APPLICATION OF HÖLDER’S INEQUALITY AND
CONVOLUTIONS

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Abstract. In this paper we introduce a new subclass \( M_p(n, \alpha, c) \) of analytic and multivalent functions in the unit disk which includes the class \( S_p(n, \alpha) \) of multivalent starlike functions of order \( \alpha \) and the class \( T_p(n, \alpha) \) of multivalent convex functions of order \( \alpha \). Using generalized Bernardi Libera integral operator and Hölder’s inequality, some interesting properties of convolution for the class \( M_p(n, \alpha, c) \) are considered.

1. Introduction

Let \( A_p(n) \) be class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N})
\]

which are analytic in the open unit disk \( \mathcal{U} = \{ z : z \in \mathbb{C}; |z| < 1 \} \). Let \( S_p(n, \alpha) \) be the subclass of \( A_p(n) \) consisting of functions \( f(z) \) which satisfy

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U})
\]

for some \( \alpha (0 \leq \alpha < p) \). A function \( f(z) \in S_p(n, \alpha) \) is known as starlike of order \( \alpha \) in \( U \).

Further, let \( T_p(n, \alpha) \) be the subclass of \( A_p(n) \) consisting of functions \( f(z) \) satisfying

\[
\frac{zf''(z)}{f'(z)} \in S_p(n, \alpha),
\]

that is,

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathcal{U})
\]
for some $\alpha$ ($0 \leq \alpha < p$). A function $f(z)$ in $T_p(n, \alpha)$ is known as convex of order $\alpha$ in $U$.

These classes $A_p(n)$, $S_p(n, \alpha)$ and $T_p(n, \alpha)$ were studied earlier by Owa [9] respectively. Also Nishiwaki and Owa [6] have given the following lemmas which provide the sufficient conditions for functions $f(z) \in A_p(n)$ to be in the classes $S_p(n, \alpha)$ and $T_p(n, \alpha)$ respectively.

**Lemma 1.1.** If $f(z) \in A_p(n)$ satisfies

$$
\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \leq p-\alpha
$$

for some $\alpha$ ($0 \leq \alpha < p$), then $f(z) \in S_p(n, \alpha)$

**Lemma 1.2.** If $f(z) \in A_p(n)$ satisfies

$$
\sum_{k=p+n}^{\infty} k(k-\alpha)|a_k| \leq p(p-\alpha)
$$

for some $\alpha$ ($0 \leq \alpha < p$), then $f(z) \in T_p(n, \alpha)$.

**Remark 1.** We note that Silverman [10] has given Lemma (1.1) and Lemma (1.2) in the case of $p = 1$ and $n = 1$. Also Srivastava, Owa and Chatterjea [11] have given the coefficient inequalities in the case of $p = 1$.

In view of lemmas (1.1) and (1.2) Nishiwaki and Owa [6] introuced the subclasses $S_p^*(n, \alpha)$, $T_p^*(n, \alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequalities (1.1) and (1.2) respectively.

Now, we introduce subclass $M_p^*(n, \alpha, c)$ consisting of functions $f(z) \in A_p(n)$ which satisfy the following coefficient inequality.

$$
\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c (k-\alpha)|a_k| \leq p-\alpha
$$

for some $c \geq 0$ and $\alpha$ ($0 \leq \alpha < p$).

Obviously $M_p^*(n, \alpha, 0) \equiv S_p^*(n, \alpha)$ and $M_p^*(n, \alpha, 1) \equiv T_p^*(n, \alpha)$.

