APPLICATION OF AN INTEGRAL OPERATOR FOR P-VALENT FUNCTIONS

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ABSTRACT. By making use of an integral operator defined in an open unit disk, we introduce and study certain new subclasses of p-valent functions. Inclusion relationships are established and integral preserving properties of functions in these subclasses are discussed.

1. Introduction and preliminaries

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, \ldots\}. \quad (1.1)$$

which are analytic in an open unit disk $U = \{z : |z| < 1\}$.

Next we define some well known subclasses of p-valent functions as follows:

$$S_p^\ast(\xi) = \left\{ f \in A_p : \Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, \quad 0 \leq \xi < p, \quad p \in \mathbb{N}, \quad z \in U \right\};$$

$$K_p(\rho, \xi) = \left\{ f \in A_p : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi, \quad 0 \leq \xi < p, \quad p \in \mathbb{N}, \quad z \in U \right\};$$

$$K_p(\rho, \xi) = \left\{ f \in A_p : \exists g(z) \in S_p^\ast(\xi) \land \Re\left(\frac{zf'(z)}{g(z)}\right) > \rho, \quad 0 \leq \rho, \quad 0 \leq \xi < p, \quad p \in \mathbb{N}, \quad z \in U \right\};$$

$$K_p^\ast(\rho, \xi) = \left\{ f \in A_p : \exists g(z) \in C_p(\xi) \land \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \rho, \quad 0 \leq \rho, \quad 0 \leq \xi < p, \quad p \in \mathbb{N}, \quad z \in U \right\};$$

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The classes $S_p^r(\xi)$, $S_p^a(\xi)$, $C_p(\xi)$, $C_p$, $K_p(\rho, \xi)$, $K_1(\rho, \xi)$, $K_p(\rho, \xi)$ and $K_1(\rho, \xi)$ were introduced by Patil and Thakare [13], Goodman [14], Owa [15], Aouf [16], Libera [17] and Noor [18, 19] respectively.

Also note that

$$f(z) \in C_p(\xi) \text{ if and only if } \frac{zf'(z)}{p} \in S_p^a(\xi), \; 0 \leq \xi < p, p \in \mathbb{N}, z \in U.$$ 

Similarly

$$f(z) \in K_p(\xi) \text{ if and only if } \frac{zf'(z)}{p} \in K_p^a(\xi), \; 0 \leq \xi < p, p \in \mathbb{N}, z \in U.$$ 

For a function $f \in A_p$, we define a differential operator as follow:

$$\mathcal{Y}^0 f(z) = f(z);$$
$$\mathcal{Y}^1_\lambda (p, \alpha, \beta, \mu) f(z) = \left(\frac{\alpha - p\mu + \beta - p\lambda}{\alpha + \beta}\right) f(z) \left(\frac{p\mu + p\lambda}{\alpha + \beta}\right) \frac{zf'(z)}{p};$$
$$\mathcal{Y}^2_\lambda (p, \alpha, \beta, \mu) f(z) = D_{p, \lambda}(\alpha, \beta, \mu) f(z);$$
$$\mathcal{Y}^3_\lambda (p, \alpha, \beta, \mu) f(z) = D(\mathcal{Y}^1_\lambda (p, \alpha, \beta, \mu) f(z));$$
$$\vdots$$
$$\mathcal{Y}^{\lambda \prime}_\lambda (p, \alpha, \beta, \mu) f(z) = D(\mathcal{Y}^{\lambda - 1}_\lambda (p, \alpha, \beta, \mu) f(z)). \tag{1.2}$$

If $f$ is given by (1.1) then from (1.2) we have

$$\mathcal{Y}^{\lambda \prime}_\lambda (p, \alpha, \beta, \mu) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\alpha + (\mu + \lambda)(k - p) + \beta}{\alpha + \beta}\right)^n a_k z^k, \tag{1.3}$$

where $f \in A_p$, $\alpha, \beta, \mu, \lambda \geq 0, \alpha + \beta \neq 0, n \in \mathbb{N}_0 = \{0, 1, \ldots\}$.

This generalizes some well known differential operators available in literature (see for examples [1]-[11]).

