SIMULTANEOUS APPROXIMATION BY A LINEAR COMBINATION OF BERNSTEIN-DURRMЕYER TYPE POLYNOMIALS

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ABSTRACT. The aim of the present paper is to study some direct results in simultaneous approximation for a linear combination of Bernstein-Durrmeyer type polynomials.

1. INTRODUCTION

For \( f \in L_B[0,1] \) (the space of bounded and Lebesgue integrable functions on \([0,1]\)), the modified Bernstein type polynomial operators

\[
P_n(f;x) = n \sum_{k=1}^{n} p_{n,k}(x) \int_{0}^{1} p_{n-1,k-1}(t)f(t) \, dt + (1-x)^n f(0),
\]

where

\[
p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq x \leq 1,
\]

were introduced by Gupta and Maheshwari [8] wherein they studied the approximation of functions of bounded variation by these operators. In [6], Gupta and Ispir studied the pointwise convergence and Voronovskaja type asymptotic results in simultaneous approximation. Gairola [5] derived direct, inverse and saturation results for an iterative combination of these operators in ordinary approximation. We [1] studied a direct theorem in the \( L_p \)-norm for these combinations of the operator \( P_n(x) \).

The operators \( P_n(f;x) \) can be expressed as

\[
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\[ P_n(f; x) = \int_0^1 W_n(t, x) f(t) \, dt, \]

where the kernel of the operators is given by

\[ W_n(t, x) = n \sum_{k=1}^{n} p_{n,k}(x) p_{n-1,k-1}(t) + (1 - x)^n \delta(t), \]

\( \delta(t) \) being the Dirac-delta function.

It turns out that the order of approximation by these operators is at best \( O(n^{-1}) \), however smooth the function may be. Following the technique of linear combination described in [3] to improve the order of approximation, we define

\[ P_n(f, k, x) = \sum_{j=0}^{k} C(j, k) P_{d_j n}(f, x), \]

where

\[ C(j, k) = \prod_{i=0, i \neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0, 0) = 1, \]

\( d_0, d_1, \ldots d_k \) being \((k + 1)\) arbitrary but fixed distinct positive integers.

The object of the present paper is to investigate some direct results in the simultaneous approximation by the operators \( P_n(., k, x) \). First we establish a Voronovskaja type asymptotic formula and then obtain an error estimate in terms of local modulus of continuity of the function involved for the operator \( P_n^{(r)}(., k, x) \).

2. Auxiliary Results

In the sequel we shall require the following results:

**Lemma 2.1.** [6] For the function \( u_{n,m}(x) \), \( m \in \mathbb{N}^0 \) (the set of non-negative integers) defined as

\[ u_{n,m}(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) \left( \frac{\nu}{n} - x \right)^m, \]

we have \( u_{n,0}(x) = 1 \) and \( u_{n,1}(x) = 0 \). Further, there holds the recurrence relation

\[ n u_{n,m+1}(x) = x \left[ u'_{n,m}(x) + m u_{n,m-1}(x) \right], m = 1, 2, 3, \ldots \]

Consequently,

(i) \( u_{n,m}(x) \) is a polynomial in \( x \) of degree \( [m/2] \), where \([\alpha]\) denotes the integral part of \( \alpha \);

(ii) for every \( x \in [0, 1] \), \( u_{n,m}(x) = O\left(n^{-[m+(m+1)/2]}\right)\).

**Remark 1.** From the above lemma, we have

\[ \sum_{\nu=0}^{n} p_{n,\nu}(x) \left( \nu - nx \right)^{2j} = O(n^j) \]  \( (2.1) \)
For $m \in \mathbb{N}^0$ (the set of non-negative integers), the $m$th order moment for the operators $P_n$ is defined as

$$\mu_{n,m}(x) = P_n \left( (t - x)^m; x \right).$$

**Lemma 2.2.** [1] For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \left( \frac{-x}{m+1} \right)$, and there holds the recurrence relation

$$(n+m+1)\mu_{n,m+1}(x) = x(1-x) \left\{ \mu'_m(x) + 2m \mu_{n,m}(x) \right\} + (m(1-2x)-x)\mu_{n,m}(x),$$

for $m \geq 1$.

