THRIAD GEODESIC COMPOSITIONS IN FOUR DIMENSIONAL SPACE WITH AN AFFINE CONNECTEDNESS WITHOUT A TORSION

(COMMUNICATED BY KRISHNAN LAL DUGAL)

MUSA AJETI

ABSTRACT. Let $A_4$ be an affinely connected space without a torsion. Following [7] we introduce the affinors $a_\alpha^\beta, b_\alpha^\beta$ and $\tilde{c}_\alpha^\beta = \iota c_\alpha^\beta = -i a_\alpha^\beta b_\alpha^\beta (i^2 = -1)$ which define the compositions $X_2 \times \bar{X}_2, Y_2 \times \bar{Y}_2$ and $Z_2 \times \bar{Z}_2$, respectively. The first two compositions are conjugate. The composition $U_2 \times \bar{U}_2$ generated by the affinor $d_\alpha^\beta = a_\alpha^\beta + b_\alpha^\beta + c_\alpha^\beta$ is considered too. We have found necessary and sufficient condition for any of the above compositions to be of the kind $(g - g)$. Characteristics of the spaces $A_4$ that contain such compositions are obtained. Connections between Richis tensor and fundamental density of $E_4 A_4$ are establish when the space is equiaffine and the compositions $X_2 \times \bar{X}_2, Y_2 \times \bar{Y}_2, Z_2 \times \bar{Z}_2$, are simultaneously of the kind $(g - g)$.

1. INTRODUCTION

Let $A_N$ be a space with a symmetric affine connectedness without a torsion, defined by $\Gamma_{\alpha\beta}^\gamma$. Let consider a composition $X_n \times X_m$ of two differentiable basic manifolds $X_n$ and $X_m (n + m = N)$ in the space $A_N$. For every point of the space of compositions $A_N(X_n \times X_m)$ there are two positions of the basic manifolds, which we denotes by $P(X_n)$ and $P(X_m)$ [3]. The defining of a composition in the space $A_N$ is equivalent to defining of a field of an affinor $a_\alpha^\beta$, that satisfies the condition [2], [3]

$\nabla [a_\alpha^\beta - a_\beta^\alpha] = 0$

The affinor $a_\alpha^\beta$ is called an affinor of the composition [2]. According to [3] and [5] the condition for integrability of the structure is $a_\alpha^\gamma \nabla_{[\alpha a_\gamma^\beta]} - a_\alpha^\gamma \nabla_{[\beta a_\gamma^\alpha]} = 0$ The projective affinors $n a_\alpha^\beta$ and $m a_\alpha^\beta$ [3], [4], defined by the equalities $n a_\alpha^\beta = \frac{1}{2} (\delta_\alpha^\beta + a_\alpha^\beta)$, $m a_\alpha^\beta = \frac{1}{2} (\delta_\alpha^\beta - a_\alpha^\beta)$ satisfy the conditions $n a_\alpha^\beta + m a_\alpha^\beta = \delta_\alpha^\beta, n a_\alpha^\beta + m a_\alpha^\beta = a_\alpha^\beta$ For

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every vector \( v^\alpha \in A_N(X_n \times X_m) \) we have \( v^\alpha = n(a^\alpha v_n^\alpha + m a^\alpha v_m^\alpha) = V^\alpha + \tilde{V}^\alpha \), where \( V^\alpha = n(a^\alpha v_n^\alpha) \in P(X_n), V^\alpha = m(a^\alpha v_m^\alpha) \in P(X_m) \).

The composition \( X_n \times X_m \in A_N(n + m = N) \) for which the positions \( P(X_n) \) and \( P(X_m) \) are parallelly translated along any line of \( X_n \) and \( X_m \), respectively is called a composition of the kind \((g - g)\) [3] or geodesic composition [6]. According to [3] the geodesic composition is characterized with the equality

\[
(2) \quad a^\alpha \nabla a^\beta + a^\beta \nabla a^\alpha = 0
\]

Let \( A_4 \) be an space with affine connectedness without a torsion, defined by \( \Gamma^\sigma_\alpha_\beta(\alpha, \beta, \sigma = 1; 2; 3; 4) \). Let \( \upsilon_1^\alpha, \upsilon_2^\alpha, \upsilon_3^\alpha, \upsilon_4^\alpha \) are independent vector fields in \( A_4 \). Following [7] we define the covectors \( \tilde{\upsilon}_\alpha \) by the equalities

\[
(3) \quad \upsilon^\beta_{\sigma} \tilde{\upsilon}_\alpha = \delta^\beta_{\sigma} \iff \upsilon^\beta_{\alpha} \upsilon_\beta = \delta^\beta_{\alpha}
\]

