A PATHOLOGICAL EXAMPLE OF A DOMINANT TERRACED MATRIX

(COMMUNICATED BY ADAM KILICMAN)

H. CRAWFORD RHALY, JR.

Dedicated to Nancy Lee Shell

Abstract. A hyponormal terraced matrix is modified to produce an example of a non-hyponormal dominant terraced matrix.

1. Introduction

This brief paper addresses a question left open at the end of [5] – Does there exist a terraced matrix, acting as a bounded linear operator on $\ell^2$, that is dominant but not hyponormal? The answer will be provided by modifying one particular entry of a known hyponormal terraced matrix.

A terraced matrix $M$ is a lower triangular infinite matrix with constant row segments. The matrix $M$ is dominant [6] if $\text{Ran}(M - \lambda) \subseteq \text{Ran}(M - \lambda)^*$ for all $\lambda$ in the spectrum of $M$, and $M$ is hyponormal if it satisfies $\langle (M^*M - MM^*)f, f \rangle \geq 0$ for all $f$ in $\ell^2$. Hyponormal operators are necessarily dominant. From [3] we know that $M$ is dominant if and only if for each complex number $\lambda$ there exists an operator $T = T(\lambda)$ on $\ell^2$ such that $(M - \lambda) = (M - \lambda)^*T$.

2. Main Results

Our first theorem involves the terraced matrix $M := M(a)$ associated with a sequence $a = \{a_n : n = 0, 1, 2, 3, \ldots\}$ of real numbers. Throughout this section we assume that $M$ acts through matrix multiplication to give a bounded linear operator on $\ell^2$.

Theorem 2.1. Suppose that $M(a)$ is the terraced matrix associated with a sequence $a = \{a_n\}$ satisfying the following conditions:

1. $\{a_n\}$ is a strictly decreasing sequence that converges to $0$;
2. $\{(n+1) a_n\}$ is a strictly increasing sequence that converges to $L < +\infty$; and
3. $\frac{1}{a_{n+1}} \geq \frac{1}{a_n} + \frac{1}{a_{n+2}}$ for all $n$.

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If the sequence \( b = \{b_n\} \) satisfies \( 0 < b_0 < 2L \) and \( b_n = a_n \) for all \( n \geq 1 \), then \( M(b) \) is dominant.

Proof. First we show that

\[
\text{Ran}(M(b) - \lambda) \subset \text{Ran}(M(b) - \lambda)^*
\]

for all \( \lambda \neq b_0 \). Since our hypothesis guarantees that \( M \equiv M(a) \) is hyponormal (see [4, Theorem 2.2]) and therefore also dominant, for each complex number \( \lambda \) there must exist an operator \( T = [t_{ij}] \) on \( \ell^2 \) such that \( (M - \lambda) = (M - \lambda)^*T \). For \( \lambda \neq b_0 \), replace the first row of \( T \) by

\[
\frac{a_0 - \lambda}{b_0 - \lambda} t_{00}, \frac{a_1 - \lambda}{b_0 - \lambda} t_{01}, \frac{a_2 - \lambda}{b_0 - \lambda} t_{02}, \frac{a_3 - \lambda}{b_0 - \lambda} t_{03}, \frac{a_4 - \lambda}{b_0 - \lambda} t_{04}, \ldots >
\]

and call the new matrix \( T' \). Clearly \( T' \) is bounded on \( \ell^2 \) since \( T \) is, and it is routine to verify that \( (M(b) - \lambda) = (M(b) - \lambda)^*T' \) for \( \lambda \neq b_0 \).

