GENERALIZED WEYL’S THEOREM FOR AN ELEMENTARY OPERATOR

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Abstract. Let $d_{A,B} \in L(L(H))$ denote either the generalized derivation
$
\delta_{A,B} = L_A - R_B$
or the elementary operator $\Delta_{A,B} = L_A R_B - I$, where
$L_A$ and $R_B$ are the left and right multiplication operators defined on $L(H)$ by
$L_A(X) = AX$ and $R_B(X) = XB$ respectively. $A$ and $B$ are bounded linear
operators on an infinite complex Hilbert space. This paper is concerned with
the transmission of polaroid and generalized Weyl’s theorem from bounded lin-
erar maps on Hilbert spaces to the elementary operator. We show that polaroid
property is preserved from $A$ and $B$ to $d_{A,B}$, we also prove that $d_{A,B}$ do not
inherit generalized Weyl’s theorem from generalized Weyl’s theorem for
$A$ and $B$. Moreover we give necessary and sufficient conditions for $d_{A,B}$ to satisfy
generalized Weyl’s theorem. Some applications for paranormal operators are
given.

1. Introduction

Let $T \in L(X)$ be a bounded linear operator on an infinite dimensional complex
Banach space $X$ and denote by $\alpha(T)$ the dimension of the kernel ker $T$, and by
$\beta(T)$ the codimension of the range $R(T)$. $T \in L(X)$ is said to be an upper semi-Fredholm operators if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is said to
be lower semi-Fredholm if $\beta(T) < \infty$. If $T \in L(X)$ is either an upper or a lower
semi-Fredholm operator, then $T$ is called a semi-Fredholm, and the index of $T$ is
defined by $ind(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called
a Fredholm operator. An operator $T \in L(X)$ is said to be Weyl operator if it is
Fredholm operator of index zero. The Weyl spectrum of $T$ is defined by

$$
\sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl operator} \}.
$$

For $T \in L(X)$ and a nonnegative integer $n$ define $T_n$ to be the restriction of $T$
to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer $n$ the range
space $R(T^n)$ is closed and $T_n$ is an upper (resp. a lower) semi-Fredholm operator,
then $T$ is called an upper (resp. a lower) semi-B-Fredholm operator. In this case

2000 Mathematics Subject Classification. 47A10, 47B20, 47B47.

Key words and phrases. Hilbert space; polaroid operators; generalized Weyl’s theorem; element-
ary operator; paranormal operators.

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Submitted April 16, 2011. Published October 24, 2011.
the index of $T$ is defined as the index of the semi-Fredholm operator $T_n$, see [5]. Moreover, if $T_n$ is a Fredholm operator, then $T$ is called a $B$-Fredholm operator. A semi-$B$-Fredholm operator is an upper or a lower semi-$B$-Fredholm operator. An operator $T \in L(X)$ is said to be $B$-Weyl operator if it is $B$-Fredholm operator of index zero. The $B$-Weyl spectrum of $T$ is defined by

$$
\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not } B\text{-Weyl operator} \}.
$$

We say that generalized Weyl’s theorem holds for $T$ if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ is the set of isolated eigenvalues of $T$.

M. Berkani [5, Theorem 4.5] has shown that every normal operator $T$ acting on a Hilbert space satisfies generalized Weyl’s theorem. This gives a generalization of the classical Weyl’s theorem. Recall that the classical Weyl’s theorem asserts that for every normal operator $T$ acting on a Hilbert space, $\sigma(T) \setminus \sigma_{W}(T) = E_0(T)$, where $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity of $T$ [21].

