ON SOME PROPERTIES OF ANALYTIC AND MEROMORPHIC FUNCTIONS

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Abstract. In the present paper, we obtain some new properties of analytic and meromorphic functions

1. Introduction

Let \( p(z) \) be analytic in the open unit disc \( U = \{ z : z \in \mathbb{C}; |z| < 1 \} \) with \( p(0) = 1 \) and suppose that there exists a point \( z_0 (|z_0| < 1) \) such that

(i) \( |p(z)| > a \) \((|z| < |z_0|; 0 < a < 1)\) and \( |p(z_0)| = a \)

or

(ii) \( \text{Re} \{ p(z) \} < M \) \((|z| < |z_0|; M > 1)\) and \( \text{Re} \{ p(z_0) \} = M \).

Also let

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1) \]

be analytic in \( U \) and

\[ F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \quad (1.2) \]

be meromorphic in \( U \).

In this paper, we shall obtain some basic results for \( z_0 p'(z_0)/p(z_0) \) under the condition (i) or (ii). Our main results will be applied to get some properties of \( z_0 f'(z_0)/f(z_0) \) or \(-z_0 F'(z_0)/F(z_0)\).

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2. Main Results

Lemma 2.1. (see [1]). Let \( w(z) \) be regular in \( U \) with \( w(0) = 0 \). If \( w(z) \) attains its maximum value on the circle \( |z| = r \) at a given point \( z_0 \in U \), then \( z_0 w'(z_0) = kw(z_0) \), where \( k \) is a a real number and \( k \geq 1 \).

Theorem 2.2. Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and suppose that there exists a point \( z_0, |z_0| < 1 \) such that \( |p(z)| > a \) (\(|z| < |z_0|\)) and \( |p(z_0)| = a \), where \( 0 < a < 1 \).

Then we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = \text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right) \leq -\frac{1-a}{1+a}.
\]

Proof. Let us put
\[
p(z) = \frac{a(1 + \psi(z))}{1 - \psi(z)}
\]
or
\[
\psi(z) = \frac{p(z) - a}{p(z) + a} \quad (|z| < |z_0|).
\]
Then \( \psi(z) \) is analytic in \( |z| < |z_0| \) and
\[
0 < \psi(0) = \frac{1-a}{1+a} < 1.
\]

By the hypothesis of the theorem, we have
\[
|p(z)| = a \left| \frac{1 + \psi(z)}{1 - \psi(z)} \right| > a \quad (|z| < |z_0|).
\]
Thus
\[
\left| \frac{1 + \psi(z)}{1 - \psi(z)} \right| > 1 \quad (|z| < |z_0|).
\]
It shows that
\[
\text{Re} \{\psi(z)\} > 0 \quad (|z| < |z_0|).
\]
On the other hand, we have
\[
|p(z_0)| = a \left| \frac{1 + \psi(z_0)}{1 - \psi(z_0)} \right| = a,
\]
this shows that
\[
\text{Re} \{\psi(z_0)\} = 0.
\]
Putting
\[
\Phi(z) = \frac{\psi(0) - \psi(z)}{\psi(0) + \psi(z)}, \quad \Phi(0) = 0,
\]
then we have \( \Phi(z) \) is analytic in \( U \), \( |\Phi(z)| < 1 \) for \( |z| < |z_0| \) and \( |\Phi(z_0)| = 1 \). Therefore, applying Lemma, we have that
\[
\frac{z_0 \Phi'(z_0)}{\Phi(z_0)} = -\frac{2\psi(0)z_0\psi'(z_0)}{(\psi(0))^2 + |\psi(z_0)|^2}
\]
\[
= k \geq 1.
\]
This means that \( z_0 \psi'(z_0) \) is real negative because \( 0 < \psi(0) < 1 \). Then we say that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 \psi'(z_0)}{1 + \psi(z_0)} + \frac{z_0 \psi'(z_0)}{1 - \psi(z_0)}
\]
\[
= \frac{2z_0'p'(z_0)}{1 - (\psi(z_0))^2} = \frac{2z_0'p'(z_0)}{1 + |\psi(z_0)|^2}
\]
\[
= -k(0) \left( \frac{(\psi(0))^2 + |\psi(z_0)|^2}{1 + |\psi(z_0)|^2} \right)
\leq -\psi(0) \left( \frac{(\psi(0))^2 + |\psi(z_0)|^2}{(\psi(0))^2 + (\psi(0))^2|\psi(z_0)|^2} \right)
< -\psi(0) = -\frac{1 - a}{1 + a}.
\]

This completes the proof.

