THE HIGHER DUALS OF A BANACH ALGEBRA INDUCED BY A BOUNDED LINEAR FUNCTIONAL

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ABSTRACT. Let $A$ be a Banach space and let $\varphi \in A^*$ be non-zero with $\|\varphi\| \leq 1$. The product $a \cdot b = \langle \varphi, a \rangle b$ makes $A$ into a Banach algebra. We denote it by $\varphi A$. Some properties of $\varphi A$ such as Arens regularity, $n$-weak amenability and semi-simplicity are investigated.

1. INTRODUCTION

This paper has its genesis in a simple example of Zhang [10, Page 507]. For an infinite set $S$ he equipped $l^1(S)$ with the algebra product $a \cdot b = a(s_0)b$ ($a, b \in l^1(S)$), where $s_0$ is a fixed element of $S$. He used this as a Banach algebra which is $(2n - 1)$-weakly amenable but is not $(2n)$-weakly amenable for any $n \in \mathbb{N}$. Here we study a more general form of this example. Indeed, we equip a non-trivial product on a general Banach space turning it to a Banach algebra. It can serve as a source of (counter-)examples for various purposes in functional analysis.

Let $A$ be a Banach space and fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \langle \varphi, a \rangle b$ turning $A$ into a Banach algebra which will be denoted by $\varphi A$. Some properties of algebras of this type are investigated in [5, 4, 1, 7]. Trivially $\varphi A$ has a left identity (indeed, every $e \in \varphi A$ with $\langle \varphi, e \rangle = 1$ is a left identity), while it has no bounded approximate identity in the case where $\dim(A) \geq 2$. Now the Zhang’s example can be interpreted as an special case of ours. Indeed, he studied $\varphi s_0 l^1(S)$, where $\varphi s_0 \in l^\infty(S)$ is the characteristic function at $s_0$. Here, among other things, we focus on the higher duals of $\varphi A$ and investigate various notions of amenability for $\varphi A$. In particular, we prove that for every $n \in \mathbb{N}$, $\varphi A$ is $(2n - 1)$-weakly amenable but it is not $(2n)$-weakly amenable for any $n$, in the case where $\dim(\ker \varphi) \geq 2$. 

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2. The results

Before we proceed for the main results we need some preliminaries. As we shall be concerned with the Arens products □ and ◊ on the bidual $A^{**}$ of a Banach algebra $A$, let us introduce these products.

Let $a, b \in A, f \in A^*$ and $m, n \in A^{**}$.

\[
\begin{align*}
\langle f \cdot a, b \rangle &= \langle f, ab \rangle & \langle b, a \cdot f \rangle &= \langle ba, f \rangle \\
\langle n \cdot f, a \rangle &= \langle n, f \cdot a \rangle & \langle a, f \cdot n \rangle &= \langle a \cdot f, n \rangle \\
\langle m \triangledown n, f \rangle &= \langle m, n \cdot f \rangle & \langle f, m \triangledown n \rangle &= \langle f \cdot m, n \rangle.
\end{align*}
\]

If □ and ◊ coincide on the whole of $A^{**}$ then $A$ is called Arens regular. For the brevity of notation we use the same symbol “·” for the various module operations linking $A$, such as $A^*$, $A^{**}$ and also as well for the $n$th dual $A^{(n)}$, $(n \in \mathbb{N})$. The main properties of these products and various $A$–module operations are detailed in [2]; see also [9].

A derivation from a Banach algebra $A$ to a Banach $A$-module $X$ is a bounded linear mapping $D : A \to X$ such that $D(ab) = D(a)b + aD(b)$ $(a, b \in A)$. For each $x \in X$ the mapping $\delta_x : a \to ax - xa$, $(a \in A)$ is a bounded derivation, called an inner derivation. The concept of $n$-weak amenability was introduced and intensively studied by Dales et al. [3]. A Banach algebra $A$ is said to be $n$-weakly amenable $(n \in \mathbb{N})$ if every derivation from $A$ into $A^{(n)}$ is inner. Trivially, 1-weak amenability is nothing else than weak amenability. A derivation $D : A \to A^*$ is called cyclic if $\langle D(a), b \rangle + \langle D(b), a \rangle = 0$ $(a, b \in A)$. If every bounded cyclic derivation from $A$ to $A^*$ is inner then $A$ is called cyclically amenable which was studied by Grønbaek [8]. Throughout the paper we usually identify an element of a space with its canonical image in its second dual.

