INCLUSION PROPERTIES FOR CERTAIN $K$-UNIFORMLY SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CERTAIN INTEGRAL OPERATOR

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Abstract. In this paper, we introduce several new $k$-uniformly classes of analytic functions defined by using the integral operator and investigate various inclusion relationships for these classes. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega$, analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in U$), such that $f(z) = g(\omega(z))$ ($z \in U$). In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [9] and [10]). For $0 \leq \gamma, \beta < 1$, we denote by $S^*(\gamma)$, $C(\gamma)$, $K(\gamma, \beta)$ and $K^*(\gamma, \beta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order $\gamma$, convex of order $\gamma$, close-to-convex of order $\gamma$, and type $\beta$ and quasi-convex of order $\gamma$, and type $\beta$ in $U$.

Now, we introduce the subclasses $US^*(k; \gamma)$, $UC(k; \gamma)$, $UK(k; \gamma, \beta)$ and $UK^*(k; \gamma, \beta)$ of the class $\mathcal{A}$ for $0 \leq \gamma, \beta < 1$, and $k \geq 0$, which are defined by

$$US^*(k; \gamma) = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} - \gamma \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\},$$

2000 Mathematics Subject Classification. 30C45.

Key words and phrases. Analytic functions, $k$-uniformly starlike functions, $k$-uniformly convex functions, $k$-uniformly close-to-convex functions, $k$-uniformly quasi-convex functions, integral operator, Hadamard product, subordination.

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where we define the function \( q \) as the following:

\[
UC (k; \gamma) = \left\{ f \in \mathcal{A} : \mathbb{R} \left( 1 + \frac{zf''(z)}{f'(z)} - \gamma \right) > k \left| \frac{zf''(z)}{f'(z)} \right| + \gamma \right\}, \tag{1.3}
\]

\[
UK (k; \gamma, \beta) = \left\{ f \in \mathcal{A} : \exists g \in US^* (k; \beta) \text{ s.t. } \mathbb{R} \left( \frac{zf'(z)}{g(z)} - \gamma \right) > k \left| \frac{zf'(z)}{g(z)} - 1 \right| \right\}, \tag{1.4}
\]

\[
UK^* (k; \gamma, \beta) = \left\{ f \in \mathcal{A} : \exists g \in UC (k; \gamma) \text{ s.t. } \mathbb{R} \left( \frac{zf'(z)}{g'(z)} - \gamma \right) > k \left| \frac{zf'(z)}{g'(z)} - 1 \right| \right\}.
\]

We note that

\[
US^* (0; \gamma) = S^* (k; \gamma), \quad UC (0; \gamma) = C (\gamma), \quad UK (0; \gamma, \beta) = K (\gamma, \beta), \quad UK^* (0; \gamma, \beta) = K^* (\gamma, \beta) \quad (0 \leq \gamma, \beta < 1).
\]

Corresponding to a conic domain \( \Omega_{k, \gamma} \) defined by

\[
\Omega_{k, \gamma} = \left\{ u + iv : u > k \sqrt{(u-1)^2 + v^2 + \gamma} \right\}, \tag{1.6}
\]

we define the function \( q_{k, \gamma} (z) \) which maps \( U \) onto the conic domain \( \Omega_{k, \gamma} \) such that \( 1 \in \Omega_{k, \gamma} \) as the following:

\[
q_{k, \gamma} (z) = \begin{cases} \frac{1+1-k^2z}{1-z} \cos \left\{ \frac{\gamma}{2} \left( \cos^{-1} k \right) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2-\gamma}{1-\sqrt{z}} & (k = 0), \\ 1 + \frac{2(1-\gamma)}{\pi z} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & (0 < k < 1), \\ \frac{1-\gamma}{\pi z} \sin \left\{ \frac{\pi}{2 \sqrt{z}} z \int_0^{\pi \sqrt{z}} \frac{dt}{\sqrt{1-t^2 \sqrt{1-k^2 t^2}}} \right\} + \frac{k^2-\gamma}{k^2-1} & (k > 1). \end{cases} \tag{1.7}
\]

where \( u(z) = z - \sqrt{k} \frac{z}{1-\sqrt{k} z} \) and \( \zeta (k) \) is such that \( k = \cosh \frac{\pi \zeta (z)}{4 \zeta (z)} \). By virtue of the properties of the conic domain \( \Omega_{k, \gamma} \), we have

\[
\mathbb{R} \{ q_{k, \gamma} (z) \} > \frac{k + \gamma}{k + 1}. \tag{1.8}
\]

