COEFFICIENT ESTIMATES FOR CERTAIN NEW SUBCLASSES OF STARLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract. In the present paper, we consider the coefficient estimates for functions in certain new subclasses of starlike and convex functions of complex order $\gamma$, which are introduced by means of a generalized differential operator and non-homogeneous Cauchy-Euler type differential equation. Several corollaries and consequences of the main results are also obtained.

1. INTRODUCTION AND DEFINITIONS

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hfill (1)

that are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$, and we note $f(z) \prec g(z)$, $(z \in U)$, if there exists a Schwarz function $\omega(z)$ analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$, $(z \in U)$. In particular, if the function $g(z)$ is univalent in $U$, then the subordination is equivalent to $f(0) = g(0)$ and $f(U) = g(U)$.

A function $f(z) \in A$ is said to be in the $S^*(\gamma)$ of starlike functions of complex order $\gamma$ if it satisfies the following inequality:

$$\text{Re} \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \ (z \in U; \ \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}).$$  \hfill (2)

Furthermore, a function $f(z) \in A$ is said to be in the $C(\gamma)$ of convex functions of complex order $\gamma$ if it satisfies the following inequality:

$$\text{Re} \left\{ 1 + \frac{1}{\gamma} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \ (z \in U; \ \gamma \in \mathbb{C}^*).$$  \hfill (3)

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The function classes $S^*(\gamma)$ and $C(\gamma)$ were considered earlier by Nasr and Aouf [1] and Wiatrowski [2], respectively, and (very recently) by Altintas et al.[3-9], Deng [10], Murugusundaramoorthy and Srivastava [11], Xu et al.[12], and Srivastava et al.[13-15].

For a function $f(z) \in \mathcal{A}$, Raducanu and Orhan [16] introduced a generalized differential operator $D^n_{\alpha,\delta}$ as follows:

$$D^0_{\alpha,\delta}f(z) = f(z)$$
$$D^1_{\alpha,\delta}f(z) = D_{\alpha,\delta}f(z) = \alpha \delta z^2 (f(z))'' + (\alpha - \delta) z (f(z))' + (1 - \alpha + \delta)f(z)$$
$$\vdots$$
$$D^n_{\alpha,\delta}f(z) = D_{\alpha,\delta}(D^{n-1}_{\alpha,\delta}f(z)), \quad (\alpha \geq \delta \geq 0, \quad n \in N_0 = N \cup \{0\}).$$  (4)

If $f$ is given by (1), then from the definition of operator $D^n_{\alpha,\delta}$ it is easy to see that

$$D^n_{\alpha,\delta}f(z) = z + \sum_{k=0}^{\infty} \Phi_k a_k z^k,$$  (5)

where $\Phi_k = [1 + (\alpha \delta k + \alpha - \delta)(k - 1)]$, $(\Phi_k^n = [\Phi_k]^n)$: $\alpha \geq \delta \geq 0$ and $n \in N_0$.

When $\alpha = 1$ and $\delta = 0$, we get the Salagean differential operator $D^n f(z)$ (see [18]), and when $\delta = 0$, we obtain the Al-Oboudi differential operator $D^n_{\alpha,0}f(z)$ (see [17]).

Next, by using the differential operator $D^n_{\alpha,\delta}$, we define new subclasses of functions belonging to the class $\mathcal{A}$.

**Definition 1.** Let $\gamma \neq 0$ be any complex number, $\alpha \geq \delta \geq 0$, $0 \leq \lambda \leq 1$, $n \in N_0$ and for the parameters $A$ and $B$ such that $-1 \leq B < A \leq 1$, we say that a function $f(z) \in \mathcal{A}$ is in the class $H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$ if it satisfies the following subordination condition:

$$1 + \frac{1}{\gamma} \left( \frac{z(F^n_{\lambda,\alpha,\delta}(z))'}{F^n_{\lambda,\alpha,\delta}(z)} - 1 \right) < \frac{1 + Az}{1 + Bz}, \quad z \in U,$$  (6)

where $F^n_{\lambda,\alpha,\delta}(z) = (1 - \lambda)D^n_{\alpha,\delta}f(z) + \lambda D^{n+1}_{\alpha,\delta}f(z)$.

