REMARKS ON THE DOMINATION THEOREM FOR SUMMING OPERATORS

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Abstract. In this note we provide some applications of the general Pietsch Domination Theorem.

1. Introduction

Pietsch Domination Theorem plays a central role in the theory of absolutely summing linear operators (see [6]). In the last years, several Pietsch-type theorems have been presented in different nonlinear settings (we mention, for example, [1, 4, 5, 7, 8, 9, 10, 12, 15]); in [3] an abstract approach to the PDT was presented as an attempt of unification (see also [13, 14]).

From now on the Banach spaces will be considered over a fixed scalar field $\mathbb{K}$ that can be $\mathbb{R}$ or $\mathbb{C}$. The topological dual of a Banach space $X$ will be denoted by $X^*$ and its closed unit ball will be represented by $B_{X^*}$.

Let us recall the General Pietsch Domination Theorem recently presented in [3, 13]:

Let $X$, $Y$ and $E$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X$ to $Y$, $G$ be a Banach space and $K$ be a compact Hausdorff topological space. Let

$$S : \mathcal{H} \times E \times G \longrightarrow [0, \infty)$$

be an arbitrary map and

$$R : K \times E \times G \longrightarrow [0, \infty)$$

be such that

$$R_{x,b} : K \longrightarrow [0, \infty)$$

defined by $R_{x,b}(\varphi) = R(\varphi, x, b)$

is continuous for every $x \in E$ and $b \in G$. 

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If $R$ and $S$ are as above and $0 < p < \infty$, a mapping $f \in \mathcal{H}$ is said to be $R$-$S$-abstract $p$-summing if there is a constant $C_1 > 0$ so that

\[
\left( \sum_{j=1}^{m} S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C_1 \sup_{\varphi \in K} \left( \sum_{j=1}^{m} R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}}, \tag{1.1}
\]

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

The general unified PDT reads as follows:

**Theorem 1.1** (General Pietsch Domination Theorem). Let $R$ and $S$ be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then $f$ is $R$-$S$-abstract $p$-summing if and only if there is a constant $C > 0$ and a Borel probability measure $\mu$ on $K$ such that

\[
S(f, x, b) \leq C \left( \int_{K} R(\varphi, x, b)^p d\mu \right)^{\frac{1}{p}} \tag{1.2}
\]

for all $x \in E$ and $b \in G$.

In [3] the following concept was introduced, as a natural adaptation of [11, Definition 3.1]:

**Definition 1.** Let $X$ and $Y$ be Banach spaces. An arbitrary mapping $f : X \to Y$ is absolutely $p$-summing at $a \in X$ if there is a $C \geq 0$ so that

\[
\sum_{j=1}^{m} \|f(a + x_j) - f(a)\|^p \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p
\]

for every natural number $m$ and every $x_1, \ldots, x_m \in X$.

Also in [3], as an application of Theorem 1.1, the following Pietsch Domination type theorem is proved:

**Theorem 1.2.** Let $X$ and $Y$ be Banach spaces. An arbitrary mapping $f : X \to Y$ is absolutely $p$-summing at $a \in X$ if and only if there is a constant $C_a \geq 0$ and a Borel probability measure $\mu_a$ on $(B_{X^*}, (\sigma(X^*, X)))$ such that

\[
\|f(a + x) - f(a)\| \leq C_a \left( \int_{B_{X^*}} |\varphi(x)|^p d\mu_a(\varphi) \right)^{\frac{1}{p}}
\]

for all $x \in X$.

From the theorem above, if $f : X \to Y$ is absolutely $p$-summing at every $a \in X$ we have a family of constants $(C_a)_{a \in X}$ and a family of probability measures $(\mu_a)_{a \in X}$ on $(B_{X^*}, (\sigma(X^*, X)))$ so that

\[
\|f(a + x) - f(a)\| \leq C_a \left( \int_{B_{X^*}} |\varphi(x)|^p d\mu_a(\varphi) \right)^{\frac{1}{p}}
\]

for all $x \in X$.

A natural question arises:

**Problem 1.3.** If $f : X \to Y$ is absolutely $p$-summing at every $a \in X$, does there exist an universal constant $C \geq 0$ and a Borel probability measure $\mu$ on $(B_{X^*}, (\sigma(X^*, X)))$ such that

\[
\|f(a + x) - f(a)\| \leq C \left( \int_{B_{X^*}} |\varphi(x)|^p d\mu(\varphi) \right)^{\frac{1}{p}} \tag{1.3}
\]
for all \((a, x) \in X \times X\)?

In this note, among other results, we solve partially this question by characterizing the maps satisfying (1.3).

2. Results

We begin this section by recalling the notion of summability at a given point and introducing some concepts related to the notion of everywhere absolutely summing multilinear operators:

Definition 2. Let \(X,Y\) be Banach spaces.

(i) A map \(f : X \rightarrow Y\) is absolutely \(p\)-summing at \(a \in X\) if there is a constant \(C \geq 0\) such that

\[
\left( \sum_{j=1}^{m} \|f(a + x_j) - f(a)\|^p \right)^{\frac{1}{p}} \leq C \|\sum_{j=1}^{m} x_j\|_{w,p}^{p}
\]

for all \(x_1, \ldots, x_m \in X\) and \(m \in \mathbb{N}\).