Making use of the principal of subordination and the definition of \( q_{k, \gamma} (z) \), we may rewrite the subclasses \( US^* (k; \gamma), UC (k; \gamma), UK (k; \gamma, \beta) \) and \( UK^* (k; \gamma, \beta) \) as the following:

\[
US^* (k; \gamma) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < q_{k, \gamma} (z) \right\}, \tag{1.9}
\]

\[
UC (k; \gamma) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < q_{k, \gamma} (z) \right\}, \tag{1.10}
\]

\[
UK (k; \gamma, \beta) = \left\{ f \in \mathcal{A} : \exists g \in US^* (k; \beta) \text{ s.t. } \frac{zf'(z)}{g(z)} < q_{k, \gamma} (z) \right\}, \tag{1.11}
\]

\[
UK^* (k; \gamma, \beta) = \left\{ f \in \mathcal{A} : \exists g \in UC (k; \gamma) \text{ s.t. } \frac{zf'(z)}{g'(z)} < q_{k, \gamma} (z) \right\}. \tag{1.12}
\]
Recently, Komatu [5] introduced a certain integral operator $L^\lambda_a : \mathcal{A} \to \mathcal{A}$ $(a > 0; \lambda \geq 0)$ as follows:

$$L^0_a f(z) = f(z) \quad (a > 0; \lambda = 0)$$

and

$$L^\lambda_a f(z) = \frac{(1 + a)^\lambda}{\Gamma(\lambda)} \int_0^1 t^{\lambda-1} \left( \log \frac{1}{t} \right)^{\lambda-1} f(tz) \, dt \quad (a > 0; \lambda > 0).$$

Thus, if $f \in \mathcal{A}$ is of the form (1.1), it is easily seen from (1.13) and (1.14) that

$$L^\lambda_a f(z) = z + \sum_{n=2}^{\infty} \left( \frac{a+1}{a+n} \right)^\lambda a_n z^n \quad (a > 0; \lambda \geq 0).$$

It is easily to deduce from (1.15) that

$$z \left( L^{\lambda+1}_a f(z) \right)' = (a + 1) L^\lambda_a f(z) - a L^{\lambda+1}_a f(z).$$

The special case $a = 1$ of the integral operator $L^\lambda_a$ is essentially the operator which considered by Jung et al. [4].

Next, by using the operator $L^\lambda_a$, we introduce the following classes of analytic functions for $a > 0, \lambda \geq 0, k \geq 0$ and $0 \leq \gamma, \beta < 1$:

$$US^* (\lambda; k; \gamma) = \{ f \in \mathcal{A} : L^\lambda_a f(z) \in US^* (k; \gamma) \},$$

$$UC (\lambda; k; \gamma) = \{ f \in \mathcal{A} : L^\lambda_a f(z) \in UC (k; \gamma) \},$$

$$UK (\lambda; k; \gamma, \beta) = \{ f \in \mathcal{A} : L^\lambda_a f(z) \in UK (k; \gamma, \beta) \},$$

$$UK^* (\lambda; k; \gamma, \beta) = \{ f \in \mathcal{A} : L^\lambda_a f(z) \in UK^* (k; \gamma, \beta) \}.$$

We also note that

$$f(z) \in US^* (\lambda; k; \gamma) \Leftrightarrow zf' (z) \in UC (\lambda; k; \gamma),$$

and

$$f(z) \in UK (\lambda; k; \gamma, \beta) \Leftrightarrow zf' (z) \in UK^* (\lambda; k; \gamma, \beta).$$

In this paper, we investigate several inclusion properties of the classes $US^* (\lambda; k; \gamma)$, $UC (\lambda; k; \gamma)$, $UK (\lambda; k; \gamma, \beta)$ and $UK^* (\lambda; k; \gamma, \beta)$ associated with the operator $L^\lambda_a$. Some applications involving integral operators are also considered.

2. INCLUSION PROPERTIES INVOLVING THE OPERATOR $L^\lambda_a$

In order to prove the main results, we shall need the following lemmas.