**Theorem 2.3.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \), suppose that there exists a point \( z_0 \ (|z_0| < 1) \) such that \( \text{Re} \{p(z)\} < M \), \( p(z) \neq 0 \) for \( |z| < |z_0| \), and \( \text{Re} \{p(z_0)\} = M \), where \( M > 1 \). Then we have
\[ \text{Re} \{z_0 p'(z_0)\} \geq \frac{M - 1}{2}. \]

**Proof.** Let us put
\[ g(z) = \frac{1}{2M - 1} \left( \frac{2M - p(z)}{p(z)} \right), \quad g(0) = 1. \]

Since \( \text{Re} \{p(z)\} < M \) for \( |z| < |z_0| \), we see that
\[ \left| \frac{2M - p(z)}{p(z)} \right| > 1 \]
for \( |z| < |z_0| \). This gives us that
\[ |g(z)| > \frac{1}{2M - 1} \quad (|z| < |z_0|) \]
and
\[ |g(z_0)| = \frac{1}{2M - 1}. \]

Now
\[ \text{Re} \left( \frac{z_0 g'(z_0)}{g(z_0)} \right) = \text{Re} \left( -\frac{z_0 p'(z_0)}{p(z_0)} - \frac{z_0 p'(z_0)}{2M - p(z_0)} \right) \tag{2.1} \]
Applying Theorem (2.2), we have
\[ \text{Re} \left( \frac{z_0 g'(z_0)}{g(z_0)} \right) \leq -\frac{1}{1 + \frac{1}{2M - 1}} = -\frac{M - 1}{M}. \tag{2.2} \]
Putting \( p(z_0) = M + ia \), where \( a \) is a real number, we have
\[ -\text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} + \frac{z_0 p'(z_0)}{2M - p(z_0)} \right) = -\text{Re} \left( \frac{2M}{M^2 + a^2 z_0 p'(z_0)} \right) \]
\[ = -\frac{2M}{M^2 + a^2} \text{Re} \{z_0 p'(z_0)\}. \tag{2.3} \]
Now (2.1) in conjunction with (2.2) and (2.3) gives

\[
- \frac{2M}{M^2 + a^2} \text{Re} \{ z_0 p'(z_0) \} \leq - \frac{M - 1}{M}.
\]

This shows that

\[
\text{Re} \{ z_0 p'(z_0) \} \geq \left( \frac{M^2 + a^2}{2M} \right) \left( \frac{M - 1}{M} \right) \geq \frac{M - 1}{2}.
\]

(2.4)

It completes the proof.

**Corollary 2.4.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and suppose that

\[
\text{Re} \left( p(z) + \frac{zp'(z)}{p(z)} \right) > \frac{a^2 + 2a - 1}{a + 1} \quad (|z| < 1; 0 < a < 1).
\]

(2.5)

Then we have \( |p(z)| > a \) in \( U \).

**Proof.** Suppose that there exists a point \( z_0 \) \((|z_0| < 1)\) such that \( a < |p(z)| \) in \( |z| < |z_0| \) and \( |p(z_0)| = a \), then Theorem (2.2) gives

\[
\frac{z_0 p'(z_0)}{p(z_0)} = \text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right) \leq - \frac{1 - a}{1 + a}.
\]

Thus it follows that

\[
\text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} + p(z_0) \right) \leq - \frac{1 - a}{1 + a} + a = \frac{a^2 + 2a - 1}{a + 1}.
\]

It contradicts the hypothesis (2.5) and it completes the proof.

