QUASI-HADAMARD PRODUCT OF ANALYTIC P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

(COMMUNICATED BY R.K RAINA)

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1. Introduction

Let \( T(p) \) denote the class of functions of the form

\[
f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0; p \in \mathbb{N}) \tag{1.1}
\]

which are analytic and p-valent in the open unit disc \( U = \{ z : |z| < 1 \} \).

A functions \( f(z) \) belonging to the class \( T(p) \) is said to be in the class \( F_p(\lambda, \alpha) \) if and only if

\[
\text{Re} \left\{ (1 - \lambda) \frac{f(z)}{z^p} + \lambda \frac{f'(z)}{pz^{p-1}} \right\} > \frac{\alpha}{p} \tag{1.2}
\]

for some \( \alpha (0 \leq \alpha < p) \), \( \lambda (\lambda \geq 0) \) and for all \( z \in U \). The class \( F_p(\lambda, \alpha) \) was studied by Lee et al. [7] and Aouf and Darwish [3].

Throughout the paper, let the functions of the form

\[
f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0; a_{p+n} \geq 0; p \in \mathbb{N}) \tag{1.3}
\]

\[
f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \quad (a_{p,i} > 0; a_{p+n,i} \geq 0; p \in \mathbb{N}) \tag{1.4}
\]

\[
g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \quad (b_p > 0; b_{p+n} \geq 0; p \in \mathbb{N}) \tag{1.5}
\]

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and
\[ g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \quad (b_{p,j} > 0; b_{p+n,j} \geq 0; p \in N) \] (1.6)
be analytic and p-valent in \( U \).

Let \( F_p^*(\lambda, \alpha) \) denote the class of functions \( f(z) \) of the form (1.3) and satisfying (1.2) for some \( \lambda, \alpha \) and for all \( z \in U \). Also let \( G_p^*(\lambda, \alpha) \) denote the class of functions of the form (1.3) such that \( f'(z) \) is in \( F_p^*(\lambda, \alpha) \).

We note that when \( a_p = 1 \), the class \( G_p^*(\lambda, \alpha) = G_p^*(\lambda, \alpha) \) was studied by Aouf [2].

Using similar arguments as given by Lee et al. [7] and Aouf and Darwish [3] we can easily prove the following analogous results for functions in the classes \( F_p^*(\lambda, \alpha) \) and \( G_p^*(\lambda, \alpha) \).

A function \( f(z) \) defined by (1.3) belongs to the class \( F_p^*(\lambda, \alpha) \) if and only if
\[ \sum_{n=1}^{\infty} (p + \lambda n)a_{p+n} \leq (p - \alpha)a_p \] (1.7)
and \( f(z) \) defined by (1.3) belongs to the class \( G_p^*(\lambda, \alpha) \) if and only if
\[ \sum_{n=1}^{\infty} \left( \frac{p + n}{p} \right) (p + \lambda n)a_{p+n} \leq (p - \alpha)a_p \] (1.8)

We now introduce the following class of analytic and p-valent functions which plays an important role in the discussion that follows:

A function \( f(z) \) defined by (1.3) belongs to the class \( F_{p,k}^*(\lambda, \alpha) \) if and only if
\[ \sum_{n=1}^{\infty} \left( \frac{p + n}{p} \right)^k (p + \lambda n)a_{p+n} \leq (p - \alpha)a_p \] (1.9)
where \( 0 \leq \alpha < p, \lambda \geq 0 \) and \( k \) is any fixed nonnegative real number.

We note that, for every nonnegative real number \( k \), the class \( F_{p,k}^*(\lambda, \alpha) \) is nonempty as the functions of the form
\[ f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(p - \alpha)a_p}{\left( \frac{p+n}{p} \right)^k (p + \lambda n)} \mu_{p+n} z^{p+n} \] (1.10)
where \( a_p > 0, \mu_{p+n} \geq 0 \) and \( \sum_{n=1}^{\infty} \mu_{p+n} \leq 1 \), satisfy the inequality (1.9). It is evident that \( F_{p,1}^*(\lambda, \alpha) \equiv G_p^*(\lambda, \alpha) \) and, for \( k = 0 \), \( F_{p,0}^*(\lambda, \alpha) \) is identical to \( F_p^*(\lambda, \alpha) \). Further, \( F_{p,k}^*(\lambda, \alpha) \subset F_{p,h}^*(\lambda, \alpha) \) if \( k > h \geq 0 \), the containment being proper. Whence, for any positive integer \( k \), we have the inclusion relation
\[ F_{p,k}^*(\lambda, \alpha) \subset F_{p,k-1}^*(\lambda, \alpha) \subset \ldots \subset F_{p,2}^*(\lambda, \alpha) \subset G_p^*(\lambda, \alpha) \subset F_p^*(\lambda, \alpha) \] .