**Lemma 1**[3]. Let $h(z)$ be convex univalent in $\mathbf{U}$ with $h(0) = 1$ and $\Re \{ \eta h(z) + \gamma \} > 0 \ (\eta, \gamma \in \mathbb{C})$. If $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} < h(z)$$

implies

$$p(z) < h(z).$$

**Lemma 2**[8]. Let $h(z)$ be convex univalent in $\mathbf{U}$ and let $w$ be analytic in $\mathbf{U}$ with $\Re \{ w(z) \} \geq 0$. If $p(z)$ is analytic in $\mathbf{U}$ and $p(0) = h(0)$, then

$$p(z) + w(z)zp'(z) < h(z)$$

implies

$$p(z) < h(z).$$
**Theorem 1.** \(US^* (\lambda; k; \gamma) \subset US^* (\lambda + 1; k; \gamma)\).

**Proof.** Let \(f \in US^* (\lambda; k; \gamma)\) and set
\[
p(z) = \frac{z (\mathcal{L}_a^{\lambda+1} f(z))'}{\mathcal{L}_a^{\lambda+1} f(z)} \quad (z \in U),
\]
where \(p(z)\) is analytic in \(U\) with \(p(0) = 1\). From (1.16) and (2.5), we have
\[
\frac{\mathcal{L}_a^\lambda f(z)}{\mathcal{L}_a^{\lambda+1} f(z)} = \frac{1}{a+1} \{p(z) + a\}. \tag{2.6}
\]
Differentiating (2.6) with respect to \(z\) and multiplying the result equation by \(z\), we obtain
\[
\frac{z (\mathcal{L}_a^\lambda f(z))'}{\mathcal{L}_a^\lambda f(z)} = p(z) + \frac{zp'(z)}{p(z) + a}. \tag{2.7}
\]
From this and the argument given in Section 1, we may write
\[
p(z) + \frac{zp'(z)}{p(z) + a} \prec q_{k, \gamma}(z) \quad (z \in U). \tag{2.8}
\]
Since \(a > 0\) and \(\Re\{q_{k, \gamma}(z)\} > \frac{k + \gamma}{k + 1}\), we see that
\[
\Re\{q_{k, \gamma}(z) + a\} > 0 \quad (z \in U). \tag{2.9}
\]
Applying Lemma 1 to (2.8), it follows that \(p(z) \prec q_{k, \gamma}(z)\), that is, \(f \in US^* (\lambda + 1; k; \gamma)\). \(\blacksquare\)

**Theorem 2.** \(UC (\lambda; k; \gamma) \subset UC (\lambda + 1; k; \gamma)\).

**Proof.** Applying (1.21) and Theorem 1, we observe that
\[
f(z) \in UC (\lambda; k; \gamma) \iff zf'(z) \in US^* (\lambda; k; \gamma)
\]
\[
\iff zf'(z) \in US^* (\lambda + 1; k; \gamma)
\]
\[
\iff f(z) \in UC (\lambda + 1; k; \gamma),
\]
which evidently proves Theorem 2. \(\blacksquare\)

**Theorem 3.** \(UK (\lambda; k; \gamma, \beta) \subset UK (\lambda + 1; k; \gamma, \beta)\).

**Proof.** Let \(f \in UK (\lambda; k; \gamma, \beta)\). Then, from the definition of \(UK (\lambda; k; \gamma, \beta)\), there exists a function \(r(z) \in US^* (k; \gamma)\) such that
\[
\frac{z (\mathcal{L}_a^\lambda f(z))'}{r(z)} \prec q_{k, \gamma}(z). \tag{2.10}
\]
Choose the function \(g(z)\) such that \(\mathcal{L}_a^\lambda g(z) = r(z)\). Then, \(g \in US^* (\lambda; k; \gamma)\) and
\[
\frac{z (\mathcal{L}_a^\lambda f(z))'}{\mathcal{L}_a^\lambda g(z)} \prec q_{k, \gamma}(z). \tag{2.11}
\]
Now let
\[
p(z) = \frac{z (\mathcal{L}_a^{\lambda+1} f(z))'}{\mathcal{L}_a^{\lambda+1} g(z)}, \tag{2.12}
\]
which evidently proves Theorem 3. \(\blacksquare\)
where \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). Since \( g \in US^*(\lambda; k; \gamma) \), by Theorem 1, we know that \( g \in US^*(\lambda + 1; k; \gamma) \). Let

\[
t(z) = \frac{z (L_a^{\lambda+1}g(z))'}{L_a^{\lambda+1}g(z)} \quad (z \in U),
\]

where \( t(z) \) is analytic in \( U \) with \( \Re \{ t(z) \} > \frac{k + \gamma}{k + 1} \). Also, from (2.13), we note that

\[
z (L_a^{\lambda+1}f(z))' = L_a^{\lambda+1}zf'(z) = (L_a^{\lambda+1}g(z))' p(z).
\]