Consequently, we have

(i) $\mu_{n,m}(x)$ is a polynomial in $x$ of degree $m$;

(ii) for every $x \in [0, 1]$, $\mu_{n,m}(x) = O \left( n^{-\left( m+1 \right)/2} \right)$.

**Remark 2.** From the above lemma, it follows that for each $x \in (0, 1)$,

$$n \sum_{k=1}^{n} \frac{1}{p_{n,k}}(x) \int_{0}^{1} p_{n-1,k-1}(t)(t-x)^m dt = O \left( n^{-\left( m+1 \right)/2} \right), \quad m \in \mathbb{N}^0.$$

**Lemma 2.3.** If $C(j, k), j = 0, 1, 2, \ldots, k$ is defined as in 1.1, then

$$\sum_{j=0}^{k} C(j, k) d_j^{-m} = \begin{cases} 1, & m = 0 \\ 0, & m = 1, 2, 3, 4, \ldots \end{cases}$$

**Lemma 2.4.** For $p \in \mathbb{N}, P_n((t - x)^p, k, x) = n^{-k+1} \{ Q(p, k, x) + o(1) \}$ where $Q(p, k, x)$ are certain polynomials in $x$ of degree at most $p$.

From Lemma 2.2 and Lemma 2.3 the above lemma easily follows hence the details are omitted.

Throughout this paper, we assume $0 < a < b < 1$, $I = [a, b]$, $0 < a_1 < a_2 < b_2 < b_1 < 1$, $I_i = [a_i, b_i]$, $i = 1, 2$, $[a_1, b_1] \subset (a, b)$, $\| \cdot \|_{C(I)}$ the sup- norm on the interval $I$ and $C$ a constant not necessarily the same at each occurrence.

Let $f \in C[a, b]$. Then, for a sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of $m-$th order corresponding to $f$ is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \left( f(t) + (-1)^{m-1} \Delta^m_h \sum_{i=1}^{m} f(t) \right) dt, \quad t \in I_1,$$

where $\Delta^m_h$ is the $m-$th forward difference operator with step length $h$.

**Lemma 2.5.** Let $f \in C[a, b]$. Then, for the function $f_{\eta,m}$, we have

(a) $f_{\eta,m}$ has derivatives up to order $m$ over $I_1$;

(b) $\| f_{\eta,m} \|_{C(I_1)} \leq C_r \omega_r(f, \eta, [a, b]), r = 1, 2, \ldots, m$;

(c) $\| f - f_{\eta,m} \|_{C(I_1)} \leq C_{m+1} \omega_m(f, \eta, [a, b])$;

(d) $\| f_{\eta,m} \|_{C(I_1)} \leq C_{m+2} \eta^{-m} \| f \|_{C[a, b]}$;

(e) $\| f_{\eta,m} \|_{C(I_1)} \leq C_{m+3} \| f \|_{C[a, b]}$.

where $C_r$'s are certain constants that depend on $i$ but are independent of $f$ and $\eta$. 
Following ([9], Theorem 18.17) or ([10], pp.163-165), the proof of the above lemma easily follows hence the details are omitted.

**Lemma 2.6.** [1] For the function $p_{n,k}(x)$, there holds the result

$$x^r(1-x)^r \frac{d^r p_{n,k}(x)}{dx^r} = \sum_{2i+j \leq r \atop i,j \geq 0} n^i(k-nx)^j q_{i,j,r}(x)p_{n,k}(x),$$

where $q_{i,j,r}(x)$ are certain polynomials in $x$ independent of $n$ and $k$.

**Theorem 2.7.** Let $f \in L_B[0,1]$ admitting a derivative of order $2k + 2$ at a point $x \in [0,1]$ then we have

$$\lim_{n \to \infty} n^{k+1} |P_n(f,k,x) - f(x)| = \sum_{\nu=1}^{2k+2} \frac{f^{(\nu)}(x)}{\nu!} Q(\nu,k,x)$$

(2.2)

and

$$\lim_{n \to \infty} n^{k+1} |P_n(f,k+1,x) - f(x)| = 0,$$

(2.3)

where $Q(\nu,k,x)$ are certain polynomials in $x$ of degree $\nu$. Further, the limits in (2.2) and (2.3) hold uniformly in $[a,b]$ if $f^{(2k+2)}$ is continuous on $(a-\eta,b+\eta) \subset (0,1)$, $\eta > 0$.