Now we come to $\varphi A$. A direct verification reveals that for $a \in A, f \in (\varphi A)^*$ and $m, n \in (\varphi A)^{**}$,

\[
\begin{align*}
f \cdot a &= \langle \varphi, a \rangle f & a \cdot f &= \langle f, a \rangle \varphi \\
n \cdot f &= \langle n, f \rangle \varphi & f \cdot n &= \langle n, f \rangle f \\
m \triangledown n &= \langle m, n \rangle \varphi & m \triangledown n &= \langle m, n \rangle.
\end{align*}
\]

The same calculation gives the $\varphi A$–module operations of $(\varphi A)^{(2n-1)}$ and $(\varphi A)^{(2n)}$ as follows,

\[
\begin{align*}
F \cdot a &= \langle \varphi, a \rangle F & a \cdot F &= \langle F, a \rangle \varphi & (F \in (\varphi A)^{(2n-1)}) \\
G \cdot a &= \langle G, \varphi \rangle a & a \cdot G &= \langle \varphi, a \rangle G & (G \in (\varphi A)^{(2n)}).
\end{align*}
\]

We commence with the next straightforward result, most parts of which are based on the latter observations on the various duals of $\varphi A$.

**Proposition 2.1.** (i) $\varphi A$ is Arens regular and $(\varphi A)^{**} = (\varphi (A^{**})$. Furthermore, for each $n \in \mathbb{N}$, $(\varphi A)^{(2n)}$ is Arens regular.

(ii) $(\varphi A)^{**} \cdot \varphi A = \varphi A$ and $\varphi A \cdot (\varphi A)^{**} = (\varphi A)^{**}$; in particular, $\varphi A$ is a left ideal of $(\varphi A)^{**}$.

(iii) $(\varphi A)^* \cdot \varphi A = (\varphi A)^*$ and $\varphi A \cdot (\varphi A)^* = \mathbb{C}\varphi$.

As $\varphi A$ has no approximate identity, in general, it is not amenable. The next result investigates $n$–weak amenability of $\varphi A$.
Theorem 2.2. For each \( n \in \mathbb{N} \), \( \mathcal{A} \) is \((2n - 1)\)-weakly amenable, while in the case where \( \dim(\ker \varphi) \geq 2 \), \( \mathcal{A} \) is not \((2n)\)-weakly amenable for any \( n \in \mathbb{N} \).

Proof. Let \( D : \mathcal{A} \rightarrow (\mathcal{A})^{(2n-1)} \) be a derivation and let \( a, b \in \mathcal{A} \). Then
\[
\langle \varphi, a \rangle D(b) = D(ab) = D(a)b + aD(b) = \langle \varphi, b \rangle D(a) + \langle \varphi, a \rangle \varphi.
\]
It follows that \( \langle \varphi, a \rangle (D(b), a) = \langle \varphi, b \rangle (D(a), a) + \langle \varphi, a \rangle (D(b), a) \), from which we have \( (D(a), a) = 0 \), or equivalently, \( (D(a + b), a + b) = 0 \). Therefore \( (D(a), b) = -D(b), a \). Now with \( e \) as a left identity for \( \mathcal{A} \) we have
\[
D(b) = D(eb) = \langle \varphi, b \rangle D(e) + \langle D(b), e \rangle \varphi = \langle \varphi, b \rangle D(e) - \langle D(e), b \rangle \varphi = \delta_{D(e)}(b).
\]
Therefore \( D \) is inner, as required.