According to [6], [7] we can define the affinor

\[
(4) \quad a^\beta_{\alpha} = \upsilon^1_{\beta} \upsilon^\alpha_1 + \upsilon^2_{\beta} \upsilon^\alpha_2 - \upsilon^3_{\beta} \upsilon^\alpha_3 - \upsilon^4_{\beta} \upsilon^\alpha_4
\]

that satisfy the equalities (1). The affinor (4) defines a composition \((X_n \times X_m)\) in \( A_4 \). The projective affinors of the composition \((X_n \times X_m)\) are [7]

\[
(5) \quad \frac{1}{a} a^\beta_{\alpha} = \upsilon^1_{\beta} \upsilon^\alpha_1 + \upsilon^2_{\beta} \upsilon^\alpha_2, \quad \frac{2}{a} a^\beta_{\alpha} = \upsilon^3_{\beta} \upsilon^\alpha_3 + \upsilon^4_{\beta} \upsilon^\alpha_4
\]

Following [7] we choose the net \((v, \upsilon, \upsilon, \upsilon)\) for a coordinate one. Then we have

\[
(6) \quad \upsilon^\alpha_1(1, 0, 0, 0), \quad \upsilon^\alpha_2(0, 1, 0, 0), \quad \upsilon^\alpha_3(0, 0, 1, 0), \quad \upsilon^\alpha_4(0, 0, 0, 1)
\]

Let consider the vectors [7]

\[
(7) \quad \upsilon^\alpha_1 = \upsilon^\alpha + \upsilon^\alpha_2, \quad \upsilon^\alpha_2 = \upsilon^\alpha + \upsilon^\alpha_4, \quad \upsilon^\alpha_3 = \upsilon^\alpha - \upsilon^\alpha_2, \quad \upsilon^\alpha_4 = \upsilon^\alpha - \upsilon^\alpha_4
\]

We define the covectors \( \tilde{\upsilon}^\alpha_\sigma \) by the equalities

\[
(8) \quad \tilde{\upsilon}^\alpha_{\beta} = \delta^\beta_{\sigma} \iff \tilde{\upsilon}^\alpha_{\sigma} = \delta^\beta_{\alpha}
\]

From (3) and (8) follow

\[
(9) \quad \frac{1}{a} \upsilon^\alpha_1 = \frac{1}{2} (\upsilon^\alpha + \upsilon^\alpha_2), \quad \frac{1}{a} \upsilon^\alpha_2 = \frac{1}{2} (\upsilon^\alpha + \upsilon^\alpha_4), \quad \frac{1}{a} \upsilon^\alpha_3 = \frac{1}{2} (\upsilon^\alpha - \upsilon^\alpha_2), \quad \frac{1}{a} \upsilon^\alpha_4 = \frac{1}{2} (\upsilon^\alpha - \upsilon^\alpha_4)
\]

Let consider the affinor...
\( b_\alpha^\beta = w_1^\alpha b_\alpha^1 + w_2^\beta b_\alpha^2 - w_3^\alpha b_\alpha^3 - w_4^\beta b_\alpha^4, \)

which according to [7] satisfies the equality \( b_\alpha^\beta b_\sigma^\alpha = \delta_\beta^\alpha \). Therefore the affinor (10) defines a composition \( Y_2 \times Y_2 \) in \( A_4 \). According to [7] the compositions \( X_2 \times X_2 \) and \( Y_2 \times Y_2 \) are conjugate. By (3), (7), (8) and (10) we obtain

\[
(11) \quad b_\alpha^\beta = \upsilon_1^\alpha 3 \upsilon_\alpha + \upsilon_2^\beta 4 \upsilon_\alpha + \upsilon_2^\beta 4 \upsilon_\alpha .
\]

Following [7] let consider the affinor \( c_\alpha^\beta = -a_\beta^\alpha b_\sigma^\alpha \), which satisfies the equality \( c_\alpha^\beta c_\alpha^\alpha = -\delta_\alpha^\alpha \). With the help of (3), (4), (11) we establish

\[
(12) \quad c_\alpha^\beta = \upsilon_3^\alpha 1 \upsilon_\alpha - \upsilon_1^\alpha 3 \upsilon_\alpha + \upsilon_2^\beta 4 \upsilon_\alpha - \upsilon_2^\beta 4 \upsilon_\alpha .
\]

The affinor \( c_\alpha^\beta = i c_\alpha^\beta \), where \( i^2 = -1 \), defines a composition \( Z_2 \times Z_2 \) in \( A_4 \).

