INEQUALITIES CONCERNING POLYNOMIALS HAVING ZEROS IN CLOSED INTERIOR OF A CIRCLE  
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Abstract. In this paper, we have obtained certain inequalities for polynomials having zeros in closed interior of a circle. Our result gives the generalization of the known result.  

1. Introduction and Statement of Results  
Let \( P(z) \) be a polynomial of degree \( n \) and let \( M(P, R) = \text{Max}_{|z|=R}|P(z)| \), \( m(P, k) = \text{Min}_{|z|=k}|P(z)| \), then by maximum modulus principle [4, p. 158 problem III 267 and 269], we have  
\[
M(P, r) \geq r^n M(P, 1), \text{ for } r < 1, 
\]
with equality only for \( P(z) = z^n, |z| = 1 \).  

Rivlin [5] obtained stronger inequality and proved that if \( P(z) \) is a polynomial of degree \( n \) having all its zeros in the disk \( |z| \geq 1 \), then  
\[
M(P, r) \geq (1 + \frac{r}{2})^n M(P, 1) \text{ for } r < 1. 
\]
Here equality holds for \( P(z) = (\alpha + \beta z)^n, |\alpha| = |\beta| \).  

For the polynomials of degree \( n \) not vanishing in \( |z| < k, k > 0 \), Aziz [1] obtained the following generalization of (1.2).  

Theorem 1.1 Let \( P(z) \) be a polynomial of degree \( n \), having no zeros in the disk \( |z| < k, k > 0 \), then  
\[
M(P, r) \geq \left( \frac{r + k}{1 + k} \right)^n M(P, 1), \text{ for } k \geq 1 \text{ and } r < 1 \text{ or } k < 1 \text{ and } r \leq k^2. 
\]
Here equality holds for \( P(z) = (z + k)^n \).  

By using Theorem 1.1 to the polynomial \( z^n P(\frac{1}{z}) \), Aziz [1] obtained the following:  

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Theorem 1.2 Let \( P(z) \) be a polynomial of degree \( n \), having all its zeros in the disk \( |z| \leq k, \ k > 0 \), then
\[
M(P, R) \geq \left( \frac{R + k}{1 + k} \right)^n M(P, 1), \ \text{for} \ k \leq 1 \ \text{and} \ R > 1 \ \text{or} \ k > 1 \ \text{and} \ R \geq k^2. \quad (1.4)
\]
Here equality holds for the polynomial \( P(z) = (z + k)^n \).

For the polynomials having all their zeros in \( |z| \leq k, \ k > 1 \), Jain [3] proved the following:

Theorem 1.3 Let \( P(z) \) be a polynomial of degree \( n \), having all its zeros in the disk \( |z| \leq k, \ k > 1 \), then for \( k < R < k^2 \),
\[
M(P, R) \geq R^n \left( \frac{R + k}{1 + k} \right) M(P, 1), \ \text{for} \ s < n. \quad (1.5)
\]
where \( s \) is the order of a possible zero of \( P(z) \) at \( z=0 \).

In this paper, we have obtained the following generalization of Theorem 1.3 by involving the coefficients of the polynomial \( P(z) := \sum_{j=0}^{n} a_j z^j \) of degree \( n \) having all its zeros in the disk \( |z| \leq k, \ k > 1 \) with \( s \)-fold zeros at the origin. In fact we prove:

Theorem 1.4 Let \( P(z) := \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) having all its zeros in \( |z| \leq k, \ k > 1 \), then for \( k < R < k^2 \),
\[
M(P, R) \geq \frac{n^{n-s}(k^n + R^n)^{|a_n|+2R|a_{n-1}|}}{(n-s)(R^n-k^n+R^n)|a_{n-1}|+R(R^n+1)|a_{n-1}|} \text{Max}_{|z|=1} |P(z)| \\
+ \frac{n^{n-s}(R^{n+1}-1)(|a_n|+|a_{n-1}|)}{R^n-k^n+R^n|a_{n-1}|+R(R^n+1)|a_{n-1}|} m(P, k),
\]
where \( s \) is the order of a possible zeros of \( P(z) \) at \( z=0 \).

2. Lemmas

The following lemma is due to Dewan, Singh and Yadav [2].

Lemma 2.1 If \( P(z) := \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) having no zeros in the disk \( |z| < k, \ k \geq 1 \), then
\[
\text{Max}_{|z|=1} |P'(z)| \leq n \frac{n^{n-s}(k^n + 2k^n|a_1|)}{n(1+k^n)|a_0|+2k^n|a_1|} \text{Max}_{|z|=1} |P(z)| - \left( 1 - \frac{n^{n-s}(k^n + 2k^n|a_1|)}{n(1+k^n)|a_0|+2k^n|a_1|} \right) m(P, k) \frac{n}{k^n},
\]
where \( m(P, k) = \text{Min}_{|z|=k} |P(z)| \).