We now consider the case \( \lambda = b_0 \). If \( x = \langle x_0, x_1, x_2, \ldots \rangle \in \ell^2 \), it must be shown that \( (M(b) - b_0)x \in \text{Ran}(M(b) - b_0)^* \). Since \( M(a) \) is dominant, we know that

\[
(M(a) - b_0)x = (M(a) - b_0)^*y
\]

for some \( y = \langle y_0, y_1, y_2, \ldots \rangle \in \ell^2 \). It can be verified that

\[
(M(b) - b_0)^*y = (a_0 - b_0)(x_0 - y_0)e_0 + (M(b) - b_0)x,
\]

where \( \{e_n : n \geq 0\} \) is the standard orthonormal basis for \( \ell^2 \). We want to find \( z = \langle z_0, z_1, z_2, z_3, \ldots \rangle \in \ell^2 \) satisfying

\[
(M(b) - b_0)^*z = (b_0 - a_0)(x_0 - y_0)e_0.
\]

Computations reveal that \( z_0 \) can be chosen arbitrarily, but we must have

\[
z_n = \frac{\prod_{j=0}^{n-1} (b_0 - a_j)}{b_0^n} (x_0 - y_0)
\]

for \( n \geq 1 \). Raabe’s Test [2, p. 396] can then be used to verify that \( z \in \ell^2 \) when \( 2L > b_0 \). This means we have \( (M(b) - b_0)^*(y + z) = (M(b) - b_0)x \), and this completes our proof that \( M(b) \) is dominant for \( 0 < b_0 < 2L \).

We note the following corollary to the proof of the theorem.

**Corollary 2.2.** Suppose that \( M(a) \) is the terraced matrix associated with a decreasing sequence \( a = \{a_n\} \) of positive numbers converging to 0 and that \( \{n+1)a_n\} \) converges to a finite number \( L > 0 \). If \( M(a) \) is a hyponormal operator on \( \ell^2 \) and the sequence \( b = \{b_n\} \) satisfies \( 0 < b_0 < 2L \) and \( b_n = a_n \) for all \( n \geq 1 \), then \( M(b) \) is dominant.

We observe that the preceding theorem and corollary have made no assertion regarding hyponormality for \( M(b) \). In the following, we let \( S_n \) denote the \( n \)-by-\( n \) section in the northwest corner of the matrix of the self-commutator \( M(b)^*M(b) - M(b)M(b)^* \).

**Example 2.1.** (Modified Cesàro Matrix). Start with \( M(a) \) given by \( a_n = \frac{1}{n+1} \) for all \( n \). Take \( b_0 \in (0, 2) \) and \( b_n = \frac{1}{n+2} \) for all \( n \geq 1 \). We observe that this example satisfies the hypothesis of Corollary 2.2 with \( L = 1 \) since the Cesàro operator \( M(a) \)
on $\ell^2$ is known to be hyponormal (see [1]), so $M(b)$ is dominant when $0 < b_0 < 2$. If $z_1, z_2$ denote the zeroes of
\[
y = -\left[\frac{1}{36} \pi^2 \frac{19}{12} + \frac{1}{108} \right] x^2 + \left[\frac{5}{36} \pi^2 - \frac{19}{12} + \frac{1}{54} \right] x - \frac{1}{72} \left[\frac{5}{6} - \frac{19}{12} + \frac{1}{108} \right],\]
then $\det(S_b) < 0$ when
\[
b_0 \in (0, z_1 \approx 0.69665) \cup (z_2 \approx 1.77128, 2),
\]
so $M(b)$ will not be hyponormal for those values of $b_0$.

We note that with a little more effort it can be demonstrated that $M(b)$ is not hyponormal for any $b_0 \in (0, 2) \setminus \{1\}$. This can be accomplished by applying an obvious sequence of elementary row and column operations to reduce $S_n$ to arrow-head form and then to upper triangular form, in which the first diagonal element is negative for $n = n(b_0)$ sufficiently large and all of the rest of the diagonal elements are positive, so $\det(S_n) < 0$ when $n$ is sufficiently large.

We will now present a result that applies to terraced matrices associated with sequences of complex numbers.

**Theorem 2.3.** Assume that $M := M(\alpha)$ is a terraced matrix associated with an injective sequence $\alpha = \{\alpha_n : n = 0, 1, 2, 3, \ldots\}$ of nonzero complex numbers, and let $M(\beta)$ denote the terraced matrix associated with the sequence $\beta = \{\beta_n\}$ given by $\beta_0 = \alpha_1$ and $\beta_n = \alpha_n$ for all $n \geq 1$. If $M(\alpha)$ is hyponormal, then $M(\beta)$ is dominant but not hyponormal.