For functions $f_j(z) \in A_p(n)$ given by

$$
f_j(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,j} z^k (j = 1, 2, ..., m),
$$

we define

$$
G_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k,
$$

and

$$
H_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} (a_{k,j})^{p_j}\right) z^k \quad (p_j > 0)
$$

where $G_m(z)$ denotes the convolution of $f_j(z)$ ($j = 1, 2, ..., m$). Therefore $H_m(z)$ is the generalization of convolutions. In the case $p_j = 1$, we have $G_m(z) := H_m(z)$. The generalization of the convolutions was considered by Choi, Kim and Owa [2], and Nishiwaki and Owa [6].
Further for functions $f_j(z) \in A_p(n)$ given by (1.4), generalized Libera integral operator is defined as follows:

$$B_{j,p}(z) = \frac{p + c_j}{z^{c_j}} \int_0^z t^{c_j-1} f(t) \, dt \quad (c_j > -p) \quad (1.7)$$

(For $c_j = 1$, we obtain multivalent Libera operator. For $p = 1$, we get generalized Bernardi-Libera-Livingston integral operator defined recently by Gordji et al. [4] and for $p = 1, c_j = 1$, Libera [5] studied the above operator. The operator (1.7) also includes the Alexander operator [1] for $p = 1$ and $c_j = 0$.)

Using the operator (1.7), we find that the convolution integral of $B_{1,p}$ and $B_{2,p}$ as

$$(B_{1,p} * B_{2,p})(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p + c_1)(p + c_2)}{(k + c_1)(k + c_2)} a_{k,1} a_{k,2} z^k$$

(1.8)

The convolution integral was studied by Duren [3]. Hence the convolution integral of $B_{1,p}, B_{2,p}, ..., B_{m,p}$ is given by

$$(B_{1,p} * B_{2,p} * ... * B_{m,p})(z) = z^p + \sum_{k=p+n}^{\infty} \left( \prod_{j=1}^{m} \frac{p + c_j}{k + c_j} a_{k,j} \right) z^k$$

For functions $f_j(z) \in A_p(n) \quad (j = 1, 2, ..., m)$ given by (1.4), the familiar H"{o}lder’s inequality assumes the form

$$\sum_{k=p+n}^{\infty} \left( \prod_{j=1}^{m} |a_{k,j}| \right) \leq \prod_{j=1}^{m} \left( \sum_{k=p+n}^{\infty} |a_{k,j}|^{p_j} \right)^{\frac{1}{p_j}}$$

(1.9)

where $p_j > 1$ and $\sum_{j=1}^{m} \frac{1}{p_j} \geq 1 \quad (j = 1, 2, ..., m)$

Nishiwaki, Owa and Srivastava [8] have given some results of H"{o}lder’s-type inequalities for subclass of uniformly starlike functions. Also applying these inequalities, Nishiwaki and Owa [6] have obtained some interesting properties of generalizations of convolutions for functions $f(z)$ in the classes $S^*_p(n,\alpha)$ and $T^*_p(n,\alpha)$ . Again Nishiwaki and Owa [7] have given application of convolution integral for certain subclasses by using H"{o}lder’s inequalities . Motivated essentially by these papers, we discuss some applications of H"{o}lder’s inequalities for $H_m(z)$ defined by (1.6) and convolution integral defined by (1.8) for the subclass $M^*_p(n,\alpha, c)$. 
2. Main Results

Theorem 1. If \( f_j(z) \in M_p^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \), then \( H_m(z) \in M_p^*(n, \beta, c) \) with

\[
\beta = \inf_{k \geq p+n} \left\{ p - \frac{k^c (k-p) \prod_{j=1}^m [p^e (p-\alpha_j)]^{p_j}}{p^e \prod_{j=1}^m [k^c (k-\alpha_j)]^{p_j} - k^e \prod_{j=1}^m [p^e (p-\alpha_j)]^{p_j}} \right\} \quad (2.1)
\]

where \( p_j \geq \frac{1}{q_j} \), \( q_j > 1 \) and \( \sum_{j=1}^m \frac{1}{q_j} \geq 1 \).

Proof. Since \( f_j(z) \in M_p^*(n, \alpha_j, c) \), therefore by equation (1.3), we get

\[
\sum_{k=p+n}^{\infty} \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) |a_{k,j}| \leq 1 \quad (j = 1, 2, \ldots, m)
\]

which implies

\[
\left\{ \sum_{k=p+n}^{\infty} \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) |a_{k,j}| \right\}^{\frac{1}{q_j}} \leq 1 \quad (2.2)
\]

with \( q_j > 1 \) and \( \sum_{j=1}^m \frac{1}{q_j} \geq 1 \).

