A GENERALIZED CLASS OF $k$-UNIFORMLY STARLIKE FUNCTIONS INVOLVING WGH OPERATORS

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Abstract. In this paper, involving Wgh operators $W^p_q([\alpha_1])$ and $W^p_q([\alpha_1 + 1])$, a generalized class of $k$-uniformly starlike functions is defined. Some results on coefficient inequalities, inclusion and convolution properties for functions belonging to this class are derived. Our results generalize some of the previously obtained results as well as generate new ones.

1. Introduction

Let $S$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

(1.1)

which are univalent analytic in the open unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. Let $S^*$ and $CV$ denote the subclasses of $S$ whose members are, respectively, starlike and convex in $\Delta$.

Subclasses $k$-$SP$ and $k$-$UCV$ of $S^*$ and $CV$, respectively, are studied by Kanas and Wiśniewska in [13], [14] (see [12], [16]) which are defined as follows:

Definition 1.1. Let $f \in S$, and $0 \leq k < \infty$. Then $f \in k$-$SP$ if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|.$$

(1.2)

Definition 1.2. Let $f \in S$, and $0 \leq k < \infty$. Then $f \in k$-$UCV$ if and only if

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|.$$

(1.3)

By Alexander property, we have $f \in k$-$UCV \iff zf' \in k$-$SP$. Note that these classes were introduced by Goodman [10] by giving a two-variable characterisation

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of I-UCV. Renning [20] and independently Ma and Minda [17] have given a more applicable one-variable characterisation for this class.

Geometrically, the class $k-\mathcal{SP}$ ($k$-UCV) is described as the family of functions $f$ such that
\[ p(z) = \frac{z f'(z)}{f(z)} \left(1 + \frac{z f''(z)}{f(z)}\right) \]
is subordinate to the univalent functions $p_k$ such that $p_k(\Delta)$ describe a conic region:
\[ \Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}, \]
with $1 \in \Omega_k$. Some explicit form of extremal functions $p_k$ are given in [12].

For $\alpha_i \in \mathbb{C}$ $\left(\frac{\alpha_i}{A_i} \neq 0, -1, -2, ..., A_i > 0; i = 1, 2, ..., p\right)$ and
\[ \beta_i \in \mathbb{C} \left(\frac{\beta_i}{B_i} \neq 0, -1, -2, ..., B_i > 0; i = 1, 2, ..., q\right) \] such that $1 + \sum_{i=1}^{p} B_i - \sum_{i=1}^{p} A_i \geq 0$,
Wright’s generalized hypergeometric (Wgh) function $p\psi_q [z]$ [24] ([23]) is defined by
\[ p\psi_q [z] = p\psi_q \left[ (\alpha_i, A_i)_{1,p} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma (\alpha_i + n A_i)}{\prod_{i=1}^{q} \Gamma (\beta_i + n B_i)} z^n, \]
which is analytic for bounded values of $|z|$. Involving Wgh function defined by (1.5) with $\alpha_i \neq 0, -1, -2, ..., i = 1, 2, ..., p$ and $\beta_i \neq 0, -1, -2, ..., i = 1, 2, ..., q$, a linear operator: $\mathbf{W}_q^p ([\alpha_1]) = \mathbf{W}_q^p ((\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}) : \mathcal{S} \rightarrow \mathcal{S}$ is defined with the use of convolution for each of the form (1.1) by
\[ \mathbf{W}_q^p ([\alpha_1]) f(z) = z \prod_{i=1}^{p} \frac{\Gamma (\alpha_i)}{\Gamma (\alpha_i)} \frac{\prod_{i=1}^{q} \Gamma (\beta_i)}{\Gamma (\beta_i)} \frac{\prod_{i=1}^{p} \Gamma (\alpha_i + n A_i)}{\prod_{i=1}^{p} \Gamma (\alpha_i)} \frac{1}{(n-1)!}, n \geq 2. \]

Also, we get
\[ \mathbf{W}_q^p ([\alpha_1 + 1]) f(z) := z + \sum_{n=2}^{\infty} \left(1 + \frac{(n-1)A_1}{\alpha_1}\right) a_n \theta_n z^n. \]
We call the operators $\mathbf{W}_q^p ([\alpha_1])$, $\mathbf{W}_q^p ([\alpha_1 + 1])$ as the Wgh operators. Note that the operator $\mathbf{W}_q^p ([\alpha_1])$ was defined by Dziok and Raina in [6] and was used in several works, see [1], [2], [4], [5], [6], [7], [18], [19], [22].