Recall that the ascent $p(T)$ of an operator $T$, is defined by $p(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\}$ and the descent $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf\emptyset = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. We denote by $\Pi(T) = \{ \lambda \in \mathbb{C} : p(T - \lambda I) = q(T - \lambda I) < \infty \}$ the set of poles of the resolvent. An operator $T \in L(X)$ is called Drazin invertible if and only if it has finite ascent and descent. The Drazin spectrum of an operator $T$ is defined by

$$
\sigma_D(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible} \}.
$$

Clearly,

$$
\sigma_{BW}(T) \subset \sigma_D(T) \text{ for all } T \in L(X).
$$

Let $H$ be an infinite complex Hilbert space and consider two bounded linear operators $A, B \in L(H)$. Let $L_A \in L(L(H))$ and $R_B \in L(L(H))$ be the left and the right multiplication operators, respectively, and denote by $d_{A,B} \in L(L(H))$ either the elementary operator $\Delta_{A,B}(X) = AXB - X$ or the generalized derivation $\delta_{A,B}(X) = AX - XB$. The main objective of the present paper is the transmission of polaroid and generalized Weyl’s theorem from $A$ and $B$ to $d_{A,B}$. In the second section of this paper, we show that polaroid property is preserved from $A$ and $B$ to $d_{A,B}$, and we give examples proving that $d_{A,B}$ do not inherit generalized Weyl’s theorem from generalized Weyl’s theorem for $A$ and $B$, moreover we give necessary and sufficient conditions for $d_{A,B}$ to satisfy generalized Weyl’s theorem.

In the third section we give an application to paranormal operators. Our results generalize the following ones [11, Theorem 3.3] and [15, Corollary 2.6] and [10, Theorem 3.4].

2. NECESSARY AND SUFFICIENT CONDITIONS FOR $d_{A,B}$ TO SATISFY GENERALIZED WEYL’S THEOREM

In the sequel we shall denote by $accD$ and $isoD$, the set of accumulation points and the set of isolated points of $D \subset \mathbb{C}$, respectively.

**Definition 2.1.** An operator $T \in L(X)$ is said to be polaroid if

$$
iso(\sigma(T)) \subseteq \Pi(T).
$$

It is easily seen that, if $T \in L(X)$ is polaroid, then $\Pi(T) = E(T)$.

An important subspace in local spectral theory is the the quasi-nilpotent part of $T$. 


is defined by
\[ H_0(T) = \{ x \in X : \lim_{n \to \infty} \| T^n(X) \|^{\frac{1}{n}} = 0 \}. \]
It is easily seen that \( \ker T^n \subset H_0(T) \) for every \( n \in \mathbb{N} \), see [1] for information on \( H_0(T) \).

**Lemma 2.2.** Suppose that \( A, B \in L(H) \) are polaroid operators, then \( d_{A,B} \) is polaroid.

**Proof.** Recall from [16] that \( \sigma(\delta_{A,B}) = \sigma(A) - \sigma(B) \) and \( \sigma(\Delta_{A,B}) = \sigma(A)\sigma(B) - \{1\} \).

If \( \lambda \in \text{iso} \sigma(d_{A,B}) \), then we have one of the following cases:

1. If \( d_{A,B} = \delta_{A,B} \), then there exist finite sequences \( \{ \mu_i \}_{i=1}^n \) and \( \{ \nu_i \}_{i=1}^n \), of isolated points in \( \sigma(A) \) and \( \sigma(B) \), respectively such that \( \lambda = \mu_i - \nu_i \), for all \( 1 \leq i \leq n \).
2. If \( d_{A,B} = \Delta_{A,B} \) and \( \lambda = -1 \), then either \( 0 \in \text{iso} \sigma(A) \) and \( 0 \in \text{iso} \sigma(B) \), or \( 0 \in \text{iso} \sigma(A) \) and \( 0 \notin \sigma(B) \), or \( 0 \in \text{iso} \sigma(B) \) and \( 0 \notin \sigma(A) \).
3. If \( d_{A,B} = \Delta_{A,B} \) and \( \lambda \neq -1 \), then there exist finite sequences \( \{ \mu_i \}_{i=1}^n \) and \( \{ \nu_i \}_{i=1}^n \), of isolated points in \( \sigma(A) \) and \( \sigma(B) \), respectively such that \( \mu_i \nu_i = 1 + \lambda \), for all \( 1 \leq i \leq n \).

