INTTEGRAL TYPE MODIFICATION FOR $q$–LAGUERRE
POLYNOMIALS

(COMMUNICATED BY PROFESSOR F. MARCELLAN)

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Abstract. Özarslan gave the approximation properties of linear positive op-
erators including the q-Laguerre polynomials in [13]. In this paper, we will
give Kantorovich type generalization for this operator with the help of Rie-
mann type q-integral. We also get approximation properties for the generalized
operator with modulus.

1. Introduction

In 1960, The Meyer-König and Zeller operators
\[ M_n(f; x) = \sum_{k=0}^{\infty} f \left( \frac{k}{k+n+1} \right) \binom{n+k}{k} x^k (1-x)^{n+1} \]
(0 ≤ x < 1) were introduced by Meyer-König and Zeller in [11].
In order to give the monotonicity properties, Cheney and Sharma [1] modified
these operators as:
\[ M_n^*(f; x) = \sum_{k=0}^{\infty} f \left( \frac{k}{k+n} \right) \binom{n+k}{k} x^k (1-x)^{n+1} \]
(0 ≤ x < 1).
In [1], they also introduced the operators
\[ P_n(f; x) = \exp \left( \frac{tx}{1-x} \right) \sum_{k=0}^{\infty} f \left( \frac{k}{k+n} \right) L_k^{(n)}(t) x^k (1-x)^{n+1} \]
(0 ≤ x < 1 and −∞ < t ≤ 0) where $L_k^{(n)}(t)$ denotes the Laguerre polynomials.
Since $L_k^{(n)}(0) = \binom{n+k}{k}$, then $M_n^*(f; x)$ is the special case of the operators
$P_n(f; x)$.

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The q-type generalization of the linear positive operators was initiated by Phillips in [14]. He introduced the q-type generalization of the classical Bernstein operator and obtained the rate of convergence and the Voronovskaja type asymptotic formula for these operators.

$q$-Laguerre polynomials were defined by (Hahn [7, p. 29], Jackson [8, p. 57] and Moak [12, p. 21, eq. 23])

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^{(k)}(1 - q)^k (q^{\alpha+1} x)^k}{(q^{\alpha+1}; q)_k (q; q)_k}$$

where

$$q^{(n)} = \begin{cases} 1 & ; n = 0 \\ (1 - \alpha) (1 - \alpha q) ... (1 - \alpha q^{n-1}) & ; n \in \mathbb{N}, \; \alpha \in \mathbb{C} \end{cases}$$

Moak gave the following recurrence relation [12, p. 29, eq. 4.14] and generating function [12, p. 29, eq. 4.17] for the $q$-Laguerre polynomials:

$$t L_{k-1}^{(\alpha+1)}(t; q) = [k + \alpha] q^{-\alpha-k} L_{k-1}^{(\alpha)}(t; q) - [k] q^{-\alpha-k} L_{k}^{(\alpha)}(t; q)$$

$$F_{\alpha}(x, t) = \frac{(x q^{\alpha+1}; q)_\infty}{(x; q)_\infty} \sum_{m=0}^{\infty} q^{m^2+\alpha m} \frac{[m]^{-1}}{(q; q)_m} (x q^{\alpha+1}; q)_m$$

$$= \sum_{k=0}^{\infty} L_{k}^{(\alpha)}(t; q) x^k \quad (\text{Re} \; \alpha > 1) \quad (1.1)$$

where

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - a q^j), \quad (a \in \mathbb{C}).$$

Trif [16] defined the Meyer-König and Zeller operators based on q-integer as follows:

$$M_{n,q}(f; x) = \prod_{j=0}^{n} (1 - q^j x) \sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right) \left[\begin{array}{c} n+k \\ k \end{array}\right] x^k$$

$$(0 \leq x < 1)$$

where

$$[k] = \begin{cases} (1 - q^k) / (1 - q) & ; q \neq 1 \\ 1 & ; q = 1 \end{cases}$$

$$[k]! = \begin{cases} [1] [2] ... [k] & ; k \geq 1 \\ 1 & ; k = 0 \end{cases}$$

and

$$\left[\begin{array}{c} n+k \\ k \end{array}\right] = \frac{[n+k]!}{[n]! [k]!}$$

$(k, \; n \in \mathbb{N})$ for $q \in (0, 1]$.