Proceeding along the lines of the proof of (Thm., [2]), the above theorem easily follows. Hence the details are omitted.

3. **Main Results**

**Theorem 3.1.** Let $f \in L_B[0,1]$ admitting a derivative of order $2k + r + 2$ at a point $x \in (0,1)$ then we have

$$\lim_{n \to \infty} n^{k+1} |P_n^{(r)}(f,k,x) - f^{(r)}(x)| = \sum_{\nu=r}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} Q_1(\nu,k,r,x)$$

(3.1)

and

$$\lim_{n \to \infty} n^{k+1} |P_n^{(r)}(f,k+1,x) - f^{(r)}(x)| = 0,$$

(3.2)

where $Q_1(\nu,k,r,x)$ are certain polynomials in $x$. Further, the limits in (3.1) and (3.2) hold uniformly in $[a,b]$ if $f^{(2k+r+2)}$ is continuous on $(a-\eta,b+\eta) \subset (0,1)$, $\eta > 0$.

**Proof.** By a partial Taylor’s expansion of $f$, we have

$$f(t) = \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} (t-x)^\nu + \epsilon(t,x)(t-x)^{2k+r+2},$$

where $\epsilon(t,x) \to 0$ as $t \to x$. Thus, we can write
\[ n^{k+1} \left[ P_n^{(r)}(f, k, x) - f^{(r)}(x) \right] = n^{k+1} \left[ \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} P_n^{(r)}((t-x)^\nu, k, x) - f^{(r)}(x) \right] + n^{k+1} \sum_{j=0}^k C(j, k) P_{d_j n}^{(r)}(\epsilon(t, x)(t-x)^{2k+r+2}; x) \]

\[ = \Sigma_1 + \Sigma_2, \text{ say.} \]

On an application of Lemma 2.2 and Theorem 2.7 we obtain

\[
\Sigma_1 = n^{k+1} \left[ \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} P_n^{(r)}((t-x)^\nu, k, x) - f^{(r)}(x) \right] + n^{k+1} \left[ \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} \sum_{i=0}^{\nu} \binom{\nu}{i} (-x)^{\nu-i} P_n^{(r)}(i^i, k, x) - f^{(r)}(x) \right]
\]

\[
= n^{k+1} \left[ \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} \sum_{i=0}^{\nu} \binom{\nu}{i} (-x)^{\nu-i} \times \left\{ D^r x^i + n^{-(k+1)} \left[ \sum_{j=1}^{2k+2} D^r \left( \frac{D^j x^i}{j!} Q(j, k, x) \right) + o(1) \right] \right\} - f^{(r)}(x) \right]
\]

\[
= n^{k+1} \left[ \sum_{\nu=0}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} r! \sum_{i=0}^{\nu} \binom{\nu}{i} \binom{i}{r} (-1)^{\nu-i}(x)^{\nu-r} - f^{(r)}(x) \right] + \sum_{\nu=r}^{2k+r+2} Q_1(\nu, k, r, x) f^{(\nu)}(x) + o(1),
\]

where we have used the identity

\[
\sum_{l=0}^{i} (-1)^l \binom{i}{l} \binom{l}{r} = \begin{cases} 
0, & i > r \\
(-1)^r, & i = r.
\end{cases}
\]

Thus, we get

\[
\Sigma_1 = \sum_{\nu=r}^{2k+r+2} Q_1(\nu, k, r, x) f^{(\nu)}(x) + o(1)
\]