Differentiating both sides of (2.14) with respect to \( z \) and multiplying the result equation by \( z \), we obtain

\[
z \frac{(L_a^{\lambda+1}zf'(z))'}{(L_a^{\lambda+1}g(z))'} = z \frac{(L_a^{\lambda+1}g(z))'}{(L_a^{\lambda+1}g(z))'} p(z) + z p'(z) = t(z) p(z) + zp'(z).
\]

Now using the identity (1.16) and (2.15), we obtain

\[
z \frac{(L_a^{\lambda}f(z))'}{L_a^{\lambda}g(z)} = \frac{L_a^{\lambda}zf'(z)}{L_a^{\lambda}g(z)} = \frac{z (L_a^{\lambda+1}zf'(z))'}{z (L_a^{\lambda+1}g(z))'} + a \frac{z (L_a^{\lambda+1}g(z))'}{z (L_a^{\lambda+1}g(z))'}
\]

\[
= \frac{z (L_a^{\lambda+1}f(z))'}{L_a^{\lambda+1}g(z)} + a
\]

\[
= \frac{t(z) p(z) + z p'(z) + ap(z)}{t(z) + a}.
\]

Since \( a > 0 \) and \( \Re \{ t(z) \} > \frac{k + \gamma}{k + 1} \), we see that

\[
\Re \{ t(z) + a \} > 0 \quad (z \in U).
\]

Hence, applying Lemma 2, we can show that \( p(z) < q_{k, \gamma}(z) \) so that \( f \in UK^*(\lambda + 1; k; \gamma, \beta) \).

This completes the proof of Theorem 3.

**Theorem 4.** \( UK^*(\lambda; k; \gamma, \beta) \subset UK^*(\lambda + 1; k; \gamma, \beta) \).

**Proof.** Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (1.22).

3. **Inclusion Properties Involving the Integral Operator** \( F_c \)

In this section, we consider the generalized Libera integral operator \( F_c \) (see [2], [6] and [7]) defined by

\[
F_c(f)(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (f \in A; c > -1).
\]

**Theorem 5.** Let \( c > -\frac{k + \gamma}{k + 1} \). If \( f \in US^*(\lambda; k; \gamma) \), then \( F_c(f) \in US^*(\lambda; k; \gamma) \).
Proof. Let \( f \in US^*(\lambda; k; \gamma) \) and set
\[
p(z) = \frac{z \left( \mathcal{L}_a^\lambda f_c (f) (z) \right)'}{\mathcal{L}_a^\lambda f_c (f) (z)} \quad (z \in U),
\]
where \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). From (3.1), we have
\[
z \left( \mathcal{L}_a^\lambda f_c (f) (z) \right)' = (c + 1) \mathcal{L}_a^\lambda f (z) - c \mathcal{L}_a^\lambda f_c (f) (z).
\]
Then, by using (3.2) and (3.3), we obtain
\[
(c + 1) \frac{\mathcal{L}_a^\lambda f (z)}{\mathcal{L}_a^\lambda f_c (f) (z)} = p(z) + c.
\]
Taking the logarithmic differentiation on both sides of (3.4) and multiplying by \( z \), we have
\[
p(z) + \frac{zp'(z)}{p(z) + c} = \frac{z \left( \mathcal{L}_a^\lambda f (z) \right)'}{\mathcal{L}_a^\lambda f (z)} < q_{k, \gamma}(z).
\]
Hence, by virtue of Lemma 1, we conclude that \( p(z) < q_{k, \gamma}(z) \) in \( U \), which implies that \( F_c (f) \in US^*(\lambda; k; \gamma) \). \[\Box\]

**Theorem 6.** Let \( c > -\frac{k + \gamma}{k + 1} \). If \( f \in UC (\lambda; k; \gamma) \), then \( F_c (f) \in UC (\lambda; k; \gamma) \).