The special classes $H^n_{0,1,0,\alpha}(1 - 2\alpha, -1)$ and $H^n_{0,1,1,\alpha}(A, B)$ were introduced and studied by Altintas et al.[4] and Srivastava et al.[14], respectively.

**Definition 2.** A function $f(z) \in \mathcal{A}$ is said to be in the class $K^{m,n}_{\gamma,\lambda,\alpha,\delta}(A, B; \mu)$ if it satisfies the following non-homogeneous Cauchy-Euler type differential equation of order $m$:

$$z^m \frac{d^m w}{dz^m} + \left( m \begin{pmatrix} m \\ 1 \end{pmatrix} (\mu + m - 1)z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \left( m \begin{pmatrix} 1 \\ m \end{pmatrix} w \prod_{i=0}^{m-1} (\mu + i) \right) = g(z) \prod_{i=0}^{m-1} (\mu + i + 1),$$  (7)

where $w = f(z) \in \mathcal{A}$, $g(z) \in H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$, $\mu \in R \setminus (-\infty, -1]$ and $m \in N^* = N \setminus \{1\} = \{2, 3, \cdots\}$.

The special cases of the class $K^{2,0}_{1,1,0}(A, B; \mu)$ and $K^{3,0}_{1,1,0}(A, B; \mu)$ were also introduced and studied by Altintas et al.[4]. The object of the present paper is to derive the coefficient estimates for functions in the classes $H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$ and $K^{m,n}_{\gamma,\lambda,\alpha,\delta}(A, B; \mu)$ employing the techniques used earlier by Srivastava et al.[14].
2. MAIN RESULTS

The first property for \( f(z) \in H_{n,\gamma,\lambda,\alpha,\delta}(A, B) \) is contained in

**Theorem 1.** Let the function \( f(z) \) given by (1) be in the class \( H_{n,\gamma,\lambda,\alpha,\delta}(A, B) \). Then

\[
|a_k| \leq \prod_{j=0}^{k-2} \left( j + 2|\frac{A-B}{1-B} \right)
\]

where \( \Phi_k = [1 + (\alpha \delta k + \alpha - \delta)(k - 1)] \), \( k \in N^* \) and \( n \in N_0 \).

**Proof.** By the definitions of \( D_{n,\alpha,\delta}f(z) \) and \( F_{n,\lambda,\alpha,\delta}(z) \), we can write

\[
F_{n,\lambda,\alpha,\delta}(z) = z + \sum_{k=2}^{\infty} A_k z^k \quad (z \in U),
\]

in which

\[
A_k = \Phi_n^k [1 + \lambda (\Phi_k - 1)] a_k \quad (k \in N^*).
\]

Then, clearly, \( F_{n,\lambda,\alpha,\delta}(z) \) is analytic in \( U \) with

\[
F_{n,\lambda,\alpha,\delta}(0) = (F_{n,\lambda,\alpha,\delta})'(0) - 1 = 0.
\]

Thus, by virtue of the subordination condition in equation (6) of Definition 1, we have

\[
1 + \frac{1}{\gamma} \left( \frac{z(F_{n,\lambda,\alpha,\delta}(z))'}{F_{n,\lambda,\alpha,\delta}(z)} - 1 \right) \subset g(U),
\]

where the function \( g(z) \) is given by

\[
g(z) = \frac{1 + A z}{1 + B z} \quad (z \in U, -1 \leq B < A \leq 1).
\]

By setting

\[
h(z) = 1 + \frac{1}{\gamma} \left( \frac{z(F_{n,\lambda,\alpha,\delta}(z))'}{F_{n,\lambda,\alpha,\delta}(z)} - 1 \right),
\]

we deduce also that \( h(0) = g(0) = 1 \) and \( h(U) \subset g(U) \) \( (z \in U) \) for the function \( g(z) \) given by (13). Therefore, we have

\[
h(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)} \quad (\omega(0) = 0, |\omega(z)| < 1)
\]

and

\[
|\omega(z)| = \left| \frac{h(z) - 1}{A - Bh(z)} \right| < 1, \quad h(z) = u + iv.
\]