Taking $A_i = 1 \ (i = 1, 2, ..., p)$ and $B_i = 1 \ (i = 1, 2, ..., q)$, Wgh operator $\mathbf{W}_q^p ([\alpha_1])$ reduces to the Dziok-Srivastava operator $\mathbf{F}_q^p ([\alpha_1])$ ([8]) which is defined for $f \in \mathcal{S}$ by
\[ \mathbf{F}_q^p ([\alpha_1]) f(z) = z \ pF_q [z] * f(z), \]
Let \( W_{\beta} \) operators involve the Ruscheweyh derivative operators \( D \) and Yaguchi [12], \( F_{\gamma} \) is the Carlson and Shaffer operator [3] and \( F_{\gamma}(1+\lambda,1;1) \) is the Ruscheweyh derivative operator [21] which is defined for \( f \in \mathcal{S} \) by

\[
D^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} f(z), \lambda > -1.
\]

We define here a new class \( UC(k, [\alpha_1]) \) involving Wgh operators \( W_{\eta}^p([\alpha_1]) \) and \( W_{\eta}^p([\alpha_1+1]) \):

**Definition 1.3.** Let Wgh operators \( W_{\eta}^p([\alpha_1]) \) and \( W_{\eta}^p([\alpha_1+1]) \) be defined by (1.6) and (1.8), respectively, then for \( 0 \leq k < \infty \), a function \( f \in \mathcal{S} \) is said to be in the class \( UC(k, [\alpha_1]) \) if it satisfies

\[
\Re \left( \frac{W_{\eta}^p([\alpha_1+1])f(z)}{W_{\eta}^p([\alpha_1])f(z)} \right) > k \left| \frac{W_{\eta}^p([\alpha_1+1])f(z)}{W_{\eta}^p([\alpha_1])f(z)} - 1 \right|.
\]

On taking \( p = q + 1 \) and \( A_i = 1 (i = 1, 2, ..., p) \), \( \alpha_i = \beta_i = B_i = 1 (i = 1, 2, ..., q) \) and if \( \alpha_{q+1} = 1 + \lambda, \lambda > -1 \), class \( UC(k, [\alpha_1]) \) reduces to the class \( UC(\lambda, k) \) which involve the Ruscheweyh derivative operators \( D^\lambda \) and \( D^{\lambda+1} \) and is studied by Kanas and Yaguchi [12]. Clearly, if \( p = q + 1 \) and \( \alpha_i = A_i = 1 (i = 1, 2, ..., p), \beta_i = B_i = 1 (i = 1, 2, ..., q) \), then the class \( UC(k, [\alpha_1]) \) reduces to the class \( k-SP \) and also to the class \( k-UCV \) if we replace \( f \) by \( zf \) in (1.9).

The purpose of this paper is to find some results for the class \( UC(k, [\alpha_1]) \) which is defined by using its subordinate condition. Coefficient inequalities, inclusion and convolution properties for this class are derived with some of the consequent results.

### 2. COEFFICIENT INEQUALITIES

**Theorem 2.1.** Let \( W_{\eta}^p([\alpha_1]) \) be the Wgh operator defined by (1.6) with \( \frac{\alpha_1}{\eta} \geq 1 \), and if for \( 0 \leq k < \infty \), the function \( f \) of the form (1.1) belongs to the class \( UC(k, [\alpha_1]) \), then there exists a convex univalent function:

\[
p_k(z) = \frac{W_{\eta}^p([\alpha_1+1])f_k(z)}{W_{\eta}^p([\alpha_1])f_k(z)}, z \in \Delta,
\]

\[
f_k(z) = z + d_2z^2 + d_3z^3 + \ldots \text{ such that}
\]

\[
|a_2| \leq |d_2|, \quad |a_3| \leq |d_3|.
\]