Let us define the quasi-Hadamard product of the functions \( f(z) \) defined by (1.3) and \( g(z) \) defined by (1.5) by
\[ f \ast g(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n} \] (1.11)
Similarly, we can define the quasi-Hadamard product of more than two functions.
In this paper, we establish certain results concerning the quasi-Hadamard product of functions in the classes $F^*_p(k, \lambda, \alpha)$, $F^*_p(\lambda, \alpha)$ and $G^*_p(\lambda, \alpha)$ analogous to the results due to Kumar ([8] and [9]), Aouf et al. [4], Aouf [1], Darwish [5] and Hossen [6].

2. The main theorems

Unless otherwise mentioned we shall assume throughout the following results that $\lambda \geq 1$, $0 < \alpha < p$ and $p \in N$.

**Theorem 1.** Let the functions $f_i(z)$ defined by (1.4) be in the class $G^*_p(\lambda, \alpha)$ for every $i = 1, 2, ..., m$; and let the functions $g_j(z)$ defined by (1.6) be in the class $F^*_p(\lambda, \alpha)$ for every $j = 1, 2, ..., q$. Then, the quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_q(z)$ belongs to the class $F^*_{p, 2m+q+1}(\lambda, \alpha)$.

**Proof.** We denote quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_q(z)$ by the function $h(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\} z^{-\sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j} \right\}} z^{p+n}. \tag{2.1}$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right) 2^{m+q-1} (p+\lambda n) \left\{ \prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j} \right\} \leq (p-\alpha) \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\} \tag{2.2}$$

Since $f_i(z) \in G^*_p(\lambda, \alpha)$, we have

$$\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right) (p+\lambda n) a_{p+n,i} \leq (p-\alpha) a_{p,i}, \tag{2.3}$$

for every $i = 1, 2, ..., m$. Therefore

$$\left( \frac{p+n}{p} \right) (p+\lambda n) a_{p+n,i} \leq (p-\alpha) a_{p,i},$$

or

$$a_{p+n,i} \leq \frac{(p-\alpha)}{(p+\lambda n)} a_{p,i},$$

for every $i = 1, 2, ..., m$. The right-hand expression of this last inequality is not greater than $\frac{a_{p,i}}{(p+n)^2}$. Hence

$$a_{p+n,i} \leq \frac{a_{p,i}}{(p+n)^2}, \tag{2.4}$$

for every $i = 1, 2, ..., q$. Similarly, for $g_j(z) \in F^*_p(\lambda, \alpha)$, we have

$$\sum_{n=1}^{\infty} (p+\lambda n) b_{p+n,j} \leq (p-\alpha) b_{p,j} \tag{2.5}$$
for every \( j = 1, 2, \ldots, q \). Whence we obtain

\[
 b_{p+n,j} \leq \frac{b_{p,j}}{(p+n)^p},
\]

(2.6)

for every \( j = 1, 2, \ldots, q \).

Using (2.4) for for every \( i = 1, 2, \ldots, m \), (2.6) for \( j = 1, 2, \ldots, q - 1 \), and (2.5) for \( j = q \), we get

\[
\sum_{n=1}^{\infty} \left( \frac{p+n}{p} \right)^{2m+q-1} \frac{1}{(p+n)^p} \left( (p+\lambda n) \prod_{i=1}^{m} a_{p+n,i} \prod_{j=1}^{q} b_{p+n,j} \right) \leq \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q-1} b_{p,j} \cdot b_{p+n,q}
\]

\[
= \sum_{n=1}^{\infty} (p+\lambda n)b_{p+n,q} \left( \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q-1} b_{p,j} \right)
\]

Hence \( b(z) \in F_{p,2m+q-1}^*(\lambda, \alpha) \). This completes the proof of Theorem 1.

We note that the required estimate can be also obtained by using (2.4) for \( i = 1, 2, \ldots, m - 1 \), (2.6) for \( j = 1, 2, \ldots, q \) and (2.3) for \( i = m \).

Now we discuss the applications of Theorem 1. Taking into account the quasi-Hadamard product of functions \( f_1(z), f_2(z), \ldots, f_m(z) \) only, in the proof of Theorem 1, and using (2.4) for \( i = 1, 2, \ldots, m - 1 \), and (2.3) for \( i = m \), we are led to

**Corollary 1.** Let the functions \( f_i(z) \) defined by (1.4) belongs to the class \( G_{p,m}^*(\lambda, \alpha) \) for every \( i = 1, 2, \ldots, m \). Then the quasi-Hadamard product \( f_1 * f_2 * \ldots * f_m(z) \) belongs to the class \( F_{p,2m-1}^*(\lambda, \alpha) \).

Next, taking into account the quasi-Hadamard product of the functions \( g_1(z), g_2(z), \ldots, g_q(z) \) only, in the proof of Theorem 1, and using (2.6) for \( j = 1, 2, \ldots, q - 1 \), and (2.5) for \( j = q \), we are led to

**Corollary 2.** Let the functions \( g_j(z) \) defined by (1.6) belongs to the class \( F_{p}^*(\lambda, \alpha) \) for every \( j = 1, 2, \ldots, q \). Then the quasi-Hadamard product \( g_1 * g_2 * \ldots * g_q(z) \) belongs to the class \( F_{p,q-1}^*(\lambda, \alpha) \).

**References**


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