(ii) A map \(f : X \rightarrow Y\) is strongly absolutely \(p\)-summing at \(A \subset X\) if there is a constant \(C \geq 0\) such that

\[
\left( \sum_{j=1}^{m} \|f(a_j + x_j) - f(a_j)\|^p \right)^{\frac{1}{p}} \leq C \|\sum_{j=1}^{m} x_j\|_{w,p}^{p}
\]

for all \(a_1, \ldots, a_m \in A, x_1, \ldots, x_m \in X\) and \(m \in \mathbb{N}\).

(iii) When \(A = X\) in (ii) \(f\) is called strongly everywhere absolutely \(p\)-summing.

The next theorem characterizes the maps satisfying (1.3):

Theorem 2.1. A map \(f : X \rightarrow Y\) is strongly absolutely \(p\)-summing at \(A \subset X\) if and only if there are a constant \(C \geq 0\) and a Borel probability measure \(\mu\) on \((B_X, (\sigma(X, X)))\) such that

\[
\|f(a + x) - f(a)\| \leq C \left( \int_{B_X} |\varphi(x)|^p d\mu(\varphi) \right)^{\frac{1}{p}}
\]

for all \((x, a) \in X \times A\).

Proof. Let \(\mathcal{H}\) be the set of all maps from \(X\) to \(Y\). Now choose the parameters

\[
E = A \times X
\]

\[
G = K
\]

\[
K = (B_X, (\sigma(X, X))).
\]

Define

\[
S : \mathcal{H} \times (A \times X) \times K \rightarrow [0, \infty)
\]

\[
R : (B_X, (\sigma(X, X))) \times (A, X) \times K \rightarrow [0, \infty)
\]

by

\[
S(f, (a, x), b) = \|f(a + x) - f(a)\|
\]

\[
R(\varphi, (a, x), b) = |\varphi(x)|.
\]
Note that (2.2) is equivalent to
\[
\left( \sum_{j=1}^{m} S(f, (a_j, x_j), b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{j=1}^{m} R(\varphi, (a_j, x_j), b_j)^p \right)^{\frac{1}{p}},
\]
for all \((a_1, x_1), \ldots, (a_m, x_m) \in E, b_1, \ldots, b_m \in G\) and \(m \in \mathbb{N}\).

From Theorem 1.1 we have
\[
S(f, (a, x), b) \leq C \left( \int_{K} R(\varphi, (a, x), b)^p \, d\mu(\varphi) \right)^{1/p}
\]
for all \(a \in A, x \in X\) and \(b \in \mathbb{K}\), i.e.,
\[
\|f(a + x) - f(a)\| \leq C \left( \int_{K} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}
\]
for all \((x, a) \in X \times A\).

\[\square\]

**Corollary 2.2.** A map \(f : X \to Y\) is strongly everywhere absolutely \(p\)-summing if and only if there are a constant \(C \geq 0\) and a Borel probability measure \(\mu\) on \(B_X\) such that
\[
\|f(a + x) - f(a)\| \leq C \left( \int_{K} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}
\]
for all \((x, a) \in X \times X\).

Now we note that the general PDT allows a local version.

**Definition 3.** Let \(X, Y\) be Banach spaces.

(i) A map \(f : X \to Y\) is locally absolutely \(p\)-summing at \(a \in X\) if there are \(C \geq 0, \delta > 0\) such that
\[
\left( \sum_{j=1}^{m} \|f(a + x_j) - f(a)\|^p \right)^{\frac{1}{p}} \leq C \|x_j\|_{w,p} \quad (2.3)
\]
for every \(x_1, \ldots, x_m \in X\) so that \(\|x_j\| \leq \delta\).

(ii) A map \(f : X \to Y\) is locally strongly absolutely \(p\)-summing at \(A \subset X\) if there are \(C \geq 0, \delta > 0\) such that
\[
\left( \sum_{j=1}^{m} \|f(a_j + x_j) - f(a_j)\|^p \right)^{\frac{1}{p}} \leq C \|x_j\|_{w,p}
\]
for every \(a \in A\) and every \(x_1, \ldots, x_m, a_1, \ldots, a_m \in X\) so that \(\|x_j\| \leq \delta\).

(iii) When \(A = X\) in (ii) \(f\) is called locally strongly everywhere absolutely \(p\)-summing.

**Theorem 2.3.** A map \(f : X \to Y\) is locally strongly absolutely \(p\)-summing at \(A\) if and only if there are \(C \geq 0, \delta > 0\) and a Borel probability measure \(\mu\) on \((B_X, (\sigma(X^*, X)))\) such that
\[
\|f(a + x) - f(a)\| \leq C \left( \int_{B_X^*} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}
\]
for all \((x, a) \in B(0, \delta) \times A\).
Proof. Let $\mathcal{H}$ be the set of all maps from $X$ to $Y$. Consider also the sets

$$E = A \times B(0, \delta)$$

$$G = K \text{ and } K = B_{X^*}.$$ 

The proof follows the lines of the proof of Theorem 2.1. 

**Corollary 2.4.** A map $f : X \to Y$ is locally strongly everywhere absolutely $p$-summing if and only if there are $C \geq 0$, $\delta > 0$ and a Borel probability measure $\mu$ on $B_{X^*}$ such that

$$\|f(a + x) - f(a)\| \leq C \left( \int_{B_{X^*}} \|\varphi(x)\|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}$$

for all $(x, a) \in B(0, \delta) \times X$.

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