,\nu) \to Z_\beta \) is compact, by Lemma 2.1 we have \( \lim_{k \to \infty} \| C_{\varphi,g}^n f_k \|_{Z_\beta} = 0 \). On the other hand, we have

\[
\| C_{\varphi,g}^n f_k \|_{Z_\beta} \geq \prod_{j=1}^n \left( \frac{1}{q} + t + j \right) |g'(z_k)| |\varphi(z_k)|^n \frac{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n+1}}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+t+1}}.
\]

Therefore, by (2.13) and (2.15) we get

\[
\lim_{k \to \infty} \frac{(1 - |z_k|^2)^t |g'(z_k)| |\varphi(z_k)|^n}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{1}{2}+n}} = 0. \tag{2.16}
\]

Then the first equality of (2.12) follows from (2.16). \( \square \)

Using the condition that polynomials is dense in \( H(p,q,\nu) \) and similarly to the proof of Theorem 9 of [4], we get the following result. We omit the details.
Theorem 2.3. Assume that $0 < p, q < \infty$, $\nu$ is a normal function, $\varphi$ is an analytic self-map of $\mathbb{D}$ and $n$ is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ is bounded if and only if $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta}$ is bounded,

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g'(z)| = \lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g(z)||\varphi'(z)| = 0. \quad (2.17)$$

Theorem 2.4. Assume that $0 < p, q < \infty$, $\nu$ is a normal function, $\varphi$ is an analytic self-map of $\mathbb{D}$ and $n$ is a nonnegative integer. Then $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n}} = \lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n+1}} = 0. \quad (2.18)$$

Proof. Assume that (2.18) holds. Let $f \in H(p,q,\nu)$. By the proof of Theorem 2.1 we have

$$(1 - |z|^2)^\beta |(C^n_{\varphi,g}f)'(z)|$$

$$\leq \frac{C(1 - |z|^2)^\beta |g(z)||\varphi'(z)||f|_{H(p,q,\nu)}}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n+1}} + \frac{C(1 - |z|^2)^\beta |g'(z)||f|_{H(p,q,\nu)}}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n}} \quad (2.19)$$

Taking the supremum in (2.19) over all $f \in H(p,q,\nu)$ such that $||f||_{H(p,q,\nu)} \leq 1$, then letting $|z| \rightarrow 1$, we get

$$\lim_{|z| \rightarrow 1} \sup_{||f||_{H(p,q,\nu)} \leq 1} (1 - |z|^2)^\beta |(C^n_{\varphi,g}f)'(z)| = 0.$$ 

From which by Lemma 2.2 we see that $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ is compact.

Conversely, suppose that $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ is compact. Then $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ is bounded and by Theorem 2.3 we get

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g'(z)| = \lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g(z)||\varphi'(z)| = 0. \quad (2.20)$$

If $||\varphi||_\infty < 1$, from (2.20), we obtain that

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n}} \leq \frac{C\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g'(z)|}{\nu(||\varphi||_\infty)(1 - ||\varphi||_\infty^2)^{1/2 + n+1}} = 0$$

and

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n+1}} \leq \frac{C\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(||\varphi||_\infty)(1 - ||\varphi||_\infty^2)^{1/2 + n+1}} = 0,$$

from which the result follows in this case.

Now we assume that $||\varphi||_\infty = 1$. Let $(\varphi(z_k))_{k \in \mathbb{N}}$ be a sequence such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$. From the compactness of $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta,0}$ we see that the operator $C^n_{\varphi,g} : H(p,q,\nu) \rightarrow \mathcal{Z}_{\beta}$ is compact. From Theorem 2.2 we get

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n}} = \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g(z)||\varphi'(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{1/2 + n+1}} = 0 \quad (2.21)$$

Using (2.20) and (2.21) we easily get the desired result. The proof is completed. □
ON AN INTEGRAL-TYPE OPERATOR......

REFERENCES


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