**Proof.** Let \( f \in UC(k, [\alpha_1]) \), we get

\[
p(z) \prec p_k(z), z \in \Delta,
\]

where

\[
p(z) = \frac{W_{\eta}^p([\alpha_1+1])f(z)}{W_{\eta}^p([\alpha_1])f(z)} = 1 + p_1z + p_2z^2 + \ldots
\]
and

\[ p_k(z) = \frac{W_q^p([\alpha_1 + 1]) f_k(z)}{W_q^p([\alpha_1]) f_k(z)} = 1 + P_1 z + P_2 z^2 + \ldots (P_j \geq 0, \ j = 1, 2, \ldots). \] (2.4)

On writing the series expansions of \( W_q^p([\alpha_1]) f(z), \ W_q^p([\alpha_1 + 1]) f(z) \) and \( W_q^p([\alpha_1]) f_k(z), \ W_q^p([\alpha_1 + 1]) f_k(z) \), equations (2.3) and (2.4) provide

\[ (1 + p_1 z + p_2 z^2 + \ldots) \left( z + \sum_{n=2}^{\infty} a_n \theta_n z^n \right) = z + \sum_{n=2}^{\infty} \left( 1 + (n - 1) \frac{A_1}{\alpha_1} \right) a_n \theta_n z^n \] (2.5)

and

\[ (1 + P_1 z + P_2 z^2 + \ldots) \left( z + \sum_{n=2}^{\infty} d_n \theta_n z^n \right) = z + \sum_{n=2}^{\infty} \left( 1 + (n - 1) \frac{A_1}{\alpha_1} \right) d_n \theta_n z^n. \] (2.6)

Hence, from (2.5) and (2.6), we get the coefficient relations:

\[ (m - 1) \frac{A_1}{\alpha_1} a_m \theta_m = \sum_{j=1}^{m-1} p_j a_{m-j} \theta_{m-j}, \ m \geq 2 \] (2.7)

and

\[ (m - 1) \frac{A_1}{\alpha_1} d_m \theta_m = \sum_{j=1}^{m-1} P_j d_{m-j} \theta_{m-j}, \ m \geq 2, \]

\[ a_1 = d_1 = \theta_1 = 1. \]

Hence, we obtain

\[ \frac{A_1}{\alpha_1} \theta_2 a_2 = p_1, \quad 2 \frac{A_1}{\alpha_1} \theta_3 a_3 = \frac{\alpha_1}{A_1} p_1^2 + p_2 \] (2.8)

and

\[ \frac{A_1}{\alpha_1} \theta_2 d_2 = P_1, \quad 2 \frac{A_1}{\alpha_1} \theta_3 d_3 = \frac{\alpha_1}{A_1} P_1^2 + P_2. \] (2.9)

As \( P_j \geq 0, \ j = 1, 2, \ldots \), from (2.9), it is clear that \( \theta_2 d_2 \) and \( \theta_3 d_3 \) are also non-negative real numbers. Further, with the use of subordination (2.2), we get \( p(z) = p_k(w(z)) \) for some analytic function \( w \) with \( w(0) = 0 \) and \( |w(z)| < 1, z \in \Delta \). If \( w(z) = \frac{q(z) - 1}{q(z) + 1} \),

where \( q(z) = 1 + q_1 z + q_2 z^2 + \ldots \), with \( \Re(q(z)) > 0 \), we write

\[ p(z) = p_k \left( \frac{q(z) - 1}{q(z) + 1} \right). \]

On using their series expansions, we obtain

\[ 1 + p_1 z + p_2 z^2 + \ldots = 1 + \frac{P_1 q_1}{2} z + \left( \frac{P_1 q_2}{2} - \frac{P_1 q_1^2}{4} + \frac{P_2 q_1^2}{4} \right) z^2 + \ldots \] (2.10)