\section{Geodesic compositions in spaces \( A_4 \)}

According to [8] we have the following derivative equations

\[
(13) \quad \nabla_\sigma v_\alpha^\beta = T_\sigma^\nu v_\alpha^\nu, \quad \nabla_\sigma v_\beta^\alpha = -T_\nu^\sigma v_\beta^\nu .
\]

Let consider the composition \( X_2 \times X_2 \) and let accept:

\( \alpha, \beta, \gamma, \sigma, \nu, \tau \in \{1, 2, 3, 4\}; i, j, k, s \in \{1, 2\}; \bar{i}, \bar{j}, \bar{k}, \bar{s} \in \{3, 4\} \).

\textbf{Theorem 1.1.} The composition \( X_2 \times X_2 \) is of the kind \( (g - g) \) if and only if the coefficients of the derivative equations (13) satisfy the conditions

\[
(14) \quad T_\sigma^\nu v_\alpha^\nu = 0, \quad T_\tau^\sigma v_\alpha^\tau = 0.
\]

\textbf{Proof:} According to (4) and (13) we have

\[
(15) \quad \nabla_\beta a_\sigma^\nu = T_\beta^\nu v_\sigma^1 + T_\beta^\nu v_\sigma^2 - T_\beta^\nu v_\sigma^3 + T_\beta^\nu v_\sigma^4 - T_\beta^\nu v_\sigma^2 .
\]

Taking into account the independence of the covectors \( v_\alpha \) and using (2), (3), (4), (15) we find the equalities

\[
(\delta_\beta^\sigma + a_\beta^\sigma)(T_\sigma^\nu v_\alpha^\nu + T_\tau^\sigma v_\alpha^\nu) = 0, \quad (\delta_\beta^\sigma + a_\beta^\sigma)(T_\sigma^\nu v_\alpha^\nu + T_\tau^\sigma v_\alpha^\nu) = 0,
\]

(16)

\[
(\delta_\beta^\sigma - a_\beta^\sigma)(T_\sigma^\nu v_\alpha^\nu + T_\tau^\sigma v_\alpha^\nu) = 0, \quad (\delta_\beta^\sigma - a_\beta^\sigma)(T_\sigma^\nu v_\alpha^\nu + T_\tau^\sigma v_\alpha^\nu) = 0.
\]

Because of the independence of the vectors \( v_\alpha^\nu \) it follows an equivalence of (16) to the following equalities.
\[
3T_\beta + a_\beta^\sigma T_\sigma = 0, \quad 4T_\beta + a_\beta^\sigma T_\sigma = 0, \quad 3T_\beta + a_\beta^\sigma T_\sigma = 0, \quad 4T_\beta + a_\beta^\sigma T_\sigma = 0,
\]
(17)
\[
3T_\beta - a_\beta^\sigma T_\sigma = 0, \quad 2T_\beta - a_\beta^\sigma T_\sigma = 0, \quad 3T_\beta - a_\beta^\sigma T_\sigma = 0, \quad 4T_\beta - a_\beta^\sigma T_\sigma = 0.
\]

Now it is easy to see that the equalities (14) follow after contraction by \(\upsilon^\beta_1\) and \(\upsilon^\beta_2\) for the first four equalities of (17) and by \(\upsilon^\beta_3\) and \(\upsilon^\beta_4\) for the last four equalities of (17). Let’s note that the equalities (17) are proved in [6] by another approach.

