NEW INEQUALITIES USING FRACTIONAL Q-INTEGRALS THEORY

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ABSTRACT. The aim of the present paper is to establish some new fractional q-integral inequalities on the specific time scale: $T_{t_0} = \{ t : t = t_0 q^n, n \in \mathbb{N}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$, and $0 < q < 1$.

1. Introduction

The integral inequalities play a fundamental role in the theory of differential equations. Significant development in this area has been achieved for the last two decades. For details, we refer to [12, 13, 16, 22, 18, 19] and the references therein. Moreover, the study of the the fractional q-integral inequalities is also of great importance. We refer the reader to [3, 15] for further information and applications. Now we shall introduce some important results that have motivated our work. We begin by [14], where Ngo et al. proved that for any positive continuous function $f$ on $[0, 1]$ satisfying

$$\int_x^1 f(\tau) d\tau \geq \int_x^1 \tau d\tau, \quad x \in [0, 1],$$

and for $\delta > 0$, the inequalities

$$\int_0^1 f^{\delta+1}(\tau) d\tau \geq \int_0^1 \tau^\delta f(\tau) d\tau \quad (1.1)$$

and

$$\int_0^1 f^{\delta+1}(\tau) d\tau \geq \int_0^1 \tau f^\delta(\tau) d\tau \quad (1.2)$$

hold.

Then [10], W.J. Liu, G.S. Cheng and C.C. Li established the following result:

$$\int_a^b f^{\alpha+\beta}(\tau) d\tau \geq \int_a^b (\tau - a)^\alpha f^{\beta}(\tau) d\tau, \quad (1.3)$$

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C50.
Key words and phrases. Fractional q-calculus, q-Integral inequalities.
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where $\alpha > 0, \beta > 0$ and $f$ is a positive continuous function on $[a, b]$ such that

$$\int_{a}^{b} f^{\gamma}(\tau) d\tau \geq \int_{x}^{b} (\tau - a)^{\gamma} d\tau; \ \gamma := \min(1, \beta), x \in [a, b].$$

Recently, Liu et al. [11] proved that for any positive, continuous and decreasing function $f$ on $[a, b]$, the inequality

$$\frac{\int_{a}^{b} f^{\beta}(\tau) d\tau}{\int_{a}^{b} f^{\gamma}(\tau) d\tau} \geq \frac{\int_{a}^{b} (\tau - a)^{\delta} f^{\beta}(\tau) d\tau}{\int_{a}^{b} (\tau - a)^{\delta} f^{\gamma}(\tau) d\tau}, \ \beta \geq 0, \ \delta > 0 \ (1.4)$$

is valid.

This result was generalized to the following [11]:

**Theorem 1.1.** Let $f \geq 0, g \geq 0$ be two continuous functions on $[a, b]$, such that $f$ is decreasing and $g$ is increasing. Then for all $\beta \geq \gamma > 0, \delta > 0,$

$$\frac{\int_{a}^{b} f^{\beta}(\tau) d\tau}{\int_{a}^{b} f^{\gamma}(\tau) d\tau} \geq \frac{\int_{a}^{b} g^{\delta}(\tau) f^{\beta}(\tau) d\tau}{\int_{a}^{b} g^{\delta}(\tau) f^{\gamma}(\tau) d\tau}. \ (1.5)$$

The same authors established the following result:

**Theorem 1.2.** Let $f \geq 0$ and $g \geq 0$ be two continuous functions on $[a, b]$ satisfying

$$\left( f^{\delta}(\tau) g^{\delta}(\rho) - f^{\delta}(\rho) g^{\delta}(\tau) \right) \left( f^{\beta - \gamma}(\tau) - f^{\beta - \gamma}(\rho) \right) \geq 0; \ \tau, \rho \in [a, b].$$

Then, for all $\beta \geq \gamma > 0, \delta > 0$ we have

$$\frac{\int_{a}^{b} f^{\delta + \beta}(\tau) d\tau}{\int_{a}^{b} f^{\delta + \gamma}(\tau) d\tau} \geq \frac{\int_{a}^{b} g^{\delta}(\tau) f^{\beta}(\tau) d\tau}{\int_{a}^{b} g^{\delta}(\tau) f^{\gamma}(\tau) d\tau}. \ (1.6)$$


Many researchers have given considerable attention to (1), (3) and (4) and a number of extensions and generalizations appeared in the literature (e.g. [4, 5, 6, 7, 9, 10]).