We state by considering **Case 1.** If \( \lambda \in \text{iso} \sigma(\delta_{A,B}) \), then there exist finite sequences \( \{ \mu_i \}_{i=1}^n \) and \( \{ \nu_i \}_{i=1}^n \) such that \( \mu_i \in \text{iso} \sigma(A) \) and \( \nu_i \in \text{iso} \sigma(B) \). Since \( A \) and \( B \) are polaroid, then from [1, Theorem 3.74] \( H_0(A - \mu_i I) = \ker(A - \mu_i I)^{p_i} \) and \( H_0(B - \nu_i I) = \ker(B - \nu_i I)^{q_i}, 1 \leq i \leq n \), for some integers \( p_i, q_i \geq 1 \), \( \lambda = \mu_i - \nu_i \). The sets \( E_1 = \{ \mu_1, \mu_2, \ldots, \mu_n \} \) and \( E_2 = \{ \nu_1, \nu_2, \ldots, \nu_n \} \) are spectral sets of \( \sigma(A) \) and \( \sigma(B) \), respectively. Hence by the Riesz decomposition theorem there exist invariant subspaces \( M_k, N_k, k = 1, 2 \) of \( A \) and \( B \) respectively such that
\[ H = M_1 \oplus M_2 = N_1 \oplus N_2, \quad \sigma(A_1) = \sigma(A|_{M_1}) = E_1, \quad \sigma(B_1) = \sigma(B|_{N_1}) = E_2, \quad \sigma(A_2) = \sigma(A|_{M_2}) = \sigma(A)\setminus E_1, \quad \sigma(B_2) = \sigma(B|_{N_2}) = \sigma(B)\setminus E_2. \]
Observe that \( \mu_i \) is a pole of \( A_1 \) of order \( p_i \) and \( \nu_i \) is a pole of \( B_1 \) of order \( q_i \), for all \( 1 \leq i \leq n \). Thus \( A_1 \) and \( B_1 \) are algebraic operators [1, Theorem 3.83]. Hence \( M_1 = \bigoplus_{i=1}^n \ker(A_1 - \mu_i I)^{p_i} \), and \( N_1 = \bigoplus_{i=1}^n \ker(B_1 - \nu_i I)^{p_i} \), let \( M_{11} = \ker(A_1 - \mu_i I)^{p_i} \) and \( N_{11} = \ker(B_1 - \nu_i I)^{p_i} \), for all \( 1 \leq i \leq n \), let \( p = \max\{p_1, p_2, \ldots, p_n\} \) and \( q = \max\{q_1, q_2, \ldots, q_n\} \), and set \( p + q = r \).

Let \( Y \in R(\delta_{A,B} - \lambda I)^{\gamma} \), then there exist \( X \in L(N_{1} \oplus N_{2}, M_{1} \oplus M_{2}) \) have the representation \( X = \{ X_{ij} \}_{k,l=1}^2 \) such that \( Y = (\delta_{A,B} - \lambda I)^{\gamma}(X) \). It will be proved that \( q(\delta_{A,B} - \lambda I) \leq r \).

\[ Y = (\delta_{A,B} - \lambda I)^{\gamma}(X) = \left( \begin{array}{c} (\delta_{A_{11},B_{11}} - \lambda I)^{\gamma}(X_{11}) \\ (\delta_{A_{12},B_{12}} - \lambda I)^{\gamma}(X_{12}) \\ (\delta_{A_{21},B_{21}} - \lambda I)^{\gamma}(X_{21}) \\ (\delta_{A_{22},B_{22}} - \lambda I)^{\gamma}(X_{22}) \end{array} \right). \]

Observe that \( \delta_{A_{ij},B_{ij}} - \lambda I \) is invertible for all \( 1 \leq i, j \leq 2 \) such that \( i, j \neq 1 \). Hence there exist operators \( Z_{ij} \) such that
\[ X_{ij} = (\delta_{A_{ij},B_{ij}} - \lambda I)Z_{ij}, \quad (2.1) \]
for all \( 1 \leq i, j \leq 2 \) such that \( i, j \neq 1 \). Let \( X_{11} = [Y_{ij}]_{1 \leq i, j \leq n} \in L(\bigoplus_{i=1}^n N_{1i}, \bigoplus_{i=1}^n M_{1i}). \)
Then for \( 1 \leq i, j \leq n \), we have
\[ (\delta_{A_{i1},B_{i1}} - \lambda I)^{\gamma}(X_{11}) = \left( \begin{array}{c} (L_{A_{i1} - \mu_i} - R_{B_{i1} - \nu_i}) + (\mu_i - \nu_i - \lambda)^{\gamma} \end{array} \right) \]