In [13], Özarslan defined the q-analogue for $P_n(f; x)$ operators as follow:

$$P_{n,q}(f; x) = \frac{1}{F_n(x, t)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right) L_{k}^{(n)}(t; q) x^k \quad (1.2)$$
For an arbitrary function \( f \), then \( M_{x}^{n,q} \) is the generating functions for the q-Laguerre polynomials. Since \( L_k^{(n)} (0; q) = \sum_{k=0}^{[n+k]} \frac{n+k}{k} \), and \( F_n (x, 0) = \prod_{j=0}^{n} (1 - q^j x) \), then \( M_{n,q} (f; x) \) is the special case of the operators \( P_{n,q} (f; x) \).

Let us recall the concepts of q-differential, q-derivative and q-integral respectively.

For an arbitrary function \( f (x) \), the q-differential is given by

\[
d_q f (x) = f (q x) - f (x).
\]

For an arbitrary function \( f (x) \), the q-derivative is defined as

\[
D_q f (x) = \frac{d_q f (x)}{d_q x} = \frac{f (q x) - f (x)}{(q - 1)x}.
\]

Now suppose that \( 0 < a < b \), \( 0 < q < 1 \) and \( f \) is a real-valued function. The q-Jackson integral of \( f \) over the interval \([0, b] \) and a general interval \([a, b] \) are defined by (see [9])

\[
\int_{0}^{a} f (t) d_q t = (1 - q) a \sum_{j=0}^{\infty} f \left( q^j a \right) q^j
\]

and

\[
\int_{a}^{b} f (t) d_q t = \int_{0}^{b} f (t) d_q t - \int_{0}^{a} f (t) d_q t
\]

respectively.

It is clear that q-Jackson integral of \( f \) over an interval \([a, b] \) contains two infinite sums, so some problems are encountered in deriving the q-analogues of some well-known integral inequalities which are used to compute order of approximation of linear positive operators containing q-Jackson integral. In order to overcome these problems Gauchman [5] and Marinković et al. [10] introduced a new type of q-integral. This new q-integral is called as Riemann type q-integral and defined by

\[
\int_{a}^{b} f (t) d_{q}^{R} t = (1 - q) (b - a) \sum_{j=0}^{\infty} f \left( a + (b - a) q^j \right) q^j
\]

where \( a, b \) and \( q \) are some real numbers such that \( 0 < a < b \) and \( 0 < q < 1 \).

Contrary to the classical definition of q-integral, this definition includes only points within the interval of integration.

Now, we give a Kantorovich type generalization of operators \( P_{n}, M_{n}^{*}, M_{n,q} \) and \( P_{n,q} \). This Kantorovich type generalization was studied by Dalmanoğlu [3], Radu [15] and etc. We consider the sequence of Kantorovich type linear positive operators as follow:

\[
(K_{n,q} f) (x, t) = \frac{1}{F_{n,q} (x, t)} \sum_{k=0}^{\infty} \left( \int_{[k+1]/[n+k]}^{[k+1]/[n+k]} f (t) d_{q}^{R} t \right) q^{-k} [n+k] \int_{[k]/[n+k]}^{[k+1]/[n+k]} L_k^{(n)} (t; q) x^k,
\]

where \( x \in [0, 1], t \in (-\infty, 0), q \in (0, 1], n > 1 \) and \( \{ F_n (x, t) \}_{n \in \mathbb{N}} \) is the generating functions for the q-Laguerre polynomials which was given in (1.3).
2. Approximation Properties of the \((K_{n,q}f)(x,t)\) Operators

We have the following theorem for the convergence of \((K_{n,q}f)(x,t)\) operators.