Now we define the integral operator for $f(z) \in A_p$ as follows:

$$\mathcal{I}_p^0(\alpha, \beta, \mu, \lambda) f(z) = f(z);$$

$$\mathcal{I}_p^1(\alpha, \beta, \mu, \lambda) f(z) = \frac{\alpha + \beta}{\mu + \lambda} z^{p-(\frac{\alpha + \beta}{\mu + \lambda})} \int_0^z t^{(\frac{\alpha + \beta}{\mu + \lambda})-p-1} f(t) \, dt;$$

$$\mathcal{I}_p^2(\alpha, \beta, \mu, \lambda) f(z) = \frac{\alpha + \beta}{\mu + \lambda} z^{p-(\frac{\alpha + \beta}{\mu + \lambda})} \int_0^z t^{(\frac{\alpha + \beta}{\mu + \lambda})-p-1} \mathcal{I}_p^1(\alpha, \beta, \mu, \lambda) f(t) \, dt;$$

$$\vdots$$

$$\mathcal{I}_p^m(\alpha, \beta, \mu, \lambda) f(z) = \frac{\alpha + \beta}{\mu + \lambda} z^{p-(\frac{\alpha + \beta}{\mu + \lambda})} \int_0^z t^{(\frac{\alpha + \beta}{\mu + \lambda})-p-1} \mathcal{I}_p^{m-1}(\alpha, \beta, \mu, \lambda) f(t) \, dt.$$ 

This implies

$$\mathcal{I}_p^m(\alpha, \beta, \mu, \lambda) f(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{\alpha + \beta}{\alpha + (\mu + \lambda)(k - p) + \beta}\right)^m a_k z^k, \tag{1.4}$$
where $\alpha \geq 0, \beta \geq 0, \mu \geq 0, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, f(z) \in A_p, z \in \mathbb{U}$.

From (1.4) we have

$$\left(\mu + \lambda\right)z \left(\mathbb{C}_p^{m+1}(\alpha, \beta, \mu, \lambda)f(z)\right)' =$$

$$(\alpha + \beta)\left(\mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f(z)\right) - \left(\alpha + \beta - p(\mu + \lambda)\right)\left(\mathbb{C}_p^{m+1}(\alpha, \beta, \mu, \lambda)f(z)\right).$$ (1.5)

Using the operator $\mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f(z)$ defined in (1.4), we introduce the following subclasses of $p$-valent functions:

$$\mathbb{S}_m^*(p, \xi, \lambda, \alpha, \beta, \mu) = \left\{ f \in A_p : \mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f \in S_p^*(\xi) \right\};$$

$$\mathbb{C}_m(p, \xi, \alpha, \beta, \mu) = \left\{ f \in A_p : \mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f \in C_p(\xi) \right\};$$

$$\mathbb{A}_m(p, \xi, \alpha, \beta, \mu) = \left\{ f \in A_p : \mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f \in K_p(p, \xi) \right\};$$

$$\mathbb{A}_m(p, \xi, \alpha, \beta, \mu) = \left\{ f \in A_p : \mathbb{C}_p^m(\alpha, \beta, \mu, \lambda)f \in K_p^*(p, \xi) \right\}.$$

2. Inclusion relationships

In this section, we establish various inclusion relationships for the functions belonging to the new subclasses of $p$-valent functions.

Lemma 2.1. [20, 21] Let $\varphi(\mu, \nu)$ be a complex function, $\phi : D \to \mathbb{C}, D \subset \mathbb{C} \times \mathbb{C}$, and let $\mu = \mu_1 + i\mu_2, \nu = \nu_1 + i\nu_2$. Suppose that $\varphi(\mu, \nu)$ satisfies the following conditions:

1. $\varphi(\mu, \nu)$ is continuous in $D$;
2. $(1, 0) \in D$ and $\Re\{\varphi(1, 0)\} > 0$;
3. $\Re\{\varphi(i\mu_2, \nu_1)\} \leq 0$ for all $(i\mu_2, \nu_1) \in D$ such that $\nu_1 \leq -\frac{1}{2}(1 + \mu_2^2)$.

Let $h(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in $\mathbb{U}$, such that $(h(z),zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re\{\varphi(h(z),zh'(z))\} > 0(z \in \mathbb{U})$, then $\Re\{h(z)\} > 0$ for $z \in \mathbb{U}$.