\[ = \left( \sum_{k=0}^r \binom{r}{k} (L_{A_{i1} - \mu_i} - R_{B_{i1} - \nu_i})^k (\mu_i - \nu_i - \lambda)^{r-k} \right) [Y_{ij}]_{1 \leq i, j, \leq n} \]
The class of polaroid operators is large, it contains:

\[ \mathcal{H}N \]

Let 
\[ \mathcal{C}H \mathcal{N} \]

The class of all operators

**Case 3.** Example 2.3. From Case 2. Is proved by E. Boisson, B.P. Duggal and I. H. Jeon \[ \text{[9, Lemma 4.7]} \].

**Case 3.** Since \( L_AR_B \) is polaroid it follows from \[ \text{[14, Lemma 3.8]} \] that \( \Delta_{A,B} \) is polaroid.

**Remark.** The class of polaroid operators is large, it contains:

1. The class of all operators \( A \in L(H) \) such that for every complex number \( \lambda \) there exists an integer \( p_\lambda \geq 1 \) for which the following condition holds

\[ H_0(A - \lambda I) = \ker(A - \lambda I)^{p_\lambda}. \]

2. \( \mathcal{H}N \) the class of hereditarily normaloid, \( A \in \mathcal{H}N \) if every part of \( A \) is normaloid, a part of \( A \) is its restriction to an invariant subspace

3. \( \mathcal{T}H\mathcal{N} \) the class of totally hereditarily normaloid. We say that \( A \in \mathcal{H}N \) is totally hereditarily normaloid if also every invertible part of \( A \) is normaloid

4. \( \mathcal{C}H\mathcal{N} \) the class of completely totally hereditarily normaloid, \( A \in \mathcal{C}H\mathcal{N} \) if either \( A \) is totally hereditarily normaloid or \( A - \lambda I \in \mathcal{H}N \) for every \( \lambda \in \mathbb{C} \)

5. \( (p, k) - Q \) the class of \((p, k)\) quasi-hyponormal, \( A \in (p, k) - Q \) if \( A^{ek}(|A|^{p} - |A|^{2p})A^{k} \geq 0 \) for some positive integer \( k \) and \( 0 < p \leq 1 \).

From [1, Theorem 3.74] we can easily prove that \( T \in L(X) \) is polaroid if and only if there exists \( p = p(\lambda) \in \mathbb{N} \) such that \( H_0(T - \lambda I) = \ker(T - \lambda I)^{p} \) for all \( \lambda \in \text{iso}(T) \). This result allows us to deduce that Lemma 2.2 generalize the following results [12, Theorem 3.2], [12, Theorem 3.3], [12, Theorem 3.4], [12, Theorem 3.5], [15, Theorem 2.3] and [10, Lemma 3.2].

\( L_A \) inherit generalized Weyl’s theorem from generalized Weyl’s theorem for \( A \in L(X) \) and \( R_B \) inherit generalized Weyl’s theorem from generalized Weyl’s theorem for \( B^* \in L(X) \) (\( B^* \) is the dual of \( B \)) see [9, Theorem 3.5] and [9, Theorem 3.6]. But \( d_{A,B} \) do not inherit generalized Weyl’s theorem from generalized Weyl’s theorem for \( A \) and \( B \) because the following examples shows that generalized Weyl’s theorem is not preserved under products and sums of commuting operators.

**Example 2.3.** Let \( I_1 \) and \( I_2 \) be the identities on \( \mathbb{C} \) and \( l^2(\mathbb{N}) \), respectively. Let \( S_1 \) and \( S_2 \) defined on \( l^2(\mathbb{N}) \) by

\[ S_1(x_1, x_2, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots), \quad S_2(x_1, x_2, \ldots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots). \]
Let $T_1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$ and $T = T_1 \oplus T_2$, from [22, Example 1] we have generalized Weyl’s theorem holds for $T$ but it does not hold for $TT = T^2$.