**Theorem 1.** Let \(q := q_n\) be a sequence satisfying \(\lim_{n \to \infty} q_n = 1\) and \(0 < q_n < 1\). If \(f \in C([0,1])\) and \(\frac{|f|}{q_n} \to 0\) \((n \to \infty)\) then \((K_{n,q}f)\) converges to \(f\) uniformly on \([0,b]\) \((0 < b < 1)\).

**Proof.** By Korovkin’s theorem, it is sufficient for us to prove that \((K_{n,q}e)\) is a positive linear operator and that the desired convergence occurs whenever \(f\) is a quadratic function. It is obvious that \((K_{n,q}f)\) is linear and positive operators.

\[
(K_{n,q}e_0)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left( \int_{[k]/[n+1]}^{[k+1]/[n+1]} dt_{q} t \right) q^{-k} [n+k] L_k^{(n)}(t;q)x^k.
\]

Since

\[
\int_{[k]/[n+1]}^{[k+1]/[n+1]} dt_{q} t = q^k \left( \frac{[k]}{n+k} + \frac{q^k}{[2]} \right),
\]

and from \([1.1]\), we get

\[
(K_{n,q}e_0)(x,t) = 1. \tag{2.1}
\]

By considering the function \(f(s) = \epsilon_1(s) = x\), we obtain

\[
(K_{n,q}e_1)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left( \int_{[k]/[n+1]}^{[k+1]/[n+1]} t dt_{q} t \right) q^{-k} [n+k] L_k^{(n)}(t;q)x^k.
\]

One can easily compute that

\[
\int_{[k]/[n+1]}^{[k+1]/[n+1]} t dt_{q} t = \frac{q^k}{n+k} \left( [k] + \frac{q^k}{[2]} \right),
\]

then we have

\[
(K_{n,q}e_1)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{n+k} \left( [k] + \frac{q^k}{[2]} \right) L_k^{(n)}(t;q)x^k.
\]

Since \(q^k < 1\) for \(0 < q < 1\) and \([k+n] \geq [n]\), we can write

\[
(K_{n,q}e_1)(x,t) \leq P_{n,q}(e_1;x) + \frac{1}{[2][n]} P_{n,q}(e_0;x).
\]

If we use \(P_{n,q}(e_0;x) = 1\) and \(P_{n,q}(e_1;x) \leq x - \frac{tx}{[n](1-bq^{n+1})}\) from \([13]\), then we get

\[
(K_{n,q}e_1)(x,t) - x \leq -\frac{tx}{[n](1-bq^{n+1})} + \frac{1}{[2][n]]. \tag{2.2}
\]

On the other hand, we see

\[
(K_{n,q}e_1)(x,t) \geq P_{n,q}(e_1;x).
\]

If we use \(P_{n,q}(e_1;x) \geq x\) from \([13]\), we obtain

\[
(K_{n,q}e_1)(x,t) \geq x. \tag{2.3}
\]
From (2.2) and (2.3), we have
\[
0 \leq (K_{n,q}e_1)(x,t) - x \leq -\frac{tx}{[n](1 - bq^{n+1})} + \frac{1}{[2][n]}.
\] (2.4)

From (2.4), it is obvious that
\[
\|(K_{n,q}e_1)(x,t) - x\|_{C[0,b]} \leq \frac{|t|b}{[n](1 - bq^{n+1})} + \frac{1}{[2][n]}.
\] (2.5)

We proceed with the consideration of the function \(f(s) = e_2(s) = s^2\).
\[
(K_{n,q}e_2)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left( \int_{[k]/[n+k]}^{[k+1]/[n+k]} q^k \right) \frac{1}{[n+k]} t^2 d_q t \left( \frac{q^k}{[k]} + \frac{q^{2k}}{[3]} \right) L_k^{(n)}(t; q) x^k.
\]