To prove that \( \mathcal{A} \) is not \((2n)\)-weakly amenable for any \( n \in \mathbb{N} \), it is enough to show that \( \mathcal{A} \) is not 2-weakly amenable, [3, Proposition 1.2]. To this end let \( f \in (\mathcal{A})^* \) be such that \( f \) and \( \varphi \) are linearly independent. It follows that \( \langle f, a_0 \rangle = \langle \varphi, b_0 \rangle = 0 \) and \( \langle f, b_0 \rangle = \langle \varphi, b_0 ] = 1 \), for some \( a_0, b_0 \in \mathcal{A} \). Define \( D : \mathcal{A} \rightarrow (\mathcal{A})^{**} \) by \( (D(a)) = (f - \varphi, a)b_0 \), then \( D \) is a derivation. If there exists \( m \in (\mathcal{A})^{**} \) with \( D(a) = am - ma \) \( (a \in \mathcal{A}) \), then by taking \( a = b_0 \), we obtain \( b_0 = -(m, \varphi)b_0 \) which follows that \( 1 = -(m, \varphi) \). Now if \( a \in \ker \varphi \), then \( \langle f, a \rangle b_0 = -(m, \varphi)a = a \). It follows that \( \dim(\ker \varphi) = 1 \) that is a contradiction. \( \square \)

As an immediate consequence of Theorem 2.2 we obtain the result of Zhang mentioned in the introduction.

Corollary 2.3 ([10, Page 507]). For each \( n \in \mathbb{N} \), \( \varphi_n \mathcal{I} \) is \((2n - 1)\)-weakly amenable, while it is not \((2n)\)-weakly amenable for any \( n \in \mathbb{N} \).

Proposition 2.4. A bounded linear map \( D : \mathcal{A} \rightarrow (\mathcal{A})^{(2n)}, (n \in \mathbb{N}) \), is a derivation if and only if \( D(\mathcal{A}) \subseteq \ker \varphi \).

Proof. A direct verification shows that \( D : \mathcal{A} \rightarrow (\mathcal{A})^{(2n)} \) is a derivation if and only if
\[
\langle \varphi, a \rangle D(b) = D(ab) = D(a)b + aD(b) = \langle \varphi, b \rangle D(a) + \langle \varphi, a \rangle \varphi \quad (a, b \in \mathcal{A}).
\]
And this is equivalent to \( (D(a), \varphi) = 0 \), \( (a \in \mathcal{A}) \); that is \( D(\mathcal{A}) \subseteq \ker \varphi \). Note that here \( \varphi \) is assumed to be an element of \( (\mathcal{A})^{(2n+1)} \). \( \square \)

The next results demonstrates that in contrast to Theorem 2.2, \( \mathcal{A} \) is \((2n)\)-weakly amenable in the case where \( \dim(\ker \varphi) < 2 \).

Proposition 2.5. If \( \dim(\ker \varphi) < 2 \) then \( \mathcal{A} \) is \((2n)\)-weakly amenable for each \( n \in \mathbb{N} \).