Now, by using of (16), we obtain that

\[
2u(1 - AB) > 1 - A^2 + (1 - B^2)(u^2 + v^2).
\]

Also, since \( (Re(h(z)))^2 \leq |h(z)|^2 \), we have \( (1 - B^2)u^2 - 2u(1 - AB) + 1 - A^2 < 0 \), which implies that

\[
\frac{1 - A}{1 - B} < u = Re(h(z)) < \frac{1 + A}{1 + B}.
\]

If

\[
Re(h(z)) > \frac{1 - A}{1 - B}, \quad h(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P,
\]

then we have that

\[
|p_k| \leq 2 \left( \frac{A - B}{1 - B} \right).
\]
By (14), we have
\[ z(F^{n}_{\alpha,\delta}(z))' - F^{n}_{\alpha,\delta}(z) = \gamma(h(z) - 1)F^{n}_{\alpha,\delta}(z). \] (20)

Then, from (9) and (18), equating the coefficient of \( z^k \) in (20), we obtain that
\[ (k - 1)A_k = \gamma(p_{k-1} + p_{k-2}A_2 + \cdots + p_1A_{k-1}). \] (21)

In particular, when \( n = 2, 3, 4 \), (21) yields
\[ |A_2| \leq 2|\gamma| \frac{A - B}{1 - B}, \quad |A_3| \leq \frac{2|\gamma|\frac{A - B}{1 - B} \left(1 + 2|\gamma|\frac{A - B}{1 - B}\right)}{2!}, \]
and
\[ |A_4| \leq \frac{2|\gamma|\frac{A - B}{1 - B} \left(1 + 2|\gamma|\frac{A - B}{1 - B}\right) \left(2 + 2|\gamma|\frac{A - B}{1 - B}\right)}{3!}, \]
respectively. Thus, by using the principle of mathematical induction, we have
\[ |A_k| \leq \frac{\prod_{j=0}^{k-2} \left(j + 2|\gamma|\frac{A - B}{1 - B}\right)}{(k - 1)!}. \] (22)

Also, since \( A_k = \Phi_k^{1}[1 + \lambda(\Phi_k - 1)]a_k \) \( (k \in N^*) \). Then, by (22), we have that inequality (8). This completes the proof of Theorem 1.

**Corollary 1.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^n_{\gamma,\lambda,1,0}(A, B) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} \left(j + 2|\gamma|\frac{A - B}{1 - B}\right)}{(k - 1)!\left[1 + \lambda(k - 1)\right]} \quad (k \in N^*). \]

**Corollary 2 ([14]).** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^0_{\gamma,\lambda,1,0}(A, B) \equiv S(\lambda, \gamma, A, B) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} \left(j + 2|\gamma|\frac{A - B}{1 - B}\right)}{(k - 1)!\left[1 + \lambda(k - 1)\right]} \quad (k \in N^*). \]

**Corollary 3.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^0_{\gamma,\lambda,1,0}(1 - 2\alpha, -1) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\gamma|(1 - \alpha))}{(k - 1)!\left[1 + \lambda(k - 1)\right]} \quad (k \in N^*). \]

**Corollary 4 ([10]).** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^0_{\gamma,\lambda,1,0}(1 - 2\alpha, -1) \equiv B(0, \lambda, \alpha, b) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|b|(1 - \alpha))}{(k - 1)!\left[1 + \lambda(k - 1)\right]} \quad (k \in N^*). \]

**Corollary 5.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^0_{\gamma,\lambda,0,0}(A, B) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} \left(j + 2|\gamma|\frac{A - B}{1 - B}\right)}{(k - 1)!\left[1 + \alpha(k - 1)\right]\left[1 + \lambda\alpha(k - 1)\right]} \quad (k \in N^*). \]