Theorem 2.2. Let $f(z) \in A_p$ and $\alpha, \beta, \mu, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}$. Then

$$\mathbb{S}_m^*(p, \lambda, \alpha, \beta, \mu) \subseteq \mathbb{S}_{m+1}^*(p, \lambda, \alpha, \beta, \mu) \subseteq \mathbb{S}_{m+2}^*(p, \lambda, \alpha, \beta, \mu).$$

Proof. To prove $\mathbb{S}_m^*(p, \lambda, \alpha, \beta, \mu) \subseteq \mathbb{S}_{m+1}^*(p, \lambda, \alpha, \beta, \mu)$, let $f(z) \in \mathbb{S}_m^*(p, \lambda, \alpha, \beta, \mu)$ and assume that

$$\frac{z(\mathbb{C}_p^{m+1}(\alpha, \beta, \mu, \lambda)f(z))'}{\mathbb{C}_p^{m+1}(\alpha, \beta, \mu, \lambda)f(z)} = \xi + (p - \xi)h(z), \ 0 \leq \xi < 1, z \in \mathbb{U}. \quad (2.1)$$

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$. 

Using (1.5) and (2.1), we have

\[ \frac{z(C_p^m(\alpha, \beta, \lambda, \mu) f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda) f(z)} - \xi = (p - \xi) h(z) + \frac{(p - \xi) zh'(z)}{(\mu + z - p) + \xi + (p - \xi)h(z)}. \]  

(2.2)

Taking \( h(z) = \mu = \mu_1 + i\mu_1 \) and \( z h'(z) = \nu = \nu_1 + i\nu_1 \), we define the function \( \varphi(\mu, \nu) \) by:

\[ \varphi(\mu, \nu) = (p - \xi)\mu + \frac{(p - \xi)\nu}{(\mu + z - p) + \xi + (p - \xi)\mu}. \]

This implies

(i) \( \varphi(\mu, \nu) \) is continuous in \( D = (\mathbb{C} - \frac{\mu + z - p + \xi}{\mu + z - p}) \times \mathbb{C} \),

(ii) \((1, 0) \in D \) and \( \Re \{ \varphi(1, 0) \} > 1 - \xi \),

(iii) For all \((i\mu_2, i\nu_1) \in D \) such that \( \nu_1 \leq -\frac{1}{2}(1 + \mu_2^2) \). Therefore

\[ \Re \{ \varphi(i\mu_2, i\nu_1) \} = \Re \{ \frac{(p - \xi)\nu}{(\mu + z - p) + \xi + (p - \xi)\mu} \} = \frac{[(\alpha + \beta) + \xi]}{(\mu + z - p) + \xi + (p - \xi)\mu}, \]

\[ \Re \{ \varphi(i\mu_2, i\nu_1) \} = \frac{[(\alpha + \beta) + \xi]}{(\mu + z - p) + \xi + (p - \xi)\mu} \leq \frac{[\alpha + \beta]}{2((\mu + z - p) + \xi) + (p - \xi)\mu} < 0. \]

Therefore, the function \( \varphi(\mu, \nu) \) satisfies the conditions of Lemma 2.1. This shows that \( \Re \{ h(z) \} > 0(z \in \mathbb{U}) \), that is, \( f(z) \in \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu) \).

**Theorem 2.3.** If \( f(z) \in A_p \) and \( \alpha, \beta, \mu, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U} \). Then

\[ \mathcal{E}_m(p, \xi, \alpha, \beta, \mu) \subseteq \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu) \subseteq \mathcal{E}_{m+2}(p, \xi, \alpha, \beta, \mu). \]

**Proof.** Let \( f \in \mathcal{E}_m(p, \xi, \alpha, \beta, \mu) \Rightarrow C_p^m(\alpha, \beta, \mu, \lambda) f \in C_p(\xi), \Rightarrow \frac{z(C_p^m(\alpha, \beta, \mu, \lambda) f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda) g(z)} \in S^p_\nu(\xi) \Rightarrow C_p^m(\alpha, \beta, \mu, \lambda) f(z) \in C_p(\xi), \Rightarrow \frac{z(f(z))'}{p} \in \mathcal{E}_m(p, \xi, \alpha, \beta, \mu) \subseteq \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu), \Rightarrow \frac{z(f(z))'}{p} \in \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu) \Rightarrow C_p^{m+1}(\alpha, \beta, \mu, \lambda) f(z) \in C_p(\xi), \Rightarrow f \in \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu).