From (2.2), we have

\[
\prod_{j=1}^m \left\{ \sum_{k=p+n}^{\infty} \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) |a_{k,j}| \right\}^{\frac{1}{q_j}} \leq 1
\]

Applying Hölder’s inequality (1.9), we find that

\[
\sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^m \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\} \leq 1 \quad (2.3)
\]

Note that we have to find the largest \( \beta \) such that

\[
\sum_{k=p+n}^{\infty} \left( \frac{k}{p} \right)^c \left( \frac{k - \beta}{p - \beta} \right) \left( \prod_{j=1}^m |a_{k,j}|^{p_j} \right) \leq 1
\]

that is,

\[
\sum_{k=p+n}^{\infty} \left( \frac{k}{p} \right)^c \left( \frac{k - \beta}{p - \beta} \right) \left( \prod_{j=1}^m |a_{k,j}|^{p_j} \right) \leq \sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^m \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\}
\]

Therefore we need to find largest \( \beta \) such that

\[
\left( \frac{k}{p} \right)^c \left( \frac{k - \beta}{p - \beta} \right) \left( \prod_{j=1}^m |a_{k,j}|^{p_j} \right) \leq \prod_{j=1}^m \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}}
\]
which is equivalent to
\[
\left( \frac{k}{p} \right)^c \left( \frac{k - \beta}{p - \beta} \right) \prod_{j=1}^{m} |a_{k,j}|^{p_j - \frac{1}{q_j}} \leq \prod_{j=1}^{m} \left( \frac{k}{p} \right)^{\alpha_j} \left( \frac{k - \alpha_j}{p - \alpha_j} \right) \]
for all \( k \geq p + n \). Since
\[
\prod_{j=1}^{m} \left\{ \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) |a_{k,j}| \right\}^{p_j - \frac{1}{q_j}} \leq 1 \quad \left( p_j - \frac{1}{q_j} \geq 0 \right)
\]
therefore
\[
\prod_{j=1}^{m} (a_{k,j})^{p_j - \frac{1}{q_j}} \leq \frac{1}{\prod_{j=1}^{m} \left\{ \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) \right\}^{p_j}}
\]
This implies that
\[
\left( \frac{k}{p} \right)^c \left( \frac{k - \beta}{p - \beta} \right) \leq \prod_{j=1}^{m} \left\{ \left( \frac{k}{p} \right)^c \left( \frac{k - \alpha_j}{p - \alpha_j} \right) \right\}
\]
for all \( k \geq p + n \). Therefore \( \beta \) should be
\[
\beta \leq p - k^c(k-p) \prod_{j=1}^{m} (p^c(p - \alpha_j))^{p_j}
\]
for all \( k \geq p + n \). This completes the proof of the theorem.

Taking \( p_j = 1 \) in Theorem 1, we obtain

**Corollary 2.1.** If \( f_j(z) \in M_p^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \), then \( G_m(z) \in M_p^*(n, \beta, c) \) with
\[
\beta = p - \frac{np^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)}{(p + n)^{m-1} \prod_{j=1}^{m} (p + n - \alpha_j) - p^{m-1} \prod_{j=1}^{m} (p - \alpha_j)} \quad (2.4)
\]

**Proof.** In view of Theorem 1, we obtain
\[
\beta \leq \inf_{k \geq p+n} \left\{ p - \frac{k^c(k-p) \prod_{j=1}^{m} p^c(p - \alpha_j)}{p^c \prod_{j=1}^{m} k^c(k - \alpha_j) - k^c \prod_{j=1}^{m} p^c(p - \alpha_j)} \right\}
\]
Let \( F(c, k, m) \) be the right-hand side of the above inequality. Further let us define \( G(c, k, m) \) be the numerator of \( F'(c, k, m) \). Then
\[
G(c, k, m) = -p^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j) \left\{ k^{(m-1)c} \prod_{j=1}^{m} (k - \alpha_j) - p^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j) \right\}
\]
Theorem 2. Due to them are also special cases of our Theorem 1 and corollary 2 obtained recently by Nishiwaki and Owa \[6\]. Corollaries 2

This means that

\[F(c, k, m)\] is increasing function for all \(k \geq p + n\).