Thus, by (2.8), (2.9) and (2.10), we get

\[ \left| \frac{A_1}{\alpha_1} \theta_2 a_2 \right| = |p_1| = \left| \frac{P_1 q_1}{2} \right| \leq |P_1| = \left| \frac{A_1}{\alpha_1} \theta_2 d_2 \right| \] (2.11)

and

\[ \left| \frac{2A_1}{\alpha_1} \theta_3 a_3 \right| = \left| \left( \frac{\alpha_1}{A_1} - 1 \right) p_1^2 + p_2 + p_1^2 \right| \leq \left( \frac{\alpha_1}{A_1} - 1 \right) P_1^2 + P_2 + P_1^2 = \left| \frac{2A_1}{\alpha_1} \theta_3 d_3 \right|. \] (2.12)
where we use the inequalities \(|q_n| \leq 2, n \geq 1\) and \(|p_2| + |p_1|^2 \leq P_2^2 + P_1^2\) \((15)\). Thus, inequalities (2.11) and (2.12) imply the desired result (2.1). This proves Theorem 2.1.

\[ \text{Theorem 2.2.} \quad \text{Let } W_{\alpha_j}^p([\alpha_1]) \text{ be the } Wgh \text{ operator defined by (1.6) with } \frac{A_1}{A_i} \geq 1, \text{ if the function } f \text{ of the form (1.1) belongs to the class } UC(k, [\alpha_1]) \text{ for } 0 \leq k < \infty, \text{ then there exists a convex univalent function:} \]

\[ p_k(z) = \frac{W_{\alpha_j}^p([\alpha_1+1])f_k(z)}{W_{\alpha_j}^p([\alpha_1])f_k(z)} = 1 + P_1z + P_2z^2 + ... (P_j \geq 0, j = 1, 2, ...), \]

such that

\[ \left| \frac{A_1}{\alpha_1} a_n \theta_n \right| \leq \frac{P_1 \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)}{(n-1)!}, n \geq 2, \]

where \(\theta_n\) is given by (1.7).

\[ \text{Proof.} \quad \text{By induction, it is shown by Theorem 2.1 that result holds for } n = 2. \text{ Let the result be true for all } j, 2 \leq j \leq n - 1. \text{ Thus, from coefficient relation (2.7) and by Rogosinski result } |p_j| \leq P_1, j = 1, 2, .., \text{ we get} \]

\[ (n-1) \left| \frac{A_1}{\alpha_1} a_n \theta_n \right| = \left| \sum_{j=2}^{n-1} p_{n-j} a_j \theta_j \right|, n \geq 2 \]

\[ \leq P_1 + \sum_{j=2}^{n-1} |p_{n-j}| |a_j \theta_j| \]

\[ \leq P_1 \left[ 1 + \sum_{j=2}^{n-1} \frac{\alpha_1}{A_1} P_1 \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)_{j-2} \right]. \quad (2.13) \]

We see that

\[ 1 + \sum_{j=2}^{n-1} \frac{\alpha_1}{A_1} P_1 \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)_{j-2} \]

\[ = \frac{1}{P_1} \left( 1 + \frac{\alpha_1}{A_1} P_1 \right) + \sum_{j=3}^{n-1} \frac{\alpha_1}{A_1} P_1 \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)_{j-2} \]

\[ = \frac{1}{P_1} \left( 1 + \frac{\alpha_1}{A_1} P_1 \right) \left( \frac{2 + \frac{\alpha_1}{A_1} P_1}{P_1} \right) + \sum_{j=4}^{n-1} \frac{\alpha_1}{A_1} P_1 \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)_{j-2} \]

\[ = \frac{1}{(n-2)!} \left( 1 + \frac{\alpha_1}{A_1} P_1 \right)_{n-2}. \]

Hence, (2.13) proves that the result is true for \(n\) also. Thus the result holds for any \(n \geq 2\). This proves Theorem 2.2. \]

Replacing \(f\) by \(zf'\), we can obtain following result on the similar lines of the proofs of Theorems 2.1 and 2.2:
Theorem 2.3. Let $\mathcal{W}_q^p(\alpha_1)$ be the Whh operator defined by (1.6) with $\frac{\alpha_1}{\alpha_1} \geq 1$, and if for $0 \leq k < \infty$, the function $f$ of the form (1.1) satisfies

$$
\frac{\mathcal{W}_q^p(\alpha_1 + 1)}{\mathcal{W}_q^p(\alpha_1)} \frac{z f'(z)}{z f(z)} < \frac{\mathcal{W}_q^p(\alpha_1 + 1)}{\mathcal{W}_q^p(\alpha_1)} \frac{z f_k'(z)}{z f_k(z)}
$$