**Theorem 2.4.** If \( f(z) \in A_p \) and \( \alpha, \beta, \mu, \lambda \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U} \). Then

\[ \Re_m(p, \rho, \xi, \alpha, \beta, \mu) \subseteq \Re_{m+1}(p, \rho, \xi, \alpha, \beta, \mu) \subseteq \Re_{m+2}(p, \rho, \xi, \alpha, \beta, \mu). \]

**Proof.** Let \( f(z) \in \Re_m(p, \rho, \xi, \alpha, \beta, \mu) \) implies

\[ \Re \left( \frac{z(C_p^m(\alpha, \beta, \mu, \lambda) f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda) g(z)} \right) > \rho, g(z) \in \mathcal{E}_m^*(p, \xi, \alpha, \beta, \mu), 0 \leq \rho < 1, z \in \mathbb{U}. \]

Since \( g(z) \in \mathcal{E}_m^*(p, \xi, \alpha, \beta, \mu) \subseteq \mathcal{E}_{m+1}(p, \xi, \alpha, \beta, \mu), \) let

\[ \frac{z(C_p^{m+1}(\alpha, \beta, \mu, \lambda)g(z))'}{C_p^{m+1}(\alpha, \beta, \mu, \lambda)g(z)} = \xi + (p - \xi)H(z). \]  

(2.3)
Suppose that
\[ \left( \frac{z(C_p^{n+1}(\alpha, \beta, \mu, \lambda)f(z))'}{C_p^n(\alpha, \beta, \mu, \lambda)g(z)} \right)' = \rho + (p - \rho)h(z), \quad 0 \leq \rho < 1, \ z \in U. \tag{2.4} \]

Where \( h(z) = 1 + c_1z + c_2z^2 + \cdots \) Using (1.5), (2.3) and (2.4) we get
\[ z(C_p^{n+1}(\alpha, \beta, \mu, \lambda)f(z))' = \frac{z(C_p^{n+1}(\alpha, \beta, \mu, \lambda)z\varphi(z))'}{C_p^n(\alpha, \beta, \mu, \lambda)g(z)} + \frac{(\alpha + \beta - \rho)(p + (p - \rho)h(z))}{\xi + (p - \xi)H(z) + \left(\frac{\alpha + \beta}{\mu + \lambda} - \rho\right)^2}. \tag{2.5} \]

Using (2.3) and (2.4) we have
\[ z(C_p^{n+1}(\alpha, \beta, \mu, \lambda)f(z))' = [\rho + (p - \rho)h(z)][\xi + (p - \xi)H(z)] + (p - \rho)zh'(z). \tag{2.6} \]

Using (2.5) and (2.6), we get
\[ z(C_p^{n+1}(\alpha, \beta, \mu, \lambda)f(z))' + \rho = (p - \rho)h(z) + \frac{(p - p - \rho)zh'(z)}{(\alpha + \beta - p + \xi) + (p - \xi)H(z)}. \tag{2.7} \]

Taking \( h(z) = \mu = \mu_1 + i\mu_1 \) and \( z\varphi(z) = \nu = \nu_1 + i\nu_1 \), we define the function \( \varphi(\mu, \nu) \) by
\[ \varphi(\mu, \nu) = (p - \rho)\mu + \frac{(p - p - \rho)(\alpha + \beta - p + \xi) + (p - \xi)H(z)}{(\alpha + \beta - p + \xi) + (p - \xi)H(z)}. \]

Clearly conditions (i) and (ii) of Lemma 2.1 in \( D = \mathbb{C} \times \mathbb{C} \) are satisfied. For (iii), we proceed as follows.

\[ \Re \{ \varphi(i\mu_2, \nu_1) \} = \nu_1(p - \rho)\left[\left(\frac{\alpha + \beta}{\mu + \lambda} - p + \xi\right) + (p - \xi)h_1(x_1, y_1)\right] + \left[\frac{(p - \xi)h_2(x_2, y_2)}{H(z)}\right]^2. \]

Where \( H(z) = h_1(x_1, y_1) + ih_2(x_2, y_2), h_1(x_1, y_1) \text{ and } h_2(x_2, y_2) \) being functions of \( x \) and \( y \) and \( \Re \{ h_1(x_1, y_1) \} > 0. \) Since \( \nu_1 \leq -\frac{1}{2}(1 + \mu_2^2) \), implies
\[ \Re \{ \varphi(i\mu_2, \nu_1) \} = -\frac{1}{2}\left(1 + \mu_2^2\right)(p - \rho)\left[\left(\frac{\alpha + \beta}{\mu + \lambda} - p + \xi\right) + (p - \xi)h_1(x_1, y_1)\right]^2 + \left[\frac{(p - \xi)h_2(x_2, y_2)}{H(z)}\right]^2 < 0. \]

Applying Lemma 2.1, on \( \varphi(\mu, \nu) \), gives \( \Re \{ h(z) \} > 0(z \in U) \). This shows that \( f(z) \in R_{m+1}(p, \rho, \xi, \lambda, \alpha, \beta, \mu) \).