One can easily see that
\[
\int_{[k]/[n+k]}^{[k+1]/[n+k]} t^2 d_q t = \frac{q^k}{[n+k]^2} \left( \frac{k^2 + 2q^k}{[k]} + \frac{q^{2k}}{[3]} \right).
\]

So, we acquire
\[
(K_{n,q}e_2)(x,t) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{[n+k]^2} \left( \frac{k^2 + 2q^k}{[k]} + \frac{q^{2k}}{[3]} \right) L_k^{(n)}(t; q) x^k.
\]

Since \(q^k < 1\) for \(0 < q < 1\) and \([k + n] \geq [n]\), we can write
\[
(K_{n,q}e_2)(x,t) \leq P_{n,q}(e_2;x) + \frac{2}{[2][n]} P_{n,q}(e_1;x) + \frac{1}{[3][n]^2} P_{n,q}(e_0;x).
\]

If we use \(P_{n,q}(e_0;x) = 1\), \(P_{n,q}(e_1;x) \leq x - \frac{tx}{[n](1 - bq^{n+1})}\) and
\[
P_{n,q}(e_2;x) \leq x^2 - \frac{tx}{[n](1 - bq^{n+1})} + \frac{x}{[n]},
\]
from [13], then we get
\[
(K_{n,q}e_2)(x,t) - x^2 \leq -\frac{tx}{[n](1 - bq^{n+1})} + \frac{x}{[n]} + \frac{1}{[3][n]^2}
+ \frac{2}{[2][n]} \left( x - \frac{tx}{[n](1 - bq^{n+1})} \right).
\] (2.6)

On the other hand, using the equality
\[
s^2 = (s - x)^2 + 2xs - x^2
\]
we may write
\[
(K_{n,q}e_2)(x,t) - x^2 = \left( K_{n,q}(e_1-x)^2 \right)(x,t) + 2x(K_{n,q}(e_1)(x, t) - x^2).
\]

By (2.4) and positivity of \(K_{n,q}\), it follows that
\[
(K_{n,q}e_2)(x,t) - x^2 \geq 0.
\] (2.7)
Thus from (2.6) and (2.7), we have
\[
0 \leq (K_{n,q}e_2)(x, t) - x^2 \leq -\frac{t(x^2 + x)}{|n| (1 - bq^{n+1})} + \frac{x}{|n|} + \frac{1}{|n|} + \frac{2}{|n|} \left(x - \frac{tx}{|n| (1 - bq^{n+1})}\right). \tag{2.8}
\]
From (2.8), it is clear that
\[
\|(K_{n,q}e_2)(x, t) - x^2\|_{C[0,b]} \leq \frac{|t|(b^2 + b)}{|n| (1 - bq^{n+1})} + \frac{2}{|n|} \left(\frac{b}{|n|} + \frac{1}{|n|^2}\right) \tag{2.9}
\]
After replacing \(q\) by a sequence \(q_n\) such that \(\lim q_n = 1\), we have from (2.4), (2.5) and (2.9) \((K_{n,q}e_i)(x, t_0) \equiv e_i(x) = x^i (i = 0, 1, 2)\) on \([0, b]\).

3. Rates of Convergence

In this section, we compute the rates of convergence by means of modulus of continuity, elements of Lipschitz class and second order modulus of smoothness.