In order to prove the assertion 3.1, it is sufficient to show that

\[ n^{k+1} P_n^{(r)}(\epsilon(t, x)(t-x)^{2k+r+2}; x) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

\[
\Sigma = P_n^{(r)}(\epsilon(t, x)(t-x)^{2k+r+2}; x)
\]

\[
= \sum_{k=1}^{n} P_{n,k}^{(r)}(x) \int_0^1 p_{n-1,k-1}(t) \epsilon(t, x)(t-x)^{2k+r+2} dt + (-1)^r \frac{n!}{(n-r)!} (1-x)^{n-r} \epsilon(0, x)(-x)^{2k+r+2}.
\]
Therefore, by using Lemma 2.6 we have

\[
|\Sigma| \leq n \sum_{2i+j \leq r, i,j \geq 0} n^{i} \frac{|q_{i,j,r}(x)|}{x^{r}(1-x)^{r}} \sum_{k=1}^{n} |k-nx|^{j} p_{n,k}(x) \times \\
\int_{0}^{1} p_{n-1,k-1}(t)|\epsilon(t,x)||t-x|^{2k+2+r} dt \\
+ \frac{n!}{(n-r)!} (1-x)^{n-r}|\epsilon(0,x)|x^{2k+r+2} \\
= J_{1} + J_{2}, \text{ say.}
\]

Since \(\epsilon(t,x) \to 0\) as \(t \to x\), for a given \(\epsilon' > 0\) we can find a \(\delta > 0\) such that \(|\epsilon(t,x)| < \epsilon'\) whenever \(0 < |t-x| < \delta\) and for \(|t-x| \geq \delta\), \(|\epsilon(t,x)| \leq K\) for some \(K > 0\). Hence

\[
|J_{1}| \leq nC_{1} \sum_{2i+j \leq r, i,j \geq 0} n^{i} \sum_{k=1}^{n} |k-nx|^{j} p_{n,k}(x) \times \\
\left[ \epsilon' \int_{|t-x|<\delta} p_{n-1,k-1}(t)|t-x|^{2k+2+r} dt + \\
\frac{1}{\delta^2} \int_{|t-x|\geq\delta} p_{n-1,k-1}(t)K|t-x|^{2k+4+r} dt \right] \\
= J_{3} + J_{4}, \text{ say,}
\]

where \(C_{1} = \sup_{2i+j \leq r, i,j \geq 0} \frac{|q_{i,j,r}(x)|}{x^{r}(1-x)^{r}}\).

Applying Schwarz inequality for integration and then summation and Lemma 2.2, 2.1 we have

\[
|J_{3}| \leq C_{1}\epsilon'n^{1/2} \sum_{2i+j \leq r, i,j \geq 0} n^{i} \left( \sum_{k=1}^{n} (k-nx)^{2j} p_{n,k}(x) \right)^{1/2} \left( \int_{0}^{1} p_{n-1,k-1}(t) dt \right)^{1/2} \\
\leq C_{1}\epsilon'n^{1/2} \sum_{2i+j \leq r, i,j \geq 0} n^{i} O(n^{j/2})O(n^{-(2k+2+r)/2}), \text{ (in view of Remark 2)},
\]

Next, again Applying Schwarz inequality for integration and then summation and Lemma 2.2, 2.1 we have
\[ |J_4| \leq \frac{C_1}{\delta^2} n^{1/2} \sum_{2^i+j \leq r, i,j \geq 0} n^i \left( \sum_{k=1}^{n} (k-n)^2 p_{n,k}(x) \right)^{1/2} \times \left( \int_0^1 p_{n-1,k-1}(t)(t-x)^{4k+8+2r} dt \right)^{1/2} \times \left( \int_0^1 p_{n-1,k-1}(t) dt \right)^{1/2} \leq \frac{C_1}{\delta^2} \sum_{2^i+j \leq r, i,j \geq 0} n^i O(n^{1/2}) O(n^{-(2k+4+r)}/2) = C_2 n^{-(k+2)}. \]

Combining the estimates of \(J_3\) and \(J_4\), we get \(J_1 = c' O(n^{-(k+1)})\). Clearly, \(J_2 = o(n^{-(k+1)})\). Combining the estimates \(J_1\) and \(J_2\), due to the arbitrariness of \(c' > 0\), it follows that \(n^{k+1} \sum c = 0\) as \(n \to \infty\). This completes the proof of the assertion 3.1.