The following theorem can be proved in a similar manner.

**Theorem 2.5.** If \( f(z) \in A_p \) and \( \alpha, \beta, \mu, \lambda \geq 0, p \in N, m \in N_0 = N \cup \{0\}, z \in U \). Then
\[ R_{m}(p, \rho, \xi, \lambda, \alpha, \beta, \mu) \subseteq R_{m+1}(p, \rho, \xi, \lambda, \alpha, \beta, \mu) \subseteq R_{m+2}(p, \rho, \xi, \lambda, \alpha, \beta, \mu). \]

**3. Integral Operator**

For \( c > -p \) and \( f(z) \in A_p \), the integral operator \( L_{c,p} : A_p \to A_p \) is defined by
\[ L_{c,p}(f) = \frac{c + p}{z^c} - \int_0^z t^{c-1} f(t) dt. \tag{3.1} \]

The operator \( L_{c,p}(f) \) was introduced by Bernardi [12].
Theorem 3.1. Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in \mathcal{S}_m^+(p, \xi, \lambda, \alpha, \beta, \mu)$, then $L_c(f) \in \mathcal{S}_m^+(p, \xi, \lambda, \alpha, \beta, \mu)$.

Proof. Using (3.1) we get

$$z\frac{(C_p^m(\alpha, \beta, \mu, \lambda)L_{c,p}f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda)L_cf(z)} = \xi + (p-\xi)h(z).$$

(3.3)

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$. By using (3.1) and (3.2) we get

$$z\frac{(C_p^m(\alpha, \beta, \mu, \lambda)f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda)f(z)} - \xi = (p-\xi)h(z) + \frac{(p-\xi)zh'(z)}{\xi + (p-\xi)h(z) + c}.$$  

Taking $h(z) = \mu = \mu_1 + i\mu_1$ and $zh'(z) = \nu = \nu_1 + i\nu_1$, we define the function $\varphi(\mu, \nu)$ by

$$\varphi(\mu, \nu) = (p-\xi)\mu + \frac{(p-\xi)\nu}{\xi + c + (p-\xi)\mu}.$$  

Clearly conditions (i) and (ii) of Lemma 2.1 in $D = \left(\mathbb{C} - \left\{\frac{\xi+c}{\xi-p}\right\}\right) \times \mathbb{C}$ are satisfied.

We proceed for (iii) as follows;

$$\Re\{\varphi(i\mu_2, \nu_1)\} = \frac{(\xi+c)(p-\xi)\nu_1}{\xi + c} \leq \frac{-(\xi+c)(p-\xi)(1+\mu_2^2)}{2(\xi + c)^2 + 2((p-\xi)\mu_2)^2} < 0.$$  

Applying Lemma 2.1, we have $\Re\{h(z)\} > 0(z \in \mathbb{U})$, that is, $L_c(f) \in \mathcal{S}_m^+(p, \xi, \lambda, \alpha, \beta, \mu)$.

Theorem 3.2. Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in \mathcal{C}_m(p, \xi, \lambda, \alpha, \beta, \mu)$, then $L_c(f) \in \mathcal{C}_m(p, \xi, \lambda, \alpha, \beta, \mu)$.

Proof. For the proof, use Theorem 3.1 and the fact that $f(z) \in C_p(\xi) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^+(\xi)$.

Theorem 3.3. Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in \mathfrak{R}_m(p, \rho, \xi, \lambda, \alpha, \beta, \mu)$, then $L_c(f) \in \mathfrak{R}_m(p, \rho, \xi, \lambda, \alpha, \beta, \mu)$.