This means that

\[
β = \frac{np^{(m-1)c} \prod_{j=1}^{m} (p - α_j)}{(p + n)^{(m-1)c} \prod_{j=1}^{m} (p + n - α_j) - p^{(m-1)c} \prod_{j=1}^{m} (p - α_j)}
\]

On taking \(c = 0\) and \(c = 1\) in Theorem 1, we get the Theorem 2.1, 2.6 and 2.8 obtained recently by Nishiwaki and Owa \[6\]. Corollaries 2.2, 2.3, 2.5, 2.7 and 2.9 due to them are also special cases of our Theorem 1 and corollary 2.1

**Theorem 2.** If \(f_j(z) \in M_p^a(n, α_j, c)\) for each \(j = 1, 2, ..., m\) then \((B_1 * B_2 * ... * B_m)(z) \in M_p^a(n, β, c)\) with

\[
β = \frac{np^{(m-1)c} \prod_{j=1}^{m} (p + c_j)(p - α_j)}{(p + n)^{(m-1)c} \prod_{j=1}^{m} (p + n + c_j)(p + n - α_j) - p^{(m-1)c} \prod_{j=1}^{m} (p + c_j)(p - α_j)}
\]

(2.5)

**Proof.** Since \(f_j(z) \in M_p^a(n, α_j, c)\), then from (2.3), we get

\[
\sum_{k=p+n}^{∞} \left\{ \prod_{j=1}^{m} \left( \frac{k}{p} \right)^{\frac{1}{q_j}} \left( \frac{k - α_j}{p - α_j} \right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\} \leq 1
\]

Note that we have to find the largest \(β\) such that

\[
\sum_{k=p+n}^{∞} \left( \frac{k}{p} \right)^{c} \left( \frac{k - β}{p - β} \right)^{\frac{1}{q_j}} \prod_{j=1}^{m} \left( \frac{p + c_j}{k + c_j} \right) |a_{k,j}| \leq 1
\]

that is

\[
\sum_{k=p+n}^{∞} \left( \frac{k}{p} \right)^{c} \left( \frac{k - β}{p - β} \right)^{\frac{1}{q_j}} \prod_{j=1}^{m} \left( \frac{p + c_j}{k + c_j} \right) |a_{k,j}| \leq \sum_{j=1}^{m} \left\{ \prod_{j=1}^{m} \left( \frac{k}{p} \right)^{\frac{1}{q_j}} \left( \frac{k - α_j}{p - α_j} \right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\}
\]
Using the same procedure of Theorem 1, we can easily prove that for all \( k \geq p + n \), \( \beta \) should be

\[
\beta \leq p - \frac{k^c(k-p) \prod_{j=1}^{m} (p + c_j)p^c(p - \alpha_j)}{p^c \prod_{j=1}^{m} (k + c_j)k^c(k - \alpha_j) - k^c \prod_{j=1}^{m} (p + c_j)p^c(p - \alpha_j)} \tag{2.6}
\]