The purpose of this paper is to derive some new inequalities on the specific time scales $T_{t_0} = \{ t : t = t_0 q^{n}, n \in N \} \cup \{ 0 \}$, where $t_0 \in R$, and $0 < q < 1$. Our results, given in section 3, have some relationships with those obtained in [11] and mentioned above.

2. Notations and Preliminaries

In this section, we provide a summary of the mathematical notations and definitions used in this paper. For more details, one can consult [1, 2].

Let $t_0 \in R$. We define

$$T_{t_0} := \{ t : t = t_0 q^{n}, n \in N \} \cup \{ 0 \}, \ 0 < q < 1. \ (2.1)$$

For a function $f : T_{t_0} \to R$, the $\nabla_q$-derivative of $f$ is:

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q - 1)t} \ (2.2)$$
for all $t \in T \setminus \{0\}$ and its $\nabla q$-integral is defined by:

$$
\int_0^t f(\tau) \nabla \tau = (1 - q)t \sum_{i=0}^{\infty} q^i f(tq^i)
$$

(2.3)

The fundamental theorem of calculus applies to the $q$-derivative and $q$-integral. In particular, we have:

$$
\nabla q \int_0^t f(\tau) \nabla \tau = f(t).
$$

(2.4)

If $f$ is continuous at 0, then

$$
\int_0^t \nabla q f(\tau) \nabla \tau = f(t) - f(0).
$$

(2.5)

Let $T_{t_1}, T_{t_2}$ denote two time scales and let $f : T_{t_1} \to \mathbb{R}$ be continuous, and $g : T_{t_1} \to T_{t_2}$ be $q$-differentiable, strictly increasing such that $g(0) = 0$. Then for $b \in T_{t_1}$, we have:

$$
\int_0^b f(t) \nabla g(t) \nabla t = \int_0^{g(b)} (f \circ g^{-1})(s) \nabla s.
$$

(2.6)

The $q$-factorial function is defined as follows:

If $n$ is a positive integer, then

$$(t - s)^{(n)} = (t - s)(t - qs)(t - q^2s)...(t - q^{n-1}s).$$

(2.7)

If $n$ is not a positive integer, then

$$(t - s)^{(n)} = t^n \prod_{k=0}^{\infty} \frac{1 - \left(\frac{s}{t}\right)q^k}{1 - \left(\frac{s}{t}\right)q^{n+k}}.$$  

(2.8)

The $q$-derivative of the $q$-factorial function with respect to $t$ is

$$
\nabla_q (t - s)^{(n)} = \frac{1 - q^n}{1 - q} (t - s)^{(n-1)}.
$$

(2.9)

and the $q$-derivative of the $q$-factorial function with respect to $s$ is

$$
\nabla_q (t - s)^{(n)} = \frac{1 - q^n}{1 - q} (t - qs)^{(n-1)}.
$$

(2.10)

The $q$-exponential function is defined as

$$
e_q(t) = \prod_{k=0}^{\infty} (1 - q^k t), e_q(0) = 1
$$

(2.11)

The fractional $q$-integral operator of order $\alpha \geq 0$, for a function $f$ is defined as

$$
\nabla_q^{-\alpha} f(t) = \frac{1}{\Gamma_q^{(\alpha)}} \int_0^t (t - q\tau)^{\alpha-1} f(\tau) \nabla \tau; \quad \alpha > 0, t > 0,
$$

(2.12)

where $\Gamma_q(\alpha) := \frac{1}{1-q} \int_0^1 (\frac{u}{1-q})^{\alpha-1} e_q(qu) \nabla u$. 

3. Main Results

In this section, we state our results and we give their proofs.

**Theorem 3.1.** Suppose that $f$ is a positive, continuous and decreasing function on $T_{t_0}$. Then for all $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$, we have

$$\frac{\nabla_q^{-\alpha}[f^\beta(t) - f^\gamma(t)]}{\nabla_q^{-\alpha}[h^\beta(t) - h^\gamma(t)]} t > 0.$$  (3.1)

**Proof.** For any $t \in T_{t_0}$, then for all $\beta \geq \gamma > 0, \delta > 0, \tau, \rho \in (0, t)$, we have

$$\left(\rho^\delta - \tau^\delta\right) \left(f^{\beta - \gamma}(\tau) - f^{\beta - \gamma}(\rho)\right) \geq 0.$$  (3.2)