then for $f_k(z) = z + d_2 z^2 + d_3 z^3 + \ldots$ and for

$$
\frac{\mathcal{W}_q^p(\alpha_1 + 1)}{\mathcal{W}_q^p(\alpha_1)} \frac{z f_k'(z)}{z f_k(z)} = 1 + P_1 z + P_2 z^2 + \ldots (P_j \geq 0, j = 1, 2, \ldots),
$$

and

$$
|a_2| \leq |d_2|, \ |a_3| \leq |d_3|
$$

and

$$
\left| \frac{A_1}{\alpha_1} \theta_n a_n \right| \leq \frac{P_1 \left(1 + \frac{\alpha_1}{\alpha_1} P_1\right)}{n!} n^{-2}, n \geq 2,
$$

where $\theta_n$ is given by (1.7).

Theorem 2.4. If for the function $f$ of the form (1.1) and for $0 \leq k < \infty$, $\theta_n$ given by (1.7), the inequality

$$
\sum_{n=2}^{\infty} \left\{ (n - 1) (k + 1) \left| \frac{A_1}{\alpha_1} + 1 \right| \right\} |a_n \theta_n| < 1
$$

holds, then $f \in \mathcal{UC}(k, [\alpha_1])$.

Proof. To prove $f \in \mathcal{UC}(k, [\alpha_1])$, we have to show from the condition (1.9) that

$$
S_1 := k \left| \frac{\mathcal{W}_q^p(\alpha_1 + 1)f(z)}{\mathcal{W}_q^p(\alpha_1)f(z)} - 1 \right| - \Re \left( \frac{\mathcal{W}_q^p(\alpha_1 + 1)f(z)}{\mathcal{W}_q^p(\alpha_1)f(z)} - 1 \right) < 1.
$$

From (1.6) and (1.8), we get

$$
S_1 \leq (k + 1) \left| \frac{\mathcal{W}_q^p(\alpha_1 + 1)f(z)}{\mathcal{W}_q^p(\alpha_1)f(z)} - 1 \right|
$$

$$
\leq (k + 1) \left[ \sum_{n=2}^{\infty} (n - 1) \left| \frac{A_1}{\alpha_1} a_n \theta_n \right| \right] \leq 1,
$$

if (2.15) holds. This proves Theorem 2.4.

3. Inclusion Property

Theorem 3.1. Let $\mathcal{W}_q^p(\alpha_1)$ be the Whh operator defined by (1.6) with $0 < \frac{\alpha_1}{\alpha_1} < (k + 1), 0 \leq k < \infty$. Then $z f'(z) \in \mathcal{UC}(k, [\alpha_1]) \Rightarrow f \in \mathcal{UC}(k, [\alpha_1])$.

Proof. Let $z f'(z) \in \mathcal{UC}(k, [\alpha_1])$, then there exists a univalent convex function $p_k(z), z \in \Delta$ describing the conic region $\Omega_k$ defined by (1.4) such that

$$
\frac{\mathcal{W}_q^p(\alpha_1 + 1)}{\mathcal{W}_q^p(\alpha_1)} \frac{z f'(z)}{z f(z)} < p_k(z).
$$

Set

$$
p(z) = \frac{\mathcal{W}_q^p(\alpha_1 + 1)f(z)}{\mathcal{W}_q^p(\alpha_1)f(z)}.
$$
Note that \( z \left( W^p_q([\alpha_1]) f(z) \right)' = W^p_q([\alpha_1]) z f'(z) \). Hence, differentiation of (3.1) provides
\[
\frac{W^p_q([\alpha_1 + 1]) z f'(z)}{W^p_q([\alpha_1]) f(z)} = \frac{W^p_q([\alpha_1]) z f'(z)}{W^p_q([\alpha_1]) f(z)} + \frac{zp'(z)}{p(z)}
\]
Using the identity:
\[
W^p_q([\alpha_1]) z f'(z) = \frac{\alpha_1}{A_1} W^p_q([\alpha_1 + 1]) f(z) - \left( \frac{\alpha_1}{A_1} - 1 \right) W^p_q([\alpha_1]) f(z),
\]
we get
\[
\frac{W^p_q([\alpha_1 + 1]) z f'(z)}{W^p_q([\alpha_1]) f(z)} = p(z) + \frac{zp'(z)}{p(z) \frac{A_1}{\alpha_1} \left( 1 - \frac{A_1}{\alpha_1} \right)} < p_k(z).
\]
Therefore, by using a well known Lemma of Eenigenburg, Miller, Mocanu and Read [9], we get
\[
p(z) < p_k(z),
\]
provided that
\[
\Re \left( p_k(z) \frac{\alpha_1}{A_1} \left( 1 - \frac{\alpha_1}{A_1} \right) \right) > 0.
\]
Since, from the definition of \( \Omega_k \), given by (1.4), we have \( \Re (p_k(z)) > \frac{k}{k+1} \) and hence, by the hypothesis we get the result. \( \square \)