**Proof.** The proof that $\text{Ran}(M(\beta) - \lambda) \subset \text{Ran}(M(\beta) - \lambda)^*$ for all $\lambda \neq \beta_0$ requires only a minor adjustment of the argument used in the proof of Theorem 2.1, so we leave that to the reader. We now show that if $\lambda = \beta_0$, then
\[
\text{Ran}(M(\beta) - \lambda) \subset \text{Ran}(M(\beta) - \lambda)^*.
\]
Recall that $\beta_0 = \alpha_1$. If $x := \langle x_0, x_1, x_2, \ldots \rangle^T \in \ell^2$, it must be shown that
\[
(M(\beta) - \alpha_1)x \in \text{Ran}(M(\beta) - \alpha_1)^*.
\]
Since $M := M(\alpha)$ is hyponormal and therefore also dominant, we know that
\[
(M - \alpha_1)x = (M - \alpha_1)^*y
\]
for some $y := \langle y_0, y_1, y_2, \ldots \rangle^T \in \ell^2$. It can be verified that
\[
(M(\beta) - \alpha_1)^*y = [(\alpha_0 - \alpha_1)x_0 - (\alpha_0 - \alpha_1)y_0]e_0 + (\det(M(\beta) - \alpha_1)x.
\]
If
\[
z := \frac{1}{\alpha_1}[(\alpha_0 - \alpha_1)y_0 - (\alpha_0 - \alpha_1)x_0]e_1,
\]
then $(M(\beta) - \alpha_1)^*z = [(\alpha_0 - \alpha_1)y_0 - (\alpha_0 - \alpha_1)x_0]e_0$. It follows that
\[
(M(\beta) - \alpha_1)^*(y + z) = (M(\beta) - \alpha_1)x,
\]
and now the proof that $M(\beta)$ is dominant is complete. Finally, since $\det(S_2) = -|\alpha_1|^4 < 0$, $M(\beta)$ cannot be hyponormal.

**Example 2.2.** Recall that for fixed $k > 0$, the generalized Cesàro matrices of order one are the terraced matrices $C_k := M(\alpha)$ that occur when $\alpha_n = \frac{1}{k+n}$ for all $n$. $C_k$ is hyponormal for $k \geq 1$. If $M_k := M(b)$ is the terraced matrix associated with the sequence defined by $b_0 = \frac{1}{k+1}$ and $b_n = \frac{1}{k+n}$ for all $n \geq 1$, then we know from Theorem 2.3 that $M_k$ is dominant but not hyponormal for $k \geq 1$. 

In closing, we are reminded of another question left open at the conclusion of [5] – Is $C_k$ dominant for $\frac{1}{2} \leq k < 1$? The next result provides a partial answer to that question.

**Proposition 2.4.** $C_k$ is not dominant when $k = \frac{1}{2}$.

**Proof.** It easily be verified that

$$(C_k - \frac{1}{k})[k(e_0 - e_1)] = e_1,$$

so $e_1 \in \text{Ran}(C_k - \frac{1}{k})$ for all $k > 0$. For $C_k$ to be dominant, then it must also be true that $e_1 \in \text{Ran}(C_k - \frac{1}{k})^*$. A straightforward calculation reveals that in order to have $e_1 = (C_k - \frac{1}{k})^*z$ for some $z = <z_0, z_1, z_2, z_3, ... >^T$, it is necessary that $z_1 = -k$ and $z_n = \frac{(n-1)!k^2}{\prod_{j=1}^{n-1}(k+j)}$ for all $n \geq 2$. However, it then follows from a refinement (see [2, Theorem III, p. 396]) of Raabe’s test that $z \notin \ell^2$ for $k = \frac{1}{2}$ and hence $e_1 \notin \text{Ran}(C_k - \frac{1}{k})^*$ for that value of $k$. \hfill $\square$

**References**


1081 Buckley Drive, Jackson, Mississippi 39206, U.S.A.

E-mail address: rhaly@alumni.virginia.edu, rhaly@member.ams.org