**Example 2.4.** Let $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be an injective quasinilpotent operator, and let $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $U(x_1, x_2, x_3, \ldots) = (-x_1, 0, 0, \ldots)$. Define on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ the operators $T$ and $F$ by $T = I \oplus S$ and $F = U \oplus 0$.

Clearly, $F$ is a finite rank operator and $FT = TF$, it is easy to check that $\sigma(T) = \{0, 1\}$, $E(T) = \{1\}$ and it follows from [7, Example 2] that $\sigma_{BW}(T) = \{0\}$. Hence $T$ satisfies generalized Weyl’s theorem. Since $F$ is a finite rank operator, then $\sigma(F) = E(F) = \{0, -1\}$ and $\sigma_{BW}(F) = \emptyset$, then generalized Weyl’s theorem holds for $F$, and $T + F$ does not satisfy generalized Weyl’s theorem.

In the following results we give necessary and sufficient condition for $d_{A,B}$ to satisfy generalized Weyl’s theorem.

**Theorem 2.5.** Suppose that $A, B \in L(H)$ are polaroid operators which satisfy generalized Weyl’s theorem, then a necessary and sufficient condition for $\delta_{A,B}$ to satisfy generalized Weyl’s theorem is

$$\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)),$$

Proof. Assume that $\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$.

Let $\lambda \in \sigma_{BW}(\delta_{A,B}) \cap \sigma_{BW}(A)$, then for $\lambda = \mu - \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, $\mu \notin \sigma(A)$ and $\nu \notin \sigma_{BW}(B)$. Consequently $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$. Let $p(A - \mu I) = q(A - \mu I) = p_1$ and $p(B - \nu I) = q(B - \nu I) = q_1$, it follows that there exist decompositions $H = \ker(A - \mu I)p_1 \oplus \ker((A - \mu I)p_1) = M_1 \oplus M_2$ and $H = \ker((B - \nu I)q_1 \oplus \ker((B - \nu I)q_1) = N_1 \oplus N_2$ such that $\sigma(A_1) = \sigma(A |_{M_1}) = \{\mu\}$, $\sigma(A_2) = \sigma(A |_{M_2}) = \sigma(A) \setminus \{\mu\}$ and $\sigma(B_2) = \sigma(B |_{N_2}) = \sigma(B) \setminus \{\nu\}$. Observe that $A_1 - \mu$ is nilpotent of order $p_1$ and $B_1 - \nu$ is nilpotent of order $q_1$. It will be proved that $p(\delta_{A,B} - \lambda I) = q(\delta_{A,B} - \lambda I) \leq r$, such that $r = p_1 + q_1$. Let $Y \in R((\delta_{A,B} - \lambda I)^r)$, then there exist $X \in L(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{kl}]^2_{k,l=1}$ such that

$$Y = (\delta_{A,B} - \lambda I)^r(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)^r(X_{11}) & \delta_{A_1,B_2} - \lambda I)^r(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)^r(X_{21}) & (\delta_{A_2,B_2} - \lambda I)^r(X_{22}) \end{pmatrix},$$

The operator $\delta_{A_1,B_1} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. We argue as in the proof of Lemma 2.2, we get that $Y \in R((\delta_{A,B} - \lambda I)^{r+1})$, i.e $q(\delta_{A,B} - \lambda I) \leq r$. With the same decompositions we can easily prove that $\ker(\delta_{A,B} - \lambda I)^{r+1} \subset \ker(\delta_{A,B} - \lambda I)^r$, i.e $p(\delta_{A,B} - \lambda I) \leq r$, consequently $\lambda \in \Pi(\delta_{A,B})$.

Hence $\sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B}) \subset \Pi(\delta_{A,B}) \subset E(\delta_{A,B})$. On the other hand the inclusion $\Pi(\delta_{A,B}) \subset \sigma(\delta_{A,B}) \cap \sigma_{BW}(\delta_{A,B})$ holds for every operator and since $\delta_{A,B}$ is polaroid from Lemma 2.2, we have $E(\delta_{A,B}) = \Pi(\delta_{A,B})$, so $E(\delta_{A,B}) = \sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B})$, it then follows that $\delta_{A,B}$ satisfies generalized Weyl’s theorem.