Let \(f \in C[0,b]\). The modulus of continuity of \(f\) denotes by \(\omega(f, \delta)\), is defined to be
\[
\omega(f, \delta) = \sup_{s, x \in [0,b]} \frac{|f(s) - f(x)|}{|s-x| < \delta}.
\]
It is well known that a necessary and sufficient condition for a function \(f \in C[0,b]\) is
\[
\lim_{\delta \to 0} \omega(f, \delta) = 0.
\]
It is also well known that for any \(\delta > 0\) and each \(s \in [0,b]\)
\[
|f(s) - f(x)| \leq \omega(f, \delta) \left(1 + \frac{|s-x|}{\delta}\right). \tag{3.1}
\]
Before giving the theorem on the rate of convergence of the operator \(K_{n,q}f\), let us first examine its second moment:
\[
\left\|(K_{n,q}(e_1 - x)^2)(x, t)\right\|_{C[0,b]} \leq \left\|(K_{n,q}e_2)(x, t) - x^2\right\|_{C[0,b]} + \left\|x\right\|_{C[0,b]} \left\|(K_{n,q}e_1)(x, t) - x\right\|_{C[0,b]}.
\]
Using (2.9) and (2.5), we can write
\[
\left\|(K_{n,q}(e_1 - x)^2)(x, t)\right\|_{C[0,b]} \leq \frac{|t|(3b^2 + b)}{|n| (1 - bq^{n+1})} + \frac{2}{|n|} \left(\frac{b}{|n|} + \frac{1}{|n|^2}\right) \tag{3.2}
\]
The following theorem gives the rate of convergence of the operator \(K_{n,q}f\) to the function \(f\) by means of modulus of continuity.
Theorem 2. Let \( q := q_n \) be a sequence satisfying \( \lim_{n} q_n = 1 \) and \( 0 < q_n < 1 \). For all \( f \in C [0, b] \) and \( \frac{|f|}{[m]} \to 0 \ (n \to \infty) \)

\[
\| (K_{n, q} f)(x, t) - f(x) \|_{C[0, b]} \leq 2 \omega (f, \delta_n) \tag{3.3}
\]

where

\[
\delta_n = \left[ \frac{|t| (3b^2 + b)}{[n]} (1 - b q^{n+1}) + \frac{2 |t| b}{[2]} \right]\frac{1}{(1 - b q^{n+1})^2} + \left( 1 + \frac{4}{[2]} \right) \frac{b}{[3]} \left( \frac{1}{[n]} \right)^2 \right]^{1/2}.
\]

Proof. Let \( f \in C [0, b] \). By using (3.1), linearity and monotonicity of \( K_{n, q} f \), we obtain

\[
\| (K_{n, q} f)(x, t) - f(x) \| \leq (K_{n, q} |f(s) - f(x)|)(x, t) \leq \omega (f, \delta) \left( K_{n, q} \left( 1 + \frac{|s - x|}{\delta} \right) \right)(x, t) = \omega (f, \delta) \left[ 1 + \frac{1}{\delta} (K_{n, q} |s - x|)(x, t) \right]. \tag{3.4}
\]

In [2], Dalmano˘ glu and Do˘ r˘u show that the Riemann type q-integral is a positive operator and it satisfies the following Hölder’s inequality:

Let \( 0 < a < b, 0 < q < 1 \) and \( \frac{1}{m} + \frac{1}{n} = 1 \). Then

\[
R_q (\|fg\|; a; b) \leq (R_q (\|f|^m; a; b))^{1/m} (R_q (\|g|^n; a; b))^{1/n}. \tag{3.5}
\]

Therefore, by using the Cauchy-Schwarz inequality for the Riemann type q-integral with \( m = 2 \) and \( n = 2 \) in (3.3), we have

\[
\int_{[k]/[n+k]} |t - x|^2 d_q^R t \leq \int_{[k]/[n+k]} (t - x)^2 d_q^R t = \left( \int_{[k]/[n+k]} d_q^R t \right)^{1/2} \left( \int_{[k]/[n+k]} (t - x)^2 d_q^R t \right)^{1/2}.
\]