**Corollary 1.2.** If the net \((\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4)\) is chosen as a coordinate one then the composition \(X_2 \times X_2\) from the kind \((g-g)\) characterizes by the following equalities for:

1.1) the coefficients of the derivative equations

\[
\frac{7}{k} T_k = 0, \quad \frac{1}{k} T_k = 0
\]
(18)

1.2) the coefficients of the connectedness

\[
\Gamma^i_{sk}, \quad \Gamma^i_{\sigma k} = 0.
\]
(19)

**Proof:** Let choose the net \((\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4)\) for a coordinate one. Then by (6) and (14) we find (18). According to [1] and (13) we can write \(\partial_\nu v^\beta_\alpha + \Gamma^\beta_\sigma v^\nu_\alpha = T^\nu_\sigma v^\beta\) from where using (6) we obtain

\[
\Gamma^\beta_\sigma = T^\beta_\sigma.
\]
(20)

The equalities (19) follow from (18) and (20). Let’s note that the equalities (19) are obtained in [3], when the coordinates are adaptive with the composition \(X_2 \times X_2\). This happens so, because the chosen coordinate net raises adaptive with the composition coordinates.

From (19) and \(R_{\alpha \beta \gamma \nu}^\kappa = 2\partial_\nu \Gamma^\beta_\sigma - 2\Gamma^\nu_\sigma \Gamma^\beta_\gamma\) [1] we establish the validity of the following statement:

**Fact 1.** When the composition \(X_2 \times X_2\) is of the kind \((g-g)\) then in the parameters of the coordinate net \((\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4)\) the tensor of the curvature satisfy the conditions \(R_{ijk} = 0\), \(R_{i|jk} = 0\).

**Theorem 1.3.** The composition \(Y_2 \times Y_2\) is of the kind \((g-g)\) if and only if the coefficients of the derivative equations satisfy the conditions

\[
\left(\frac{1}{3} T_\sigma - \frac{3}{3} T_\sigma\right) v^\sigma = 0, \quad \left(\frac{1}{3} T_\sigma - \frac{3}{3} T_\sigma\right) v^\sigma = 0, \quad \left(\frac{1}{3} T_\sigma - \frac{3}{3} T_\sigma\right) v^\sigma = 0,
\]
(18)
If the net \( Y \) Now after contraction by \( \gamma \)

Transforming the condition \( b \) and using the independence of the covectors \( \gamma \)

Proof: Because of the equalities (11) and (14) we have

\[
\nabla_\sigma b^\beta_\alpha = \frac{\nu}{1} \nu^3 v^\beta_\alpha - \frac{3}{\nu} v^\beta_\nu v_\alpha + \frac{\nu}{3} v^\beta_\nu v_\alpha - \frac{1}{\nu} v^\beta_\nu v_\alpha
\]

(22)

Transforming the condition \( b^\beta_\alpha \nabla_\beta b^\sigma_\alpha + b^\beta_\alpha \nabla_\beta b^\sigma_\alpha = 0 \) with the help of (3), (11), (22)

and using the independence of the covectors \( v_\alpha \) we obtain the following equalities

\[
\frac{1}{1} T_{\beta} - \frac{3}{4} T_{\beta} + b^\beta_\sigma \left( \frac{1}{3} T_{\sigma} - \frac{3}{1} T_{\sigma} \right) = 0, \quad \frac{2}{1} T_{\beta} - \frac{4}{2} T_{\beta} + b^\beta_\sigma \left( \frac{2}{1} T_{\sigma} - \frac{4}{2} T_{\sigma} \right) = 0,
\]

(23)

Now after contraction by \( v_\sigma \) it is easy to see the equivalence of (23) to (21).

**Corollary 1.4.** If the net \( \left( v_1, v_2, v_3, v_4 \right) \) is chosen as a coordinate one then the composition \( Y_2 \times Y_2 \) from the kind \( (g - g) \) characterizes by the following equalities for:

2.1) the coefficients of the derivative equations

\[
\frac{1}{1} T_{\alpha} - \frac{3}{3} T_{\alpha} = T_{\pi} - \frac{1}{3} T_{\pi}, \quad \frac{2}{1} T_{\alpha} - \frac{4}{2} T_{\alpha} = T_{\pi} - \frac{1}{3} T_{\pi},
\]

(24)

2.2) the coefficients of the connectedness
The composition $Z_2 \times \overline{Z}_2$ is of the kind $(g - g)$ if and only if the coefficients of the derivative equations (13) satisfy the conditions

\[
(\begin{array}{c}
\frac{1}{4} - \frac{3}{4} \\
\frac{3}{4} - \frac{1}{4}
\end{array}) v^\sigma = \left( \begin{array}{c}
\frac{1}{4} - \frac{3}{4} \\
\frac{3}{4} - \frac{1}{4}
\end{array} \right) v^\rho,
\]