Conversely suppose that $\delta_{A,B}$ satisfies generalized Weyl’s theorem, then $E(\delta_{A,B}) = \sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B})$. If $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$, then for $\lambda = \mu - \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, we have $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$, we argue as above we get, $\lambda \in \Pi(\delta_{A,B}) = E(\delta_{A,B}) = \sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B})$. Hence $\sigma_{BW}(\delta_{A,B}) \subset \sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B})$. For the reverse inclusion, let $\lambda \in \sigma(\delta_{A,B}) \setminus \sigma_{BW}(\delta_{A,B})$, then $E(\delta_{A,B})$, which implies that there exist finite sequences $\{\nu_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n}$ of values $\mu_i \in isoc(A) \cap E(A)$ and $\nu_i \in isoc(B) \cap E(B)$ such that $\lambda = \mu_i - \nu_i$ for all $1 \leq i \leq n$, then $\mu_i \notin \sigma_{BW}(A)$ and $\nu_i \notin \sigma_{BW}(B)$, for all $1 \leq i \leq n$, consequently $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$. □
Theorem 2.6. Suppose that $A,B \in L(H)$ are polaroid operators which satisfy generalized Weyl’s theorem, then a necessary and sufficient condition for $\Delta_{A,B}$ to satisfy generalized Weyl’s theorem, is

$$\sigma_{BW}(\Delta_{A,B}) = \sigma(A) \sigma_{BW}(B) \cup \sigma_{BW}(A) \sigma(B) - \{1\}.$$

Proof. Assume that $\sigma_{BW}(\Delta_{A,B}) = \sigma(A) \sigma_{BW}(B) \cup \sigma_{BW}(A) \sigma(B) - \{1\}$.

Let $\lambda \in \sigma(\Delta_{A,B}) \backslash \sigma_{BW}(\Delta_{A,B})$ such that $\lambda \neq -1$, then for $\lambda = \mu \nu - 1$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, it follows that $\mu \notin \sigma_{BW}(A)$ and $\nu \notin \sigma_{BW}(B)$, hence $\mu \in \Pi(A)$ and $\nu \in \Pi(B)$, we argue as in the proof of Theorem 2.5, we get

$$E(\Delta_{A,B}) = \sigma(\Delta_{A,B}) \backslash \sigma_{BW}(\Delta_{A,B}).$$

it then follows that $\Delta_{A,B}$ satisfies generalized Weyl’s theorem.

Conversely suppose that $\Delta_{A,B}$ satisfies generalized Weyl’s theorem, then $E(\Delta_{A,B}) = \sigma(\Delta_{A,B}) \backslash \sigma_{BW}(\Delta_{A,B})$. If $\lambda \notin \sigma(A) \sigma_{BW}(B) \cup \sigma_{BW}(A) \sigma(B) - \{1\}$, then for $\lambda = \mu \nu - 1$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, hence $\mu \in \Pi(A)$ and $\nu \in \Pi(B)$, we argue as in proof of Theorem 2.5, we get $\lambda \in \Pi(\Delta_{A,B}) = \sigma(\Delta_{A,B}) \backslash \sigma_{BW}(\Delta_{A,B})$. Hence $\sigma_{BW}(\Delta_{A,B}) \subset \sigma(A) \sigma_{BW}(B) \cup \sigma_{BW}(A) \sigma(B) - \{1\}$. For the reverse inclusion and the case $\lambda = -1$, will be proved similarly. \qed