Lemma 2.2 If \( P(z) := \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) having all its zeros in the disk \( |z| \geq k, \ k > 0 \), then for \( r \leq k \leq R \),
Proof of lemma 2.2 Let \( r \leq k \leq R \), then the polynomial \( G(z) = P(rz) \) has no zeros in \( |z| < \frac{k}{n} \frac{M}{r} \). As \( \frac{k}{n} \geq 1 \), we have by lemma 2.1,

\[
M(G', 1) \leq \frac{n|a_0| + \frac{k^2}{n}r|a_1|}{n(1 + \frac{k}{n})|a_0| + 2 \frac{k^2}{n} r|a_1|} M(G, 1)
\]

or

\[
M(P', r) \leq \frac{n|a_0| + \frac{k^2}{n}r|a_1|}{n(r^2 + k^2)|a_0| + 2k^2 r|a_1|} M(P, r)
\]

Since \( P'(z) \) is a polynomial of degree \( n-1 \), we have by maximum modulus principle [4],

\[
M(P', t) \cdot \frac{n}{t^{n-1}} \leq \frac{M(P', r)}{r^{n-1}}, \quad t \geq r
\]

Combining (2.1) and (2.2), we get

\[
M(P', t) \leq \frac{r^{n-1}}{t^{n-1}} \left[ \frac{n|a_0| + k^2|a_1|}{n(r^2 + k^2)|a_0| + 2k^2 r|a_1|} M(P, r)
\]

\[- \left( 1 - \frac{n|a_0| + k^2 r|a_1|}{n(r^2 + k^2)|a_0| + 2k^2 r|a_1|} \right) \frac{m(P, k) n^{n-1}}{k^n} \right], \quad t \geq r.
\]

Now we have, for \( 0 \leq \theta < 2\pi \)

\[
|P(Re^{i\theta}) - P(re^{i\theta})| \leq \int_r^R |P(te^{i\theta})| dt
\]

\[
\leq n \frac{nr|a_0| + k^2|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} M(P, r) \int_r^R t^{n-1} dt
\]

\[- \frac{n}{k^n} \left( 1 - \frac{nr^2|a_0| + k^2 r|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} \right) m(P, k) \int_r^R t^{n-1} dt
\]

\[= (R^n - r^n) \frac{nr|a_0| + k^2|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} M(P, r)
\]

\[- \frac{(R^n - r^n)}{k^n} \left( 1 - \frac{nr^2|a_0| + k^2 r|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} \right) m(P, k),
\]

which is equivalent to

\[
M(P, R) \leq (R^n - r^n) \frac{nr|a_0| + k^2|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} M(P, r)
\]

\[- \frac{(R^n - r^n)}{k^n} \left( 1 - \frac{nr^2|a_0| + k^2 r|a_1|}{nr^{n-1}(r^2 + k^2)|a_0| + 2k^2 r^{n-1}|a_1|} \right) m(P, k) + M(P, r).
\]

After by simple calculation, we get
\[ M(P, r) \geq \frac{n^{n-1}(r^2 + k^2) |a_n| + 2k^2 r^n |a_1|}{n(rR^n + r^n + R^n)|a_n| + k^2 (rR^n + R^n)|a_1|} M(P, R) \]
\[ + \frac{r^{n-1}(R^n - r^n)(n|a_n| + r|a_1|)}{n(rR^n + r^n + R^n)|a_n| + k^2 (rR^n + R^n)|a_1|} m(P, k). \]
This completes the proof of Lemma 2.2.

3. Proof of the Theorem 1.4

The polynomial \( Q(z) = z^n \frac{P(z)}{z} \) has all its zeros in \(|z| \geq \frac{1}{k}, \frac{1}{k} < 1 \) and is of degree \( n - s \). By applying Lemma 2.2 to the polynomial \( Q(z) \) with \( R = 1 \), we have
\[ M(Q, r) \geq \frac{(n-s)^{n-s-1}(r^2 + \frac{1}{k^2})|a_n| + \frac{2k}{r^n} r^n - 1|a_1|}{(n-s)(r + \frac{1}{k^2})|a_n| + \frac{k}{r^n} (1 + r^n - r)|a_1|} M(Q, 1) \]
\[ + \frac{k^{n-s-2} r^{n-s-1}(1 - r^n - r)(n|a_n| + r|a_1|)}{(n-s)(r + \frac{1}{k^2})|a_n| + \frac{k}{r^n} (1 + r^n - r)|a_1|} \min_{|z|=\frac{1}{k}} |Q(z)|, \quad \frac{1}{k} < r < \frac{1}{k}, \]
which is equivalent to
\[ r^n \max_{|z|=\frac{1}{k}} |P(z)| \geq \frac{(n-s)^{n-s-1}(r^2 + \frac{1}{k^2})|a_n| + \frac{2k}{r^n} r^n - 1|a_1|}{(n-s)(r + \frac{1}{k^2})|a_n| + \frac{k}{r^n} (1 + r^n - r)|a_1|} \max_{|z|=1} |P(z)| \]
\[ + \frac{k^{n-s-2} r^{n-s-1}(1 - r^n - r)(n|a_n| + r|a_1|)}{(n-s)(r + \frac{1}{k^2})|a_n| + \frac{k}{r^n} (1 + r^n - r)|a_1|} \frac{k^2}{k^2} \min_{|z|=k} |P(z)|, \quad \frac{1}{k} < r < \frac{1}{k}, \]
which on simplification and by replacing \( r \) by \( \frac{1}{k} \), we get
\[ M(P, R) \geq \frac{R^n (n-s)(k^2 + R^2)|a_n| + 2kR|a_{n-1}|}{(n-s)(R^n - k^2 + R^2)|a_n| + R(R^n - k^n + R^2)|a_1|} \max_{|z|=1} |P(z)| \]
\[ + \frac{R^{n-1} R^n - 1)(n-s)R|a_n| + |a_{n-1}|}{k^2 (n-s)(R^n - k^2 + R^2)|a_n| + R(R^n - k^n + R^2)|a_1|} \min_{|z|=k} |P(z)|, \quad k > 1 \text{ and } k < R < k^2. \]
This completes the proof of the Theorem 1.4.

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References