(26)

\[
(\begin{array}{c}
\frac{1}{4} - \frac{3}{4} \\
\frac{3}{4} - \frac{1}{4}
\end{array}) v^\sigma = \left( \begin{array}{c}
\frac{1}{4} - \frac{3}{4} \\
\frac{3}{4} - \frac{1}{4}
\end{array} \right) v^\rho,
\]

(27)
\[
\frac{3}{4}T_\beta - \frac{1}{2}T_\beta + c_\beta^\sigma \left( \frac{3}{2}T_\sigma + \frac{1}{4}T_\pi \right) = 0, \quad \frac{4}{4}T_\beta - \frac{2}{2}T_\beta + c_\beta^\sigma \left( \frac{2}{4} + \frac{4}{2}T_\sigma \right) = 0.
\]

Now after contraction by \( \eta^\alpha \) it is easy to see the equivalence of (28) to (26).

**Corollary 1.6.** If the net \((v, v, v, v)\) is chosen as a coordinate one then the composition \(Z_2 \times \bar{Z}_2\) from the kind \((g - g)\) characterizes by the following equalities for: 3.1) the coefficients of the derivative equations

\[
(29) \quad T_1^\alpha - T_3^\alpha = \epsilon \left( \frac{3}{1} \frac{\pi}{1} + \frac{1}{3} \pi \right), \quad T_3^\alpha - T_4^\alpha = \epsilon \left( \frac{4}{1} \frac{\pi}{1} + \frac{2}{3} \pi \right),
\]

\[
T_2^\alpha - T_4^\alpha = \epsilon \left( \frac{2}{2} \frac{\pi}{2} + \frac{1}{4} \pi \right), \quad T_4^\alpha - T_2^\alpha = \epsilon \left( \frac{4}{2} \frac{\pi}{2} + \frac{2}{4} \pi \right)
\]

3.2) the coefficients of the connectedness

\[
(30) \quad \Gamma_1^\alpha - \Gamma_3^\alpha = 2\epsilon \Gamma_1^\alpha, \quad \Gamma_2^\alpha - \Gamma_4^\alpha = 2\epsilon \Gamma_2^\alpha, \quad \Gamma_1^\alpha - \Gamma_3^\alpha = \Gamma_1^\alpha + \Gamma_2^\alpha
\]

as when \( \alpha \) accepts consecutively the values 1; 2; 3; 4 then \( \bar{\alpha} \) accepts the values 3; 4; 1; 2; respectively and \( \epsilon = 1 \) for \( \alpha = 1, 2 \), \( \epsilon = -1 \) for \( \alpha = 3, 4 \).

**Proof.** Let choose the net \((v, v, v, v)\) for a coordinate net. Then the equalities (29) follow by (6) and (26), and the equalities (30) follow by (20) and (29).

Using (19) (25) and (30) we establish the validity of the following statement:

**Fact 2.** If two of the compositions \(X_2 \times \bar{X}_2, Y_2 \times \bar{Y}_2, Z_2 \times \bar{Z}_2\) are from the kind \((g - g)\) then and the third composition is of the kind \((g - g)\).

Since from (19), (25) and (30) it follows

\[
(31) \quad \Gamma_1^\alpha = \Gamma_3^\alpha = 0, \Gamma_4^\alpha = \Gamma_2^\alpha = 0, \Gamma_1^\alpha + \Gamma_2^\alpha = 0
\]