**Remark.** Above result confirms that \( k \mathcal{UCV} \subset k \mathcal{SP} \).

4. Convolution Property

**Theorem 4.1.** Let \( W^p_q([\alpha_1]) \) be the \( \mathcal{W}gh \) operator defined by (1.6), then for \( 0 \leq k < \infty \), \( f \in \mathcal{UC}(k, [\alpha_1]) \) if and only if
\[
\frac{1}{2} \left( H_k * W^p_q([\alpha_1]) f \right) (z) \neq 0, \ z \in \Delta,
\]
where
\[
H_k(z) = \frac{1}{(1 - C(t)) (1 - z)} \left( 1 - \frac{1 - \frac{A_1}{\alpha_1}}{1 - z} \right) - C(t), \ z \in \Delta
\]
and \( C(t) = kt + i \sqrt{t^2 - (kt - 1)^2} \), \( t \geq 0, t^2 - (kt - 1)^2 \geq 0 \).

**Proof.** Let
\[
p(z) = \frac{W^p_q([\alpha_1 + 1]) f(z)}{W^p_q([\alpha_1]) f(z)}, \ z \in \Delta.
\]
Since \( p(0) = 1 \), we have from the definition of conic region (1.4)
\[
f \in \mathcal{UC}(k, [\alpha_1]) \iff p(z) \notin \partial \Omega_k, \ z \in \Delta,
\]
where
\[
\partial \Omega_k = \left\{ u + iv : u = k \sqrt{(u - 1)^2 + v^2} \right\}.
\]
Note that \( \partial \Omega_k = C(t) = kt + i \sqrt{t^2 - (kt - 1)^2} \), for \( t \geq 0, t^2 - (kt - 1)^2 \geq 0 \). Thus, we have
\[
\frac{1}{2} \left( \frac{W^p_q([\alpha_1 + 1]) f(z) - C(t) W^p_q([\alpha_1]) f(z)}{(1 - C(t))} \right) \neq 0,
\]
By series expansions of $W_p^q([\alpha_1 + 1])f(z)$ and $W_p^q([\alpha_1])f(z)$, given in (1.8) and (1.6), we note that

$$W_p^q([\alpha_1 + 1])f(z) = \frac{z \left(1 - \left(1 - \frac{\alpha_1}{1} \right) \frac{z}{1-z} \right)}{(1-z)^2} \ast W_p^q([\alpha_1])f(z)$$

and

$$W_p^q([\alpha_1])f(z) = \frac{z}{1-z} \ast W_p^q([\alpha_1])f(z).$$

Hence, by (4.5), we get

$$\frac{1}{z} \left( H_t \ast W_p^q([\alpha_1])f(z) \right) = \frac{1}{z} \left\{ \frac{1}{1-C(t)} \left( \frac{1}{1-z} - \frac{z}{1-z} \right) - C(t) \frac{z}{1-z} \right\} \neq 0.$$

Thus,

$$\frac{1}{z} \left( H_t \ast W_p^q([\alpha_1])f(z) \right) \neq 0 \leftrightarrow p(z) \notin \partial \Omega_k \leftrightarrow p(z) \in \Omega_k, \ z \in \Delta.$$ 

This proves the convolution property. 