Proof. The only reasonable case that we need to verify is \( \dim(\ker \varphi) = 1 \). In this case we have \( \dim(\mathcal{A}) = 2 \). Therefore one may assume that \( A \) is generated by two elements \( e, a \in A \) such that \( \langle \varphi, e \rangle = 1 \) and \( \langle \varphi, a \rangle = 0 \). Let \( f \in (\mathcal{A})^* \) satisfy \( \langle f, e \rangle = 0 \) and \( \langle f, a \rangle = 1 \). Then \( f \) and \( \varphi \) are linearly independent and generate \( A^* \); indeed, every non-trivial element \( g \in A^* \) has the form \( g = (g, e) \varphi + (g, a) f \). Let \( D : \mathcal{A} \rightarrow (\mathcal{A})^{(2n)} \) be a derivation then as Proposition 2.5 demonstrates \( D(\mathcal{A}) \subseteq \ker \varphi \). Therefore \( D(x) = (g, x)a, (x \in \mathcal{A}) \), for some \( g \in (\mathcal{A})^* \). As for every \( x \in \mathcal{A}, x = \langle \varphi, x \rangle e + \langle f, x \rangle a \), a direct calculation reveals that \( D = \delta_{(g, e)a - (g, a)e} \); as required. \( \square \)
Remark. (i) If we go through the proof of Theorem 2.2 we see that the range of the derivation $D(a) = (f - \varphi, a)b_0$ lies in $\varphi A$, and the same argument may be applied to show that it is not inner as a derivation from $\varphi A$ to $\varphi A$. This shows that $\varphi A$ is not 0–weakly amenable, i.e. $H^1(\varphi A, \varphi A) \neq 0$; see a remark just after [3, Proposition 1.2]. The same situation has occurred in the proof of Proposition 2.5.

(ii) As $\varphi A$ has a left identity and it is a left ideal in $(\varphi A)^\ast$, it is worthwhile mentioning that, to prove the $(2n - 1)$–weak amenability of $\varphi A$ it suffices to show that $\varphi A$ is weakly amenable; [10, Theorem 3], and this has already done by Dales et al. [6, Page 713].

We have seen in the first part of the proof of Theorem 2.2 that if $D : \varphi A \to (\varphi A)^{(2n-1)}$ is a derivation then $\langle D(a), b \rangle + \langle D(b), a \rangle = 0$, $(a, b \in \varphi A)$; and the latter is known as a cyclic derivation for the case $n = 1$. Therefore as a consequence of Theorem 2.2 we get:

**Corollary 2.6.** A bounded linear mapping $D : \varphi A \to (\varphi A)^\ast$ is a derivation if and only if it is a cyclic derivation. In particular, $\varphi A$ is cyclicly amenable.

We conclude with the following list consisting of some miscellaneous properties of $\varphi A$ which can be verified straightforwardly.

- If $\varphi = \lambda \psi$ for some $\lambda \in \mathbb{C}$ then trivially $\varphi A$ and $\psi A$ are isomorphic; indeed, the mapping $a \to \lambda a$ defines an isomorphism. However, the converse is not valid, in general. For instance, let $A$ be generated by two elements $a, b$. Choose $\varphi, \psi \in A^\ast$ such that $\langle \varphi, a \rangle = \langle \psi, b \rangle = 0$ and $\langle \varphi, b \rangle = \langle \psi, a \rangle = 1$, then $\varphi A$ and $\psi A$ are isomorphic (indeed, $\alpha a + \beta b \to \alpha b + \beta a$ defines an isomorphism), however $\varphi$ and $\psi$ are linearly independent.

- It can be readily verified that $\{0\} \cup \{a \in \varphi A, \varphi(a) = 1\}$ is the set of all idempotents of $\varphi A$. Moreover, this is actually the set of all minimal idempotents of $\varphi A$.

- It is obvious that every subspace of $\varphi A$ is a left ideal, while a subspace $I$ is a right ideal if and only if either $I = \varphi A$ or $I \subseteq \ker \varphi$. In particular, $\ker \varphi$ is the unique maximal ideal in $\varphi A$.

- A subspace $I$ of $\varphi A$ is a modular left ideal if and only if either $I = \varphi A$ or $I = \ker \varphi$. In particular, $\ker \varphi$ is the unique primitive ideal in $\varphi A$ and this implies that $\text{rad}(\varphi A) = \ker \varphi$ and so $\varphi A$ is not semi-simple. Furthermore, for every non-zero proper closed ideal $I$, $\text{rad}(I) = I \cap \text{rad}(\varphi A) = I \cap \ker \varphi = I$.

- A direct verification reveals that $LM(\varphi A) = CI$ and $RM(\varphi A) = B(\varphi A)$, where $LM$ and $RM$ stand for the left and right multipliers, respectively.

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**References**


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