*Proof.* By applying Theorem 5, it follows that
\[
f(z) \in UC (\lambda; k; \gamma) \iff zf'(z) \in US^*(\lambda; k; \gamma)
\]
\[
\implies F_c \left( zf' \right)(z) \in US^*(\lambda; k; \gamma) \quad \text{(by Theorem 5)}
\]
\[
\iff z \left( F_c (f) (z) \right)' \in US^*(\lambda; k; \gamma)
\]
\[
\iff F_c (f) (z) \in UC (\lambda; k; \gamma),
\]
which proves Theorem 6. \[\Box\]

**Theorem 7.** Let \( c > -\frac{k + \gamma}{k + 1} \). If \( f \in UK (\lambda; k; \gamma, \beta) \), then \( F_c (f) \in UK (\lambda; k; \gamma, \beta) \).

*Proof.* Let \( f \in UK (\lambda; k; \gamma, \beta) \). Then, in view of the definition of the class \( UK (\lambda; k; \gamma, \beta) \), there exists a function \( g \in US^*(\lambda; k; \gamma) \) such that
\[
z \left( \mathcal{L}_a^\lambda f (z) \right)'< \frac{q_{k, \gamma}(z)}{\mathcal{L}_a^\lambda g (z)}.
\]
Thus, we set
\[
p(z) = \frac{z \left( \mathcal{L}_a^\lambda f_c (f) (z) \right)'}{\mathcal{L}_a^\lambda f_c (g) (z)} \quad (z \in U),
\]
where \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). Since \( g \in US^*(\lambda; k; \gamma) \), we see from Theorem 5 that \( F_c (g) \in US^*(\lambda; k; \gamma) \). Using (3.3) and let
\[
t(z) = \frac{z \left( \mathcal{L}_a^\lambda f_c (g) (z) \right)'}{\mathcal{L}_a^\lambda f_c (g) (z)},
\]
where \( t(z) \) is analytic in \( U \) with \( \Re \{ t(z) \} > \frac{k + \gamma}{k + 1} \). Using (3.8), we have
\[
\mathcal{L}_a^\lambda z F_c'(f)(z) = \left( \mathcal{L}_a^\lambda F_c (g) (z) \right) p(z). \]
Differentiating both sides of (3.10) with respect to $z$ and multiplying by $z$, we obtain

$$
\frac{z \left( L_0^\lambda \frac{z f'(z)}{g(z)} \right)'}{L_0^\lambda f'(g(z))} = \frac{z \left( L_0^\lambda \frac{z f'(z)}{g(z)} \right)'}{L_0^\lambda f'(g(z))} \cdot \frac{p(z) + z p'(z)}{t(z) p(z) + z p'(z)}.
$$

(3.11)

Now using the identity (3.3) and (3.11), we obtain

$$
\frac{z \left( L_0^\lambda f(z) \right)'}{L_0^\lambda g(z)} = \frac{L_0^\lambda f(z)}{L_0^\lambda g(z)} = \frac{z \left( L_0^\lambda z f'(z) \right)'}{L_0^\lambda f'(g(z))} + c \frac{z \left( L_0^\lambda f'(z) \right)'}{L_0^\lambda f'(g(z))} + c \frac{z (L_0^\lambda f'(g(z)))'}{L_0^\lambda F_c'(g(z))} + c
$$

$$
= \frac{t(z)p(z) + z p'(z) + c p(z)}{t(z) + c}.
$$

(3.12)

Since $c > \frac{k + \gamma}{k + 1}$ and $\Re \{t(z)\} > \frac{k + \gamma}{k + 1}$, we see that

$$
\Re \{t(z) + c\} > 0 \quad (z \in U).
$$

(3.13)

Applying Lemma 2 to (3.12), it follows that $p(z) \prec q_k \gamma(z)$, that is $F_c(f) \in UK(\lambda; k; \gamma, \beta)$.

**Theorem 8.** Let $c > \frac{k + \gamma}{k + 1}$. If $f \in UK^\star(\lambda; k; \gamma, \beta)$, then $F_c(f) \in UK^\star(\lambda; k; \gamma, \beta)$.

**Proof.** Just as we derived Theorem 6 as consequence of Theorem 5 and (1.21), we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7 and (1.22).

**Remark.** Putting $a = 1$ in the above results, we obtain the results of Aghalary and Jahangiri [1].

**References**


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