Let \( F(c, k, m) \) be the right hand side of the above inequality. Further let us define \( G(c, k, m) \) be the numerator of \( F'(c, k, m) \). Then

\[
G(c, k, m) = -p^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)(p + c_j) \left\{ k^{(m-1)c} \prod_{j=1}^{m} (k - \alpha_j)(k + c_j) - p^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)(p + c_j) \right\}
\]

\[
+ (k - p)(pk)^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)(p + c_j) \left[ \frac{(m-1)c}{k} \prod_{j=1}^{m} (k - \alpha_j)(k + c_j) \right.
\]

\[
+ \left\{ \sum_{j=1}^{m} \frac{\prod_{j=1}^{m} (k - \alpha_j)(k + c_j)}{k - \alpha_j} + \sum_{j=1}^{m} \frac{\prod_{j=1}^{m} (k - \alpha_j)(k + c_j)}{k + c_j} \right\}
\]

\[
= (pk)^{(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)(p + c_j)(k - \alpha_j)(k + c_j)
\]

\[
\times \left\{ (k - p) \left( \sum_{j=1}^{m} \frac{1}{k - \alpha_j} \sum_{j=1}^{m} \frac{1}{k + c_j} + \frac{(m-1)c}{k} \right) - 1 \right\} + p^{2(m-1)c} \prod_{j=1}^{m} (p - \alpha_j)^2(p + c_j)^2 \geq 0.
\]

Thus \( F(c, k, m) \) is increasing function for all \( k \geq p + n \).

This means that

\[
\beta = F(p+n) = p - \frac{np^{(m-1)c} \prod_{j=1}^{m} (p + c_j)(p - \alpha_j)}{(p + n)^{(m-1)c} \prod_{j=1}^{m} (p + n + c_j)(p + n - \alpha_j) - p^{(m-1)c} \prod_{j=1}^{m} (p + c_j)(p - \alpha_j)}
\]
Corollary 2.2. If \( f_j(z) \in M_\rho^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \) then \((B_1 * B_2 \ldots) * B_m)(z) \in M_\rho^*(n, \beta, c) \) with
\[
\beta = p - \frac{np^{(m-1)c}(p + 1)m \prod_{j=1}^{m} (p - \alpha_j)}{(p + n)^{(m-1)c}(p + n + 1)^m \prod_{j=1}^{m} (p + n - \alpha_j) - p^{(m-1)c}(p + 1)^m \prod_{j=1}^{m} (p - \alpha_j)}
\]

If we take \( p = 1 \) in Theorem 2 and Corollary 2.2, we deduce that
Corollary 2.3. If \( f_j(z) \in M_\rho^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \) then \((B_1 * B_2 \ldots) * B_m)(z) \in M_\rho^*(n, \beta, c) \) with
\[
\beta = 1 - \frac{n \prod_{j=1}^{m} (1 + c_j)(1 - \alpha_j)}{(n + 1)^{(m-1)c} \prod_{j=1}^{m} (n + 1 + c_j)(n + 1 - \alpha_j) - \prod_{j=1}^{m} (1 + c_j)(1 - \alpha_j)}
\]

Corollary 2.4. If \( f_j(z) \in M_\rho^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \) then \((B_1 * B_2 \ldots) * B_m)(z) \in M_\rho^*(n, \beta, c) \) with
\[
\beta = 1 - \frac{n2^m \prod_{j=1}^{m} (1 - \alpha_j)}{(n + 1)^{(m-1)c}(n + 2)^m \prod_{j=1}^{m} (n + 1 - \alpha_j) - 2^m \prod_{j=1}^{m} (1 - \alpha_j)}
\]

Further taking \( c_j = 0 \) in Corollary 2.3, we get
Corollary 2.5. If \( f_j(z) \in M_\rho^*(n, \alpha_j, c) \) for each \( j = 1, 2, \ldots, m \) then \((B_1 * B_2 \ldots) * B_m)(z) \in M_\rho^*(n, \beta, c) \) with
\[
\beta = 1 - \frac{n \prod_{j=1}^{m} (1 - \alpha_j)}{(n + 1)^{(m-1)c+m} \prod_{j=1}^{m} (n + 1 - \alpha_j) - \prod_{j=1}^{m} (1 - \alpha_j)}
\]

Remark. All results due to Nishiwaki and Owa [7] can be deduced as special cases if we put \( c = 0 \) and \( c = 1 \) in Theorem 2 and Corollaries 2.2-2.5.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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