On taking $p = q + 1$ and $\alpha_i = A_i = 1 (i = 1, 2, ..., p), \ \beta_i = B_i = 1 (i = 1, 2, ..., q)$, in Theorem 4.1, we get following result for the class $k-SP$

**Corollary 4.2.** Let $0 \leq k < \infty$, then $f \in k-SP$, if and only if

$$\frac{1}{z} \left( G_t \ast f(z) \right) \neq 0, \ z \in \Delta,$$

where

$$G_t(z) = \frac{1}{1-C(t)} \left( \frac{1}{1-z} - C(t) \right), \ z \in \Delta,$$

$$C(t) = kt \pm i\sqrt{t^2 - (kt - 1)^2}, \ t \geq 0, \ t^2 - (kt - 1)^2 \geq 0.$$

Note that for $f, g \in S$

$$\left( g \ast zf' \right)(z) = \left( zg' \ast f \right)(z).$$

Hence, on replacing $f$ by $zf'$ in Corollary 4.2, we get following result of Kanas and Wiśniowska [14, Theorem 3.5, p. 336] for the class $k-UCV$.

**Corollary 4.3.** [14] Let $0 \leq k < \infty$, then $f \in k-UCV$, if and only if

$$\frac{1}{z} \left[ zG_t' \ast f(z) \right] \neq 0,$$

where

$$zG_t' = \frac{1}{1-C(t)} \left( \frac{1}{1-z} + \frac{z}{1-z} - C(t) \right), \ z \in \Delta,$$

$$C(t) = kt \pm i\sqrt{t^2 - (kt - 1)^2}, \ t \geq 0, \ t^2 - (kt - 1)^2 \geq 0.$$ 

Further, Theorem 4.1 yields following result of Kanas and Yaguchi [12]:
Corollary 4.4. [12] Let \(0 \leq k < \infty\) and \(\lambda > -1\), then \(f \in UC(k, \lambda)\) if and only if

\[
\frac{1}{z} (R_t * f)(z) \neq 0, \tag{4.6}
\]

where

\[
R_t(z) = \frac{z}{(1 - z)^{\lambda + 2}} \left(1 - \frac{C(t) z}{C(t) - 1}\right)
\]

and \(C(t) = kt + i\sqrt{t^2 - (kt - 1)^2}\), \(t \geq 0\), \(t^2 - (kt - 1)^2 \geq 0\).

Proof. Applying the argument similar to the argument applied in the proof of Theorem 4.1, we get

\[
f \in UC(k, \lambda) \iff \frac{1}{z} \left\{ \frac{1}{1 - C(t)} (D^{\lambda + 1} f(z) - C(t) D^{\lambda} f(z)) \right\} \neq 0.
\]

Using the definition of Ruscheweyh derivative operators, we get

\[
D^{\lambda + 1} f(z) - C(t) D^{\lambda} f(z) = \left(\frac{z}{(1 - z)^{\lambda + 2}} - C(t) \frac{z}{(1 - z)^{\lambda + 1}}\right) * f(z),
\]

which proves the condition (4.6). \(\square\)

Theorem 4.5. Let \(W_q^p([\alpha_1])\) be the Wgh operator defined by (1.6) with \(\frac{\alpha_1}{\alpha_1 + 1} \geq 1\), then for \(0 < k < \infty\), \(f \in UC(k, [\alpha_1])\) if and only if

\[
\frac{1}{z} \left(W_q^p([\alpha_1]) H_t * f\right)(z) \neq 0, z \in \Delta, \tag{4.7}
\]

where for \(H_t(z)\) given by (4.2),

\[
W_q^p([\alpha_1]) H_t(z) = z + \sum_{n=2}^{\infty} h_n z^n, z \in \Delta,
\]

\[
|h_n| \leq \left\{ 1 + \frac{(n - 1) A_1}{\alpha_1} (1 + k) \right\} |\theta_n|, n \geq 2,
\]

\(\theta_n\) is given by (1.7).