Proof. As $f(z) \in \mathfrak{R}_m(p, \rho, \xi, \lambda, \alpha, \beta, \mu)$ gives

$$\Re\left(z\frac{(C_p^m(\alpha, \beta, \mu, \lambda)L_c(g(z))'}{C_p^m(\alpha, \beta, \mu, \lambda)L_c(g(z))}\right) > \rho.$$  

Since $g(z) \in \mathcal{S}_m^+(p, \xi, \lambda, \alpha, \beta, \mu)$ implies $L_c(g(z)) \in \mathcal{S}_m^+(p, \xi, \lambda, \alpha, \beta, \mu)$. Let

$$z\frac{(C_p^m(\alpha, \beta, \mu, \lambda)L_c(g(z))'}{C_p^m(\alpha, \beta, \mu, \lambda)L_c(g(z))} = \xi + (p-\xi)H(z), \quad \Re(H(z)) > 0, \quad z \in \mathbb{U}.$$  

Also let

$$\left(z\frac{(C_p^m(\alpha, \beta, \mu, \lambda)L_c(f(z))'}{C_p^m(\alpha, \beta, \mu, \lambda)L_c(g(z))}\right) = \rho + (p-\rho)h(z), \quad z \in \mathbb{U}.$$  

(3.4)

Where $h(z) = 1 + c_1z + c_2z^2 + \cdots$.  

After doing calculations, we get
\[
\left( \frac{zL^m(\alpha, \beta, \mu, \lambda)f(z)'}{L^m(\alpha, \beta, \mu, \lambda)g(z)} \right)' = \frac{z(C^m_\rho(\alpha, \beta, \mu, \lambda)L_\rho(zf'(z))'}{L^m(\alpha, \beta, \mu, \lambda)L_\rho(g(z))} + c \frac{(C^m_\rho(\alpha, \beta, \mu, \lambda)L_\rho(zf'(z))}{L^m(\alpha, \beta, \mu, \lambda)L_\rho(g(z))} + c
\]
Also
\[
\frac{z(C^m_\rho(\alpha, \beta, \mu, \lambda)L_\rho(zf'(z))'}{L^m(\alpha, \beta, \mu, \lambda)L_\rho(g(z))} = [\rho + (\rho - \rho)h(z)[\xi + (\rho - \xi)H(z)] + [(\rho - \rho)zh'(z)]
\]
Hence
\[
\left( \frac{z(C^m_\rho(\alpha, \beta, \mu, \lambda)f(z)'}{L^m(\alpha, \beta, \mu, \lambda)g(z)} \right)' - \rho = (\rho - \rho)h(z) + \frac{(\rho - \rho)zh'(z)}{\xi + (\rho - \xi)H(z) + c}
\]  \hspace{1cm} (3.5)
Taking \( h(z) = \mu = \mu_1 + i\mu_2 \) and \( zh'(z) = \nu = \nu_1 + i\nu_2 \), we define the function \( \varphi(\mu, \nu) \) by
\[
\varphi(\mu, \nu) = (\rho - \rho)\mu + \frac{(\rho - \rho)\nu}{\xi + (\rho - \xi)H(z) + c}.
\]  \hspace{1cm} (3.6)
It is easy to see that the function \( \varphi(\mu, \nu) \) satisfies the conditions (i) and (ii) of Lemma 2.1 in \( D = \mathbb{C} \times \mathbb{C} \). To verify the condition (iii), we proceed as follows.
\[
\Re\{\varphi(i\mu_2, \nu_1)\} = \frac{\nu_1(\rho - \rho)[(\xi + c) + (\rho - \xi)h_1(x_1, y_1)]}{[(\xi + c) + (\rho - \xi)h_1(x_1, y_1)]^2 + [(\rho - \xi)h_2(x_2, y_2)]^2}.
\]
Where \( H(z) = h_1(x_1, y_1) + i\mu_2(x_2, y_2) \), and \( h_1(x_1, y_1) \) and \( h_2(x_2, y_2) \) being functions of \( x \) and \( y \) and \( \Re(h_1(x_1, y_1)) > 0 \). By putting \( \nu_1 \leq -\frac{1}{2}(1 + \mu_2^2) \), we obtain
\[
\Re\{\varphi(i\mu_2, \nu_1)\} = -\frac{1}{2} \frac{(1 + \mu_2^2)(\rho - \rho)[(\xi + c) + (\rho - \xi)h_1(x_1, y_1)]}{[(\xi + c) + (\rho - \xi)h_1(x_1, y_1)]^2 + [(\rho - \xi)h_2(x_2, y_2)]^2} < 0.
\]
By applying Lemma 2.1 we get \( \Re\{h(z)\} > 0(z \in \mathbb{U}) \), that is, \( L_\rho(f) \in C_m(p, \xi, \lambda, \alpha, \beta, \mu) \).

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