Now applying the Cauchy-Schwarz inequality for the sum with \( p = \frac{1}{2} \) and \( q = \frac{1}{2} \), and taking into consideration (3.2), one can write

\[
(K_{n, q} |s - x|)(x, t) \leq \left( \sum_{k=0}^{[n+k]} \frac{1}{F_{n, q}(x, t)} \left( \int_{[k]/[n+k]} (t - x)^2 d_q^R t \right)^{1/2} \right) q^{-k} [n+1] L_k^{(n)} (t; q) x^k \times \left( \sum_{k=0}^{[n+k]} \frac{1}{F_{n, q}(x, t)} \left( \int_{[k]/[n+k]} d_q^R t \right)^{1/2} \right)^{1/2} q^{-k} [n+1] L_k^{(n)} (t; q) x^k \right)
\]

\[
= (K_{n, q} (e_1 - x^2))(x, t) \right)^{1/2} ((K_{n, q} e_0)(x, t))^{1/2}
\]

\[
\leq \left[ -\frac{t (3b^2 + b)}{[n]} (1 - b q^{n+1}) - \frac{2 t b}{[2]} (1 - b q^{n+1}) + \left( 1 + \frac{4}{[2]} \right) \frac{b}{[3]} + \left( \frac{1}{[n]} \right)^2 \right]^{1/2} \tag{3.6}
\]

If we write (3.6) in (3.4) and choose \( \delta = \delta_n \), then we arrive at the desired result.
Next, we compute the approximation order of operator $K_{n,q}f$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,b]$ and $0 < \alpha \leq 1$. We recall that $f$ belongs to $\text{Lip}_M(\alpha)$ if the inequality

$$|f(\zeta) - f(\eta)| \leq M|\zeta - \eta|^\alpha; \quad \zeta, \eta \in [0,b]$$

holds.

**Theorem 3.** Let $q := q_n$ be a sequence satisfying $\lim_{n} q_n = 1$ and $0 < q_n < 1$. For all $f \in \text{Lip}_M(\alpha)$ and $\frac{2}{|\alpha|} \to 0$ $(n \to \infty)$

$$||(K_{n,q}f)(x,t) - f(x)||_{C[0,b]} \leq M\delta^\alpha_n$$

where $\delta_n$ is the same as in Theorem 2.

**Proof.** Let $f \in C[0,b]$. By (3.7), linearity and monotonicity of $K_{n,q}f$, we have

$$|(K_{n,q}f)(x,t) - f(x)| \leq (K_{n,q}|f(s) - f(x)|)(x,t) \leq \frac{M}{F_{n,q}(x,t)} \times \sum_{k=0}^{\infty} \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} (t - x)^{\alpha} d_q^R t) q^{-k} [n + k] L_k^{(n)}(t; q) x^k.$$ (3.9)

On the other hand, by using the Hölder inequality for the Riemann type $q$-integral with $n = \frac{2}{\alpha}$ and $n = \frac{2}{2-\alpha}$, we have

$$\int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} (t - x)^{\alpha} d_q^R t \leq \left( \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} (t - x)^{2} d_q^R t \right)^{\alpha/2} \left( \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} d_q^R t \right)^{(2-\alpha)/2}.$$ (3.2)

If we write above inequality in (3.9) and then apply the Hölder inequality for the sum with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get

$$||(K_{n,q}f)(x,t) - f(x)|| \leq M \sum_{k=0}^{\infty} \frac{1}{F_{n,q}(x,t)} \left( \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} (t - x)^{2} d_q^R t \right)^{\alpha/2} q^{-k} [n + k] L_k^{(n)}(t; q) x^k \left( \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} d_q^R t \right)^{(2-\alpha)/2} \left( \int_{\frac{[k]}{[n+k]}}^{\frac{[k+1]}{[n+k]}} d_q^R t \right)^{(2-\alpha)/2}; \quad \zeta, \eta \in [0,b]$$

and so we have

$$|(K_{n,q}f)(x,t) - f(x)| \leq M \left( \left( K_{n,q}(\epsilon_1 - x)^2 \right)(x,t) \right)^{\alpha/2}.$$ (3.2)