Proof. Mentioning the proof of Theorem 4.1 and the convolution property: \(H_t * W_q^p([\alpha_1]) f\) \((z) = (W_q^p([\alpha_1]) H_t * f)(z)\), we get

\[
f \in UC(k, [\alpha_1]) \iff \frac{1}{z} \left(W_q^p([\alpha_1]) H_t * f\right)(z) \neq 0, z \in \Delta,
\]

where the coefficient \(h_n\) in the series expansion of \(W_q^p([\alpha_1]) H_t(z)\) is given by

\[
h_n = \left\{ \frac{\alpha_1}{A_1} - 1 \right\} + \frac{C(t) - n}{C(t) - 1} \frac{A_1}{\alpha_1} \theta_n.
\]

Hence, maximum of \(|h_n|\) for each \(n \geq 2\), depends upon the maximum of \(|\frac{C(t) - n}{C(t) - 1}|\).

Since,

\[
\left|\frac{C(t) - n}{C(t) - 1}\right|^2 = \left(\frac{C(t) - n}{C(t) - 1}\right) \left(\frac{C(t) - n}{C(t) - 1}\right) = 1 - \frac{2k(n - 1)}{t} + \frac{(n^2 - 1)}{t^2} := s(t),
\]

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which decreases in the interval \( \left[ \frac{1}{k+1}, t_0 \right) \) and increases in \((t_0, \infty)\) with a minima at \( t_0 = \frac{n+1}{k} \). But \( s(\frac{1}{k+1}) = [n + k(n - 1)]^2 > 1 \). Thus,
\[
|h_n| \leq \left( \frac{a_1}{A_1} - 1 \right) + n + k(n - 1) \frac{A_1}{a_1} |\theta_n|.
\]
This proves the result of Theorem 4.5.

**Corollary 4.6.** The function \( g(z) = z + C z^n \in \mathcal{UC}(k, [\alpha_1]) \) if and only if
\[
|C| \leq \frac{n}{1 + \frac{(n-1)A_1}{\alpha_1} (1 + k)} |\theta_n|, n \geq 2, \tag{4.8}
\]
\( \theta_n \) is given by (1.7).

**Proof.** Let (4.8) holds. To prove the result by Theorem 4.5, we have to show
\[
S_2 := \frac{1}{z} \left( W_q^p([\alpha_1]) H_t * g \right)(z) \neq 0, z \in \Delta.
\]
Since,
\[
|S_2| = |1 + h_n C z^{n-1}| > 1 - |h_n C z| \geq 1 - |z| > 0, z \in \Delta.
\]
This proves \( g \in \mathcal{UC}(k, [\alpha_1]) \). Conversely, let \( g \in \mathcal{UC}(k, [\alpha_1]) \) and let
\[
W_q^p([\alpha_1]) H(z) = z + \sum_{n=2}^{\infty} \left( 1 + \frac{(n-1)A_1}{\alpha_1} (1 + k) \right) |\theta_n| \ z^n.
\]
Then
\[
\frac{1}{z} \left( W_q^p([\alpha_1]) H * g \right)(z) = 1 + \left( 1 + \frac{(n-1)A_1}{\alpha_1} (1 + k) \right) |\theta_n| \ C z^{n-1}, z \in \Delta.
\]
Thus, if
\[
|C| > \frac{1}{\left( 1 + \frac{(n-1)A_1}{\alpha_1} (1 + k) \right) |\theta_n|},
\]
then there exists a point \( \zeta \in \Delta \) such that \( \frac{1}{\zeta} \left( W_q^p([\alpha_1]) H * g \right)(\zeta) = 0 \). This proves that the inequality (4.8) must hold. \( \square \)

5. **Concluding Remark**

It is noted that taking \( A_i = 1 (i = 1, 2, \ldots, p) \) and \( B_i = 1 (i = 1, 2, \ldots, q) \), in our results (obtained in previous Sections 2-4), similar results can also be derived for Dziok-Srivastava operators \( F^p_q([\alpha_1]) \) and \( F^p_q([\alpha_1 + 1]) \) and for the special cases of these operators discussed in Introduction. In fact, involving Ruscheweyh derivative operators \( D^k f(z) \) and \( D^{k+1} f(z) \), some of the results have been obtained by Kanas and Yaguchi in [12]. Also, our results verify some of the results of Kanas and Wiśniowska [13] for the class \( k-SP \) and also for \( k-UCV \) [14] if \( f \) is replaced by \( zf' \).

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References


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