3. Application

A bounded linear operator $T$ on a complex Hilbert space $H$, is said to be $p$-hyponormal if $(T^* T)^p \geq (TT^*)^p$. Especially, a $p$-hyponormal operator $T$ is said to be hyponormal and semi-hyponormal if $p = 1$ and $p = \frac{1}{2}$, respectively. For positive numbers $s$ and $t$, an operator $T$ belongs to class $A(s,t)$ if $(|T|^s |T^*|^t \downarrow T^2)^{\frac{1}{s+t}} \geq |T|^2$. Especially, we denote class $A(1,1)$ by class $A$. $A(\frac{1}{2}, \frac{1}{2})$ is the class of w-hyponormal operators it was introduced by Aluthge and Wang [3], the class of w-hyponormal operators contains the class of $p$-hyponormal $(0 < p \leq 1)$ and log-hyponormal operators. An operator $T \in L(H)$ is said to be log-hyponormal if $T$ is invertible and satisfies $\log(T^* T) \geq \log(TT^*)$. Recall that $T \in L(H)$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2 x\|\|x\|$, for all $x \in H$. Inclusion relations among these classes are known as follows:

$$\{\text{hyponormal}\} \subset \{p\text{-hyponormal}, 0 < p < 1\}$$

$$\subset \{\text{class}A(s,t), s,t \in [0,1]\}$$

$$\subset \{\text{class}A\}$$

$$\subset \{\text{paranormal}\}.$$ 

It is proved in [11] that if $A, B^* \in L(H)$ are hyponormal, then generalized Weyl’s theorem holds for $f(d_{A,B})$ for every $f \in \mathcal{H}(\sigma(d_{A,B}))$, where $\mathcal{H}(\sigma(d_{A,B}))$ is the set of all analytic functions defined on a neighborhood of $\sigma(d_{A,B})$, this result was extended to log-hyponormal or $p$-hyponormal operators in [15] and [19]. Also in [10] it is shown that if $A, B^* \in L(H)$ are w-hyponormal operators, then Weyl’s theorem holds for $f(d_{A,B})$ for every $f \in \mathcal{H}(\sigma(d_{A,B}))$. In the next result we can give more, before that we recall the following definitions.

Definition 3.1. For $T \in L(X)$ and a closed subset $F$ of $\mathbb{C}$ the glocal spectral subspace $X_T(F)$ defined as the set of all $x \in X$ such that there is an analytic $X$-valued function $f : \mathbb{C} \setminus F \rightarrow X$ for which $(T - \lambda I)x = f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. $T \in L(X)$ is said to have Dunford property $(C)$ if every glocal spectral subspace is
closed for every closed set $F \subseteq \mathbb{C}$ and $T \in L(X)$ is said to be decomposable if $T$ has both property $(C)$ and property $(\delta)$, where the last property means that for every open covering $(U, V)$ of $\mathbb{C}$ we have $X = X_T(\overline{U}) + X_T(\overline{V})$.

**Definition 3.2.** An operator $T \in L(X)$ has Bishop’s property $(\beta)$ if for every open set $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$, with the property that $(T - \lambda I)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$, it follows that $f_n \to 0$, again locally uniformly on $U$.

Bishop’s property $(\beta)$ implies Dunford property $(C)$, also $T$ satisfies property $(\beta)$ if and only if $T^*$ satisfies property $(\delta)$ [18, Theorem 2.5.5]. For more information on property $(\beta)$, property $(\delta)$ and Dunford’s condition $(C)$ we refer the interested reader to [18].

**Definition 3.3.** An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$), if for every open disc $D$ centered at $\lambda_0$, the only analytic function $f : D \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if $T$ has SVEP at every $\lambda \in \mathbb{C}$.

Evidently, every operator $T$, as well as its dual $T^*$, has SVEP at every point in $\partial \sigma(T)$, where $\partial \sigma(T)$ is the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$.

**Lemma 3.4.** Suppose that $A, B^* \in L(H)$ are paranormal operators, then $d_{A,B}$ has SVEP.

**Proof.** From [20] we have $A$ satisfies property $(\beta)$ and $B$ satisfies property $(\delta)$. Hence both $L_A$ and $R_B$ satisfy condition $(C)$ by [18, Corollary 3.6.11]. Since $L_A$ and $R_B$ commute, it follows by [18, Theorem 3.6.3] and [18, Note 3.6.19] that $L_A - R_B$ and $L_AR_B$ have SVEP, then SVEP holds for $d_{A,B}$.