Let us consider

$$H(\tau, \rho) := f^\gamma(\tau)f^\delta(\rho)\left(\rho^\delta - \tau^\delta\right) \left(f^{\beta - \gamma}(\tau) - f^{\beta - \gamma}(\rho)\right).$$  (3.3)

Hence, we can write

$$2^{-1} \int_0^t \int_0^t \frac{(t - q\tau)^{(\alpha - 1)}}{\Gamma_q(\alpha)} \frac{(t - q\rho)^{(\alpha - 1)}}{\Gamma_q(\alpha)} H(\tau, \rho)\nabla_q\tau \nabla_q\rho = \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\alpha}[f^\gamma(t)]$$

$$- \nabla_q^{-\alpha}[h^\beta(t)]\nabla_q^{-\alpha}[h^\gamma(t)] \geq 0.$$  (3.4)

The proof of Theorem 3.1 is complete.

We have also the following result:

**Theorem 3.2.** Let $f, g$ and $h$ be positive and continuous functions on $T_{t_0}$, such that

$$\left(g(\tau) - g(\rho)\right)\frac{f(\rho)}{h(\rho)}\frac{f(\tau)}{h(\tau)} \geq 0; \tau, \rho \in (0, t), t > 0.$$  (3.5)

Then we have

$$\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))} \geq \frac{\nabla_q^{-\alpha}(gf(t))}{\nabla_q^{-\alpha}(gh(t))},$$  (3.6)

for any $\alpha > 0, t > 0$.

**Proof.** Let $f, g$ and $h$ be three positive and continuous functions on $T_{t_0}$. By (3.5), we can write

$$g(\tau)\frac{f(\rho)}{h(\rho)} + g(\rho)\frac{f(\tau)}{h(\tau)} - g(\rho)\frac{f(\rho)}{h(\rho)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0,$$  (3.7)

where $\tau, \rho \in (0, t), t > 0$.

Therefore,

$$g(\tau)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) \geq 0, \tau, \rho \in (0, t), t > 0.$$  (3.8)

Multiplying both sides of (3.8) by $\frac{(t - q\tau)^{(\alpha - 1)}}{\Gamma_q(\alpha)}$, then integrating the resulting inequality with respect to $\tau$ over $(0, t)$, yields

$$f(\rho)\nabla_q^{-\alpha}gh(t) + g(\rho)h(\rho)\nabla_q^{-\alpha}f(t) - g(\rho)f(\rho)\nabla_q^{-\alpha}h(t) - h(\rho)\nabla_q^{-\alpha}gf(t) \geq 0,$$  (3.9)

and so,

$$\nabla_q^{-\alpha}f(t)\nabla_q^{-\alpha}gh(t) - \nabla_q^{-\alpha}h(t)\nabla_q^{-\alpha}gf(t) \geq 0.$$  (3.10)
This ends the proof of Theorem 3.2.

Using two fractional parameters, we have a more general result:

**Theorem 3.3.** Let $f, g$ and $h$ be positive and continuous functions on $T_{t_0}$, such that

\[
(g(\tau) - g(\rho))(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}) \geq 0; \tau, \rho \in (0, t), t > 0.
\]  

Then the inequality

\[
\frac{\nabla^{-\alpha}_q (f(t))\nabla^{-\omega}_q (gh(t)) + \nabla^{-\omega}_q (f(t))\nabla^{-\alpha}_q (gh(t))}{\nabla^{-\alpha}_q (h(t))\nabla^{-\omega}_q (gf(t)) + \nabla^{-\omega}_q (h(t))\nabla^{-\alpha}_q (gf(t))} \geq 1
\]

holds, for all $\alpha > 0, \omega, t > 0$.

**Proof.** As before, from (3.9), we can write

\[
\frac{(t - q\rho)^{\omega-1}}{\Gamma_q(\omega)} \left(f(\rho)\nabla^{-\alpha}_q gh(t) + g(\rho)h(\rho)\nabla^{-\alpha}_q f(t)
\right.
\]

\[
- g(\rho)f(\rho)\nabla^{-\alpha}_q h(t) - h(\rho)\nabla^{-\alpha}_q gf(t)) \geq 0
\]

which implies that

\[
\nabla^{-\omega}_q (f(t))\nabla^{-\alpha}_q (gh(t)) + \nabla^{-\omega}_q (f(t))\nabla^{-\alpha}_q (gh(t))
\]

\[
\geq \nabla^{-\alpha}_q (h(t))\nabla^{-\omega}_q (gf(t)) + \nabla^{-\omega}_q (h(t))\nabla^{-\alpha}_q (gf(t)).
\]

Theorem 3.3 is thus proved.