**Corollary 2.5.** Let \( f(z) \) defined by (1.1) be analytic and \( f'(z) \neq 0 \) in \( U \), suppose that

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{a^2 + 2a - 1}{a + 1} \quad (z \in U; 0 < a < 1).
\]

Then we have

\[
\left| \frac{zf'(z)}{f(z)} \right| > a \quad \text{in} \quad U.
\]

**Proof.** Let us put

\[
p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1.
\]

Then we have

\[
1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.
\]

Applying Corollary 2.4, we have Corollary 2.5.

**Corollary 2.6.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and suppose that there exists a point \( z_0 \) \((|z_0| < 1)\) such that \( \text{Re} \{ p(z) \} < M, p(z) \neq 0 \) for \( |z| < |z_0| \), \( \text{Re} \{ p(z_0) \} = M \) where \( M > 1 \).

Then we have

\[
\text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right) \geq \frac{M - 1}{2M}.
\]
Proof. Applying (2.4) of Theorem (2.3), we have
\[ \text{Re} \left\{ z_0 p'(z_0) \right\} \geq \left( \frac{M^2 + a^2}{2M} \right) \left( \frac{M - 1}{M} \right) \]
where \( p(z_0) = M + ia \), \( a \) is a real number.
Then it follows that
\[ \text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right) = \text{Re} \left( \frac{z_0 p'(z_0)}{M + ia} \right) \]
\[ = \text{Re} \left\{ \left( \frac{M - ia}{M^2 + a^2} \right) z_0 p'(z_0) \right\} \]
\[ = \left( \frac{M}{M^2 + a^2} \right) z_0 p'(z_0) \]
\[ \geq \left( \frac{M}{M^2 + a^2} \right) \left( \frac{M^2 + a^2}{2M^2} \right) (M - 1) \]
\[ = \frac{M - 1}{2M} \cdot \]
It completes the proof.

Applying the same method as the proof of Corollary (2.5) and (2.6), we have the following result.

**Corollary 2.7.** Let \( f(z) \) defined by (1.1) be analytic and in \( f'(z) \neq 0 \) in \( U \). Also suppose that
\[ 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{2M^2 + M - 1}{2M} \quad (|z| < 1; M > 1). \]
Then we have
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < M \quad \text{in} \ U. \]

Corollary 2.7 is equivalent to Corollary 2.7.

**Corollary 2.8.** Let \( f(z) \) defined by (1.1) be analytic and \( f'(z) \neq 0 \) in \( U \). Also suppose that
\[ 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \beta \quad (|z| < 1; \beta > 1). \]
Then we have
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \frac{2\beta - 1 + \sqrt{4\beta^2 - 4\beta + 9}}{4} \quad \text{in} \ U. \]

**Corollary 2.9.** Let \( F(z) \) defined by (1.2) be meromorphic and \( F'(z) \neq 0 \) in \( U \). Also suppose that
\[ \left\{ 1 + \text{Re} \left( \frac{zf''(z)}{F'(z)} \right) \right\} > \frac{2M^2 - M + 1}{2M} \quad (|z| < 1; M > 1). \quad (2.6) \]
Then we have
\[ -\text{Re} \left( \frac{zf'(z)}{F(z)} \right) < M \quad \text{in} \ U. \]
Proof. Let us put
\[ p(z) = - z \frac{F'(z)}{F(z)}, \quad p(0) = 1, \]
then \( p(z) \) is analytic in \( U \) and it follows that
\[ - \left( 1 + z \frac{F''(z)}{F'(z)} \right) = p(z) - \frac{zp'(z)}{p(z)}. \tag{2.7} \]
If there exists a point \( z_0 (|z_0| < 1) \) such that \( \text{Re} \{ p(z_0) \} < M \) for \( |z| < |z_0| \) and \( \text{Re} \{ p(z_0) \} = M \), then from Corollary 2.5, we have
\[ \text{Re} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right) \geq \frac{M - 1}{2M}. \]
It follows that
\[ - \left\{ 1 + \text{Re} \left( \frac{z_0 F''(z_0)}{F'(z_0)} \right) \right\} = \text{Re} \left( p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)} \right) \leq M - \frac{M - 1}{2M} = \frac{2M^2 - M + 1}{2M}. \]
This contradicts (2.6) and it completes the proof of Corollary 2.9.

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