If we use (3.2) and choose $\delta = \delta_n$, then the proof is completed. ■
Finally, we establish a local approximation theorem for the operator $K_{n,q}f$. Let $\Omega^2 := \{ g \in C[0,b] : g', g'' \in C[0,b] \}$. For any $\delta > 0$, the Peetre’s K-functional is defined by

$$K_2 (\varphi; \delta) = \inf_{g \in \Omega^2} \{ \| \varphi - g \| + \delta \| g'' \| \}$$

where $\| . \|$ is the uniform norm on $C[0,b]$ (see [6]). From [4] (p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2 (f; \delta) \leq C \omega_2 (f; \sqrt{\delta})$$

(3.10)

where the second order modulus of smoothness of $f \in C[0,b]$ is denoted by

$$\omega_2 (f; \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x, x+h \in [0,b]} | f(x + 2h) - 2f(x + h) + f(x) | .$$

We recall the usual modulus of continuity of $f \in C[0,b]$ by

$$\omega (f; \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x, x+h \in [0,b]} | f(x + h) - f(x) | .$$

Now consider the following operator

$$(L_{n,q}f)(x,t) = (K_{n,q}f)(x,t) - f \left( x - \frac{tx}{[n]} \frac{1}{1 - b q^{n+1}} + \frac{1}{[n]} \right) + f(x)$$

(3.11)

for $x \in [0,1]$.

**Lemma 4.** Let $g \in \Omega^2$. Then we have

$$| (L_{n,q}g)(x,t) - g(x) | \leq \left\{ \frac{-t (3x^2 + x)}{[n]} (1 - b q^{n+1}) - \frac{2tx}{[n]} \frac{1}{1 - b q^{n+1}} + \frac{1 + \frac{4}{[2]}}{[n]} \frac{t}{[n]} \right\} + \frac{1}{[n]} \left\{ \frac{t}{[n]} (1 - b q^{n+1}) + \frac{1}{[2]} \right\} \| g'' \| .$$

(3.12)

**Proof.** By definition of the operator $L_{n,q}f$, (2.1) and (2.4), it is seen that

$$(L_{n,q} (s - x))(x,t) = (K_{n,q} (s - x))(x,t) + \frac{tx}{[n]} \frac{1}{1 - b q^{n+1}} - \frac{1}{[n]} \frac{t}{[2]} \frac{1}{[n]}$$

(3.13)

Let $x \in [0,1]$ and $g \in \Omega^2$. Then by using the Taylor formula

$$g(s) - g(x) = (s - x) g'(x) + \int_{x}^{s} (s - u) g''(u) du$$

and (3.13), we have
\[(L_{n,q}g)(x, t) - g(x) =
\]
\[g'(x)(L_{n,q}(s - x))(x, t) + (L_{n,q}\left(\int_x^s (s - u)g''(u) \, du\right))(x, t)\]
\[\leq (L_{n,q}\left(\int_x^s (s - u)g''(u) \, du\right))(x, t)\]
\[= (K_{n,q}\left(\int_x^s (s - u)g''(u) \, du\right))(x, t)\]
\[+ \left(\frac{tx}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n - u}\right)g''(u) \, du.\]

The monotonicity of \(K_{n,q}f\) gives

\[\left|\left(L_{n,q}g\right)(x, t) - g(x)\right| \leq \left|\left(K_{n,q}\int_x^s (s - u)g''(u) \, du\right)(x, t)\right| + \left(\frac{tx}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n} - u\right)g''(u) \, du.\]

On the other hand, it is clear that

\[\left|\int_x^s (s - u)g''(u) \, du\right| \leq (s - x)^2 \|g''\|.\]

Now let

\[I := \int_x^s \left(x - \frac{tx}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n - u}\right)g''(u) \, du.\]

Then we may write

\[I \leq \left(\frac{tx}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n} + \frac{1}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n}\right)^2 \|g''\|.\]