**Theorem 3.5.** Suppose that $A, B^* \in L(H)$ are paranormal operators, then

$$\sigma_{BW}(\delta_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)),$$

and

$$\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}.$$ 

**Proof.** We know from [13, Proposition 2.1] and [17, P. 229] that paranormal operators are polaroid. Since a Hilbert space operator is polaroid if and only if its adjoint is polaroid, it follows that $B$ is polaroid and from [14, Theorem 4.2] that generalized Weyl’s theorem holds for $A, A^*, B^*$ and $B$.

Let $\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$, then for every $\lambda = \mu + \nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, we have $\mu \notin \sigma_{BW}(A)$ and $\nu \notin \sigma_{BW}(B)$ which implies that $\mu \in E(A) = \Pi(A)$ and $\nu \in E(B) = \Pi(B)$. We argue as in the proof of Theorem 2.5, we get $\lambda \in \Pi(\delta_{A,B})$, it follows from [6, Theorem 2.3] that $\delta_{A,B} - \lambda I$ is $B$-Fredholm of index zero. Hence

$$\sigma_{BW}(\delta_{A,B}) \subset (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).$$

For the reverse inclusion, let $\lambda \in \sigma(\delta_{A,B})$ and $\lambda \notin \sigma_{BW}(\delta_{A,B})$. Since $d_{A,B}$ has SVEP from Lemma 3.4, then from [8, Theorem 3.3] $\sigma_{BW}(\delta_{A,B}) = \sigma_D(\delta_{A,B})$ and $\delta_{A,B}$ is polaroid from Lemma 2.2, therefore $\sigma_D(\delta_{A,B}) = acc\sigma(\delta_{A,B})$, it follows that $\lambda \in isoc(\delta_{A,B})$, then there exist finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ of values
\(\mu_i \in isos(A)\) and \(\nu_i \in isos(B)\) such that \(\lambda = \mu_i - \nu_i\) for all \(1 \leq i \leq n\), so \(\mu_i \in \Pi(A) = E(A)\) and \(\nu_i \in \Pi(A) = E(B)\), for all \(1 \leq i \leq n\), which implies that \(\mu_i \notin \sigma_{BW}(A)\) and \(\nu_i \notin \sigma_{BW}(B)\), for all \(1 \leq i \leq n\), consequently \(\lambda \notin (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))\). Hence

\[
\sigma_{BW}(d_{A,B}) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).
\]

The equality \(\sigma_{BW}(\Delta_{A,B}) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) - \{1\}\), will be proved similarly. \(\square\)

**Corollary 3.6.** Suppose that \(A, B^* \in L(H)\) are paranormal operators, then generalized Weyl’s theorem holds for \(f(d_{A,B})\) and \(f(d_{A,B}')\) for every \(f \in \mathcal{H}(\sigma(d_{A,B}))\), where \(d_{A,B}'\) is the dual of \(d_{A,B}\).

**Proof.** By Theorem 3.5 we get generalized Weyl’s theorem holds for \(d_{A,B}\). To show that generalized Weyl’s theorem holds for \(f(d_{A,B})\), observe first from Lemma 2.2 that \(d_{A,B}\) is polaroid, then \(d_{A,B}\) is isoloid, i.e. every isolated point of the spectrum is an eigenvalue of \(d_{A,B}\). From [22, Theorem 2.2] it follows that generalized Weyl’s theorem holds for \(f(d_{A,B})\).

Since \(d_{A,B}\) is polaroid and has SVEP, then from [4, Theorem 2.10] and [4, Theorem 2.4] \(d_{A,B}'\) (the dual of \(d_{A,B}\)) satisfies generalized Weyl’s theorem. Since \(d_{A,B}\) is polaroid, then by [2, Lemma 2.3] \(d_{A,B}'\) is polaroid, hence \(d_{A,B}'\) is isoloid, From [7, Lemma 2.9] it follows that \(f(d_{A,B}')\) satisfies generalized Weyl’s theorem for every \(f \in \mathcal{H}(\sigma(d_{A,B}))\).

\(\square\)

**Acknowledgments.** The author is grateful to the referee for several helpful remarks and suggestions concerning this paper.

**References**


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