then we can formulate

**Fact 3.** When the compositions \(X_2 \times \bar{X}_2, Y_2 \times \bar{Y}_2, Z_2 \times \bar{Z}_2\) are of the kind \((g-g)\) then in the parameters of the coordinate net \((v, v, v, v)\) the tensor of the curvature satisfy the conditions \(R_{ijkl} = R_{ijkl} = 0, R_{133} = R_{244} = R_{311} = R_{422} = R_{143} = R_{234} = R_{321} = R_{412} = 0.\)
Let consider the affinor
\[
d_{\alpha}^\beta = a_{\alpha}^\beta + b_{\alpha}^\beta + c_{\alpha}^\beta.
\]
According to (3), (4), (10) and (12) we have
\[
a_{\alpha}^\beta b_{\alpha}^\beta + b_{\alpha}^\beta a_{\alpha}^\beta = 0, b_{\alpha}^\beta c_{\alpha}^\beta + c_{\alpha}^\beta b_{\alpha}^\beta = 0, c_{\alpha}^\beta a_{\alpha}^\beta + a_{\alpha}^\beta c_{\alpha}^\beta = 0.
\]
From (32) and (33) it follows
\[
d_{\alpha}^\beta = a_{\alpha}^\beta a_{\alpha}^\beta + b_{\alpha}^\beta b_{\alpha}^\beta + c_{\alpha}^\beta c_{\alpha}^\beta = \delta_{\alpha}^\beta + \delta_{\alpha}^\beta - \delta_{\alpha}^\beta = \delta_{\alpha}^\beta,
\]
which means that the affinor \(d_{\alpha}^\beta\) defines a composition \(U_2 \times U_2\) with the positions \(P(U_2)\) and \(P(U_2)\):

**Theorem 1.7.** The composition \(U_2 \times U_2\) is of the kind \((g-g)\) if and only if the coefficients of the derivative equations (13) satisfy the conditions
\[
\frac{s}{k} - d_{\beta}^\sigma \frac{s}{k} = 0
\]
and
\[
\frac{\pi}{k} + \frac{\pi}{k+2} - \frac{\pi-2}{k} \beta - 2 \frac{\pi-2}{k+2} \beta + d_{\beta}^\sigma \left(\frac{\pi}{k} \sigma + \frac{\pi}{k+2} \sigma - \frac{\pi-2}{k} \sigma\right) = 0
\]

**Proof** According to (2) the composition \(U_2 \times U_2\) will be of the kind \((g - g)\) if and only if
\[
d_{\alpha}^\gamma \nabla_{\beta} d_{\sigma}^\gamma + d_{\beta}^\gamma \nabla_{\sigma} d_{\alpha}^\gamma = 0
\]
With the help of (4), (10), (12) and (32) we find
\[
d_{\sigma}^\gamma = \alpha_{\sigma}^\gamma + 2(v_i^\nu v_{\nu}^i + v_i^\nu v_{\nu}^i) = v_i^\nu v_{\nu}^i - v_i^\nu v_{\nu}^i + 2(v_i^\nu v_{\nu}^i).
\]
Then (36) can be written in the form
\[
d_{\alpha}^\gamma (T_\beta^\nu v_i^\sigma - i_\delta^\nu v_i^\sigma - T_\beta^\nu v_i^\sigma + i_\delta^\nu v_i^\sigma + 2 T_\beta^\nu v_i^\sigma + T_\sigma^\nu v_i^\sigma + 2 T_\beta^\nu v_i^\sigma - 2 T_\beta^\nu v_i^\sigma) +
\]
\[
d_{\beta}^\gamma (T_\sigma^\nu v_i^\alpha - i_\delta^\nu v_i^\alpha - T_\sigma^\nu v_i^\alpha + i_\delta^\nu v_i^\alpha + 2 T_\sigma^\nu v_i^\alpha + T_\sigma^\nu v_i^\alpha + 2 T_\sigma^\nu v_i^\alpha - 2 T_\sigma^\nu v_i^\alpha) = 0
\]
The equalities received from (38) after contraction by \(v_i^\alpha\) and \(v_i^\alpha\) are contracted once again by \(s_{\nu}^\nu\) and \(\pi_{\nu}^\nu\). As a result of these operations we reach to (34) and (35).

**Corollary 4** If the net \((v, v, v, v)\) is chosen as a coordinate one then the composition \(U_2 \times U_2\) from the kind \((g - g)\) characterizes by the following
isolations for:

4.1) the coefficients of the derivative equations

\[ T^i_k = 0 \]

\[ T^i_k + T^i_{k+2} - T^{i-2}_k + T^{i+2}_{k+2} + T^{i+2}_{k+2} - T^{-2}_{i+2} = 0 \]

4.2) the coefficients of the connectedness

\[ \Gamma^i_k = 0, \]

\[ \Gamma^i_k + \Gamma^i_{k+2} - \Gamma^{i-2}_k + \Gamma^{i+2}_{k+2} + \Gamma^{i+2}_{k+2} - \Gamma^{-2}_{i+2} = 0 \]