Substituting (3.15) and (3.16) into (3.14), we have

\[\left|\left(L_{n,q}g\right)(x, t) - g(x)\right| \leq \left\{\left(K_{n,q}(s - x)^2\right)(x, t) + \left(\frac{tx}{\left\lfloor \frac{x}{n}\right\rfloor (1 - bq^{n+1}) + \left\lfloor \frac{x}{2}\right\rfloor n}\right)^2 \|g''\|\right\}.\]
Using (3.2) in (3.17), it follows that
\[
| (L_{n,q} g)(x,t) - g(x) | \leq \left\{ \frac{-t (3x^2 + x)}{[n] (1 - bq^{n+1})} + \frac{2tx}{[2] [n]^2 (1 - bq^{n+1})} + \frac{1}{[3] [n]^2} + \left( 1 + \frac{4}{[2]} \right) \frac{x}{[n]} \right. \\
+ \left. \left( \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)^2 \right\} \|g''\|.
\]
This completes the proof. \(\blacksquare\)

**Theorem 5.** Let \(q := q_n\) be a sequence satisfying \(\lim_{n} q_n = 1\) and \(0 < q_n < 1\). For each \(f \in C[0,1]\) and \(x \in [0,1]\), we have
\[
|(K_{n,q} f)(x,t) - f(x)| \leq C \omega_2 \left( f; \sqrt{\delta_n (x)} \right) + \omega \left( f; \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)
\]
where
\[
\delta_n (x) = \frac{-t (3x^2 + x)}{[n] (1 - bq^{n+1})} - \frac{2tx}{[2] [n]^2 (1 - bq^{n+1})} + \left( 1 + \frac{4}{[2]} \right) \frac{x}{[n]} \\
+ \frac{1}{[3] [n]^2} + \left( \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)^2
\]
and \(C\) is a positive constant.

**Proof.** From (3.11), we have
\[
| (L_{n,q} f)(x,t) | \leq 3 \|f\|.
\] (3.18)
In view of (3.12) and (3.18), the equality (3.11) implies that
\[
| (K_{n,q} f)(x,t) - f(x) | \leq | (L_{n,q} (f - g))(x,t) | + | (f - g)(x) | + | (L_{n,q} g - g(x))(x,t) |
\]
\[
+ \left( f \left( x - \frac{tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right) - f(x) \right) \\
\leq 4 \|f - g\| + | (L_{n,q} g)(x,t) - g(x) |
\]
\[
+ \omega \left( f; \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)
\]
\[
\leq 4 \|f - g\| + \left( \frac{-t (3x^2 + x)}{[n] (1 - bq^{n+1})} - \frac{2tx}{[2] [n]^2 (1 - bq^{n+1})} \\
+ \left( 1 + \frac{4}{[2]} \right) \frac{x}{[n]} + \frac{1}{[3] [n]^2} \\
+ \left( \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)^2 \left\|g''\right\| \\
+ \omega \left( f; \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right)
\]
\[
\leq 4 \|f - g\| + 4 \delta_n (x) \|g''\| + \omega \left( f; \frac{-tx}{[n] (1 - bq^{n+1})} + \frac{1}{[2] [n]} \right).
\]
Hence taking infimum on two-hand side of above inequality over all \( g \in \Omega^2 \) and considering (5.10), we get

\[
| (K_{n,q}f)(x,t_0) - f(x) | \leq 4K_2 \left( f; \delta_n(x) \right) + \omega \left( f; \left[ \frac{-tx}{n(1-bq^{n+1})} + \frac{1}{2} \right] \left[ n \right] \right)
\]

\[
\leq C \omega_2 \left( f; \sqrt{\delta_n(x)} \right) + \omega \left( f; \left[ \frac{-tx}{n(1-bq^{n+1})} + \frac{1}{2} \right] \left[ n \right] \right)
\]

which is the desired result.

REFERENCES


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