**Corollary 6.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in H^0_{\gamma,\lambda,0,0}(1 - 2\alpha, -1) \), then
\[ |a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\gamma|(1 - \alpha))}{(k - 1)!\left[1 + \alpha(k - 1)\right]\left[1 + \lambda\alpha(k - 1)\right]} \quad (k \in N^*). \]
**Theorem 2.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in K_{\gamma, \lambda, \alpha, \delta}^{m,n}(A, B; \mu) \), then
\[
|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\frac{A-B}{1-B}|) \prod_{i=0}^{m-1} (\mu + i + 1)}{(k - 1)! \Phi_k^n[1 + \lambda(\Phi_k - 1)] \prod_{i=0}^{m-1} (\mu + i + k)} (k, m \in \mathbb{N}^*), \tag{23}
\]
where \( \lambda \in [0, 1] \) and \( \alpha, \beta, \gamma, \delta \) are given by (1). If \( \gamma \in C^* \), then \( -1 < B < A \leq 1; \mu \in R \setminus (-\infty, -1] \).

**Proof.** Suppose that the function \( f(z) \in \mathcal{A} \) be given by (1). Let
\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in H_{\gamma, \lambda, \alpha, \delta}^{n}(A, B).
\]
By Theorem 1, we have
\[
|b_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\frac{A-B}{1-B}|) \prod_{i=0}^{m-1} (\mu + i + 1)}{(k - 1)! \Phi_k^n[1 + \lambda(\Phi_k - 1)] \prod_{i=0}^{m-1} (\mu + i + k)} (k \in \mathbb{N}^*, \ n \in \mathbb{N}_0), \tag{24}
\]
Then we deduce from (7) that
\[
a_k = \left( \frac{\prod_{i=0}^{m-1} (\mu + i + 1)}{\prod_{i=0}^{m-1} (\mu + i + k)} \right) b_k (k, m \in \mathbb{N}^*; \ \mu \in R \setminus (-\infty, -1]). \tag{25}
\]
Using (24) and (25), we have the assertion (23) of Theorem 2. This completes the proof.

**Corollary 7.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in K_{\gamma, \lambda, \alpha, 0}^{m,n}(A, B; \mu) \), then
\[
|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\frac{A-B}{1-B}|) \prod_{i=0}^{m-1} (\mu + i + 1)}{(k - 1)! [1 + \alpha(k - 1)] \prod_{i=0}^{m-1} (\mu + i + k)} (k, m \in \mathbb{N}^*).
\]

**Corollary 8.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in K_{\gamma, \lambda, 1, 0}^{m,n}(A, B; \mu) \equiv K(\lambda, \gamma, A, B, m; \mu) \), then
\[
|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\frac{A-B}{1-B}|) \prod_{i=0}^{m-1} (\mu + i + 1)}{(k - 1)! [1 + \lambda(k - 1)] \prod_{i=0}^{m-1} (\mu + i + k)} (k, m \in \mathbb{N}^*).
\]

**Corollary 9.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in K_{\gamma, \lambda, 0, 0}^{m,n}(1 - 2\alpha, -1; \mu) \equiv T(0, \lambda, \alpha, b; \mu) \), then
\[
|a_k| \leq \frac{\prod_{j=0}^{k-2} (j + 2|\frac{A-B}{1-B}|) \prod_{i=0}^{m-1} (\mu + i + 1)}{(k - 1)! [1 + \alpha(k - 1)] \prod_{i=0}^{m-1} (\mu + i + k)} (k, m \in \mathbb{N}^*).
\]

**Corollary 10.** Let the function \( f(z) \in \mathcal{A} \) be given by (1). If \( f(z) \in K_{\gamma, \lambda, 1, 0}^{2,0}(1 - 2\alpha, -1; \mu) \equiv T(0, \lambda, \alpha, b, \mu) \), then
\[
|a_k| \leq \frac{(1 + \mu)(2 + \mu) \prod_{j=0}^{k-2} (j + 2|1 - \alpha|)}{(k - 1)!(k + \mu)(k + \mu + 1)[1 + \lambda(k - 1)]} (k \in \mathbb{N}^*).
\]

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