**Remark 3.4.** It is clear that Theorem 3.2 would follow as a special case of Theorem 3.3 for $\alpha = \omega$.

We further have

**Theorem 3.5.** Suppose that $f$ and $h$ are two positive continuous functions such that $f \leq h$ on $T_{t_0}$. If $\frac{f}{h}$ is decreasing and $f$ is increasing on $T_{t_0}$, then for any $p \geq 1, \alpha > 0, t > 0$, the inequality

\[
\frac{\nabla^{-\alpha}_q (f(t))}{\nabla^{-\alpha}_q (h(t))} \geq \frac{\nabla^{-\alpha}_q (f^p(t))}{\nabla^{-\alpha}_q (h^p(t))}
\]

holds.

**Proof.** Thanks to Theorem 3.2, we have

\[
\frac{\nabla^{-\alpha}_q (f(t))}{\nabla^{-\alpha}_q (h(t))} \geq \frac{\nabla^{-\alpha}_q (f^{p-1}(t))}{\nabla^{-\alpha}_q (h^{p-1}(t))}.
\]

The hypothesis $f \leq h$ on $T_{t_0}$ implies that

\[
\frac{(t - q\tau)^{\alpha-1}}{\Gamma_q(\alpha)} h^{p-1}(\tau) \leq \frac{(t - q\tau)^{\alpha-1}}{\Gamma_q(\alpha)} h^p(\tau), \tau \in (0, t), t > 0.
\]

Then by integration over $(0, t)$, we get

\[
\nabla^{-\alpha}_q (h^{p-1}(t)) \leq \nabla^{-\alpha}_q (h^p(t)),
\]
and so,
\[ \frac{\nabla_q^{-\alpha}(f f^p(t))}{\nabla_q^{-\alpha}(h f^p(t))} \geq \frac{\nabla_q^{-\alpha}(f^p(t))}{\nabla_q^{-\alpha}(h^p(t))}. \] (3.19)

Then thanks to (3.16) and (3.19), we obtain (3.15).

Another result is given by the following theorem:

**Theorem 3.6.** Suppose that \( f \) and \( h \) are two positive continuous functions such that \( f \leq h \) on \( T_0 \). If \( \frac{f}{h} \) is decreasing and \( f \) is increasing on \( T_0 \), then for any \( p \geq 1, \alpha > 0, \omega > 0, t > 0 \), we have
\[ \frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(h^p(t)) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(h^p(t))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(f^p(t)) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(f^p(t))} \geq 1. \] (3.20)

**Proof.** We take \( g := f^{p-1} \) in Theorem 3.5. Then we obtain
\[ \frac{\nabla_q^{-\alpha}(f(t))\nabla_q^{-\omega}(h f^{p-1}(t)) + \nabla_q^{-\omega}(f(t))\nabla_q^{-\alpha}(h f^{p-1}(t))}{\nabla_q^{-\alpha}(h(t))\nabla_q^{-\omega}(f^p(t)) + \nabla_q^{-\omega}(h(t))\nabla_q^{-\alpha}(f^p(t))} \geq 1. \] (3.21)

The hypothesis \( f \leq h \) on \( T_0 \) implies that
\[ \frac{(t - q \rho)^{-\omega - 1}}{\Gamma_q(\omega)} h f^{p-1}(\rho) \leq \frac{(t - q \rho)^{-\omega - 1}}{\Gamma_q(\omega)} h^p(\rho), \rho \in (0, t), t > 0. \] (3.22)

Integrating both sides of (3.22) with respect to \( \rho \) over \( (0, t) \), we obtain
\[ \nabla_q^{-\omega}(h f^{p-1}(t)) \leq \nabla_q^{-\omega}(h^p(t)). \] (3.23)

Hence by (3.18) and (3.23), we have
\[ \nabla_q^{-\alpha} f(t) \nabla_q^{-\omega}(h f^{p-1}(t)) + \nabla_q^{-\omega} f(t) \nabla_q^{-\alpha}(h f^{p-1}(t)) \leq \nabla_q^{-\alpha} f(t) \nabla_q^{-\omega}(h^p(t)) + \nabla_q^{-\omega} f(t) \nabla_q^{-\alpha}(h^p(t)). \] (3.24)

By (3.21) and (3.24), we complete the proof of this theorem.

**Remark 3.7.** Applying Theorem 3.6, for \( \alpha = \omega \), we obtain Theorem 3.5.

**References**


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