**Proof:** Let choose the net \((v_1, v_2, v_3, v_4)\) as a coordinate one. Then taking into account (4), (6) and (37) we find the following presentation of the affinor \((d^\beta_\alpha)\)

\[
(d^\beta_\alpha) = \begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

From (34), (35) and (41) we obtain the equalities (39), from where according to (20) follow (40) From [2] and the first equation of (39) it follows the validity of the statement:

**Fact 4** If the composition \(U_2 \times U_2\) is of the kind \((g - g)\), then the composition \(X_2 \times X_2\) is of the kind \((X_3 - g)\), i.e. the positions \(P(X_2)\) are parallely translated along any line of \(X_2\).

3. Geodesic compositions in spaces \(EqA_4\)

Let \(A_4\) by an equiaffine space with a fundamental 4-vector \(e_{\alpha\beta\gamma\delta}\). The quantity \(e_{\alpha\beta\gamma\delta}v^\alpha v^\beta v^\gamma v^\delta\) is called an extent, defined by the vector fields \(v^\alpha\). It is known that the extent \(V\) preserves when the vectors \(v^\alpha\) are parallely translated along any line in \(EqA_4\) [1]. We denote by \(e = e_{1234}\) the fundamental density of the equiaffine space \(EqA_4\). The following characteristics for the equiaffine spaces are known: \(\Gamma^i_k = \partial - \alpha \ln e, \nabla_v e_{\alpha\beta\gamma\delta} = 0, R_{\alpha\beta} = R_{\beta\alpha}[1]\)

**Proposition 1** If in the equiaffine space \(EqA_4\) with a fundamental 4-vector \(e_{\alpha\beta\gamma\delta} = \frac{1}{1,2,3,4}(v^\alpha_{\beta} v^\gamma v^\delta)\), where \(v^\alpha\) are defined by (3), the compositions \(X_2 \times X_2, Y_2 \times Y_2\) are the kind of \((g - g)\), then the space \(EqA_4\) is affine

**Proof:** From \(\nabla e_{\alpha\beta\gamma\delta} = \nabla^1 v^\alpha_{\beta} v^\gamma v^\delta = 0\) and derivative equations (13)
we obtain $\nabla_{\sigma} \nu = 0$ which, according to (20), is equivalent to $\Gamma_{\nu \delta}^\delta = 0$. Since the compositions $X_2 \times X_2, Y_2 \times Y_2$ are the kind of $(g-g)$, then on basis of Fact 2, (31) and $\Gamma_{\nu \delta}^\delta = \partial_\nu \ln e = 0$ we establish $\Gamma_{\nu \delta}^\delta = 0$, i.e. the space $EqA_4$ is affine.

**Proposition 2** If in the equiaffine $EqA_4$ with a fundamental density $e$ the compositions $X_2 \times X_2, Y_2 \times Y_2$ are the kind of $(g-g)$ then on basis of Fact 2, (31) and $\Gamma_{\nu \delta}^\delta = \partial_\nu \ln e = 0$ we establish $\Gamma_{\nu \delta}^\delta = 0$, i.e. the space $EqA_4$ is affine.

\[ R_{13} = -\sigma_{13} \ln e, \quad R_{14} = -\sigma_{14} \ln e + \Gamma_{13}^\alpha \partial_\alpha \ln e + \partial_\alpha \Gamma_{14}^\alpha, \]
\[ R_{24} = -\sigma_{24} \ln e, \quad R_{23} = -\sigma_{23} \ln e - \Gamma_{23}^\alpha \partial_\alpha \ln e - \partial_\alpha \Gamma_{23}^\alpha. \]

(42)

\[ R_{ij} = -\delta_{ij} \ln e - \epsilon_{i} \Gamma_{14}^i \Gamma_{14}^j, \quad R_{ij} = -\delta_{ij} \ln e - \epsilon_{j} \Gamma_{14}^i \Gamma_{14}^j \]

where $\epsilon = 1$ for $i = j$ or $i = j$, $\epsilon = -1$ for $i \neq j$ or $i \neq j$.

Now using (31) and (42) we obtain

\[ R_{14} + R_{23} = -\delta_{14} \ln e - \delta_{23} \ln e. \]

\[ \text{References} \]


MUSA AJETI, PRESHEVE, SERBIA

E-mail address: m-ajeti@hotmail.com