The assertion 3.2 can be proved along similar lines by noting that

\[ M_n((t-x)^i, k+1, x) = O(n^{-(k+2)}), \quad i = 1, 2, 3, \ldots \]

which follows from Lemma 2.4.

Uniformity assertion follows easily from the fact that \(\delta(c)\) in the above proof can be chosen to be independent of \(x \in [a, b]\) and all the other estimates hold uniformly on \([a, b]\).

In the following theorem, we study an error estimate for \(P_n^{(r)}(f, k, x)\).

**Theorem 3.2.** Let \(p \in \mathbb{N}, 1 \leq p \leq 2k+2\) and \(f \in L_B[0,1]\). If \(f^{(p+r)}\) exists and is continuous on \((a-\eta, b+\eta) \subset [0,1], \eta > 0\) then

\[ ||P_n^{(r)}(f, k, .) - f^{(r)}|| \leq \max \left\{ C_1 n^{-p/2} \omega \left( f^{(p+r)}, n^{-1/2} \right), C_2 n^{-(k+1)} \right\}, \quad (3.3) \]

where \(C_1 = C_1(k, p, r), C_2 = C_2(k, p, r, f)\) and \(\omega \left( f^{(p+r)}, \delta \right)\) is the modulus of continuity of \(f^{(p+r)}\) on \((a-\eta, b+\eta)\).

**Proof.** By our hypothesis, we may write for all \(t \in [0,1]\) and \(x \in [a, b]\)

\[ f(t) = \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!} (t-x)^\nu + \frac{f^{(p+r)}(\xi)}{(p+r)!} (t-x)^{p+r} \chi(t) + F(t, x)(1-\chi(t)), \quad (3.4) \]

where \(\chi(t)\) is the characteristic function of \((a-\eta, b+\eta)\), \(\xi\) lies between \(t\) and \(x\) and \(F(t, x)\) is defined as

\[ F(t, x) = f(t) - \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!} (t-x)^\nu, \quad \forall t \in [0,1] \setminus (a-\eta, b+\eta) \text{ and } x \in [a, b]. \]
Now operating by $P_n^{(r)}(.,k,x)$ on both sides of (3.4) and breaking the right hand side into three parts $I_1, I_2$ and $I_3$, say, corresponding to the three terms on the right hand side of (3.4), we get

$$P_n^{(r)}(f,k,x) - f^{(r)}(x) = I_1 + I_2 + I_3.$$

To estimate

$$I_1 = \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!} P_n^{(r)}((t-x)^\nu, k, x),$$

proceeding as in the estimate of $\Sigma_1$ of Theorem 3.1, we obtain

$$I_1 = O(n^{-(k+1)}),$$

uniformly in $x \in [a, b]$. For every $\delta > 0$, we have

$$|f^{(p+r)}(\xi) - f^{(p+r)}(x)| \leq \omega_{f^{(p+r)}}(|\xi-x|) \leq \omega_{f^{(p+r)}}(|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{f^{(p+r)}}(\delta).$$

Consequently,

$$|I_2| \leq \frac{\omega(f^{(p+r)}, \delta)}{(p+r)!} \sum_{j=0}^{k} |C(j,k)| \left[ \sum_{\nu=1}^{d_j n} |p^{(r)}_{d_j n, \nu}(x)| \right] \times \int_0^1 p_{d_j n-1, \nu-1}(t) |t-x|^{p+r}(1 + |t-x|^{d-j}) dt$$

$$= \frac{d_j n!}{(d_j n - r)!} (1-x)^{d_j n-r} \left[ |x|^{p+r} + \delta^{-1} |x|^{p+r+1} \right]$$

$$= I_4 + I_5, \text{ say.}$$

In order to estimate $I_2$, we proceed as follows:

Using Lemma 2.6 and Schwarz inequality for integration and then for summation we have

$$n \sum_{\nu=1}^{n} |p^{(r)}_{n, \nu}(x)| \int_0^1 p_{n-1, \nu-1}(t) |t-x|^s dt$$

$$\leq n \sum_{\nu=1}^{n} \sum_{2j+1 \leq r} \sum_{i, j \geq 0} n^i |\nu - nx|^j \left[ \frac{q_{i,j,r}(x)}{x^r (1-x)^r} p_{n, \nu}(x) \int_0^1 p_{n-1, \nu-1}(t) |t-x|^s dt \right]$$

$$\leq K \sum_{2j+1 \leq r} \sum_{i, j \geq 0} n^i O(n^{(j-s)/2}) = O(n^{(r-s)/2}), \quad (3.5)$$

uniformly in $x \in [a, b]$, where $K = \sup_{2j+1 \leq r} \sup_{x \in [a, b]} |q_{i,j,r}(x)| \frac{1}{x^r (1-x)^r}$. Choosing $\delta = n^{-1/2}$ and using 3.5 we lead to,
\[ |I_4| \leq \frac{\omega \left( f^{(p+r)}, n^{-1/2} \right)}{(p + r)!} \left[ O \left( n^{-p/2} \right) + n^{1/2}O \left( n^{-(p+1)/2} \right) \right] \]
\[ = \omega \left( f^{(p+r)}, n^{-1/2} \right) O \left( n^{-p/2} \right), \text{ uniformly for all } x \in [a, b]. \]

Now, \( I_5 = O \left( n^{-s} \right) \), for any \( s > 0 \), uniformly for all \( x \in [a, b] \). Choosing \( s > k + 1 \), \( I_5 = o \left( n^{-(k+1)} \right) \), uniformly for all \( x \in [a, b] \).

To estimate \( I_3 \), we note that \( t \in [0, 1] \setminus (a - \eta, b + \eta) \), we can choose \( \delta > 0 \) in such a way that \( |t - x| \geq \delta \) for all \( x \in [a, b] \).

Thus, by Lemma 2.6, we obtain

\[ |I_3| \leq \sum_{j=0}^{k} |C(j, k)| \left[ d_n \sum_{\nu=1}^{d_n} |p_{d_n, \nu}(x)| \right. \]
\[ \times \left. \int_{|t-x|\geq \delta} p_{d_n, \nu-1}(t)|F(t, x)| dt + \frac{d_n!}{(d_n - r)!} (1 - x)^n r |F(0, x)| \right] \]

For \( |t - x| \geq \delta \), we can find a constant \( C > 0 \) such that \( |F(t, x)| \leq C \), therefore using 3.5 it easily follows that \( I_3 = O \left( n^{-s} \right) \) for any \( s > 0 \), uniformly on \( [a, b] \).

Choosing \( s > k + 1 \) we obtain \( I_3 = o \left( n^{-(k+1)} \right) \), uniformly on \( [a, b] \). Now combining the estimates of \( I_1, I_2, I_3 \), the required result is immediate.

This completes the proof. \( \square \)

In the following theorem, we study an error estimate for \( P_n^{(r)}(f, k, x) \) in terms of higher order modulus of continuity in simultaneous approximation.

**Theorem 3.3.** Let \( f \in L_B[0, 1] \). If \( f^{(r)} \) exists and is continuous on \( I_1 \), then for sufficiently large \( n \),

\[ \| P_n^{(r)}(f, k, \cdot) - f^{(r)}(\cdot) \|_{C(I_2)} \leq C \left\{ n^{-k} \| f \|_{L_B[0, 1]} + \omega_{2k+2}(f^{(r)}; n^{-1/2}; I_1) \right\}, \]

where \( C \) is independent of \( f \) and \( n \).

**Proof.** We can write

\[ I = \| P_n^{(r)}(f, k, \cdot) - f^{(r)} \|_{C(I_2)} \]
\[ \leq \| P_n^{(r)}(f - f_{n, 2k+2}, k, \cdot) \|_{C(I_2)} + \| P_n^{(r)}(f_{n, 2k+2}, k, \cdot) - f_{n, 2k+2}^{(r)} \|_{C(I_2)} \]
\[ + \| f^{(r)}(x) - f_{n, 2k+2}^{(r)}(x) \|_{C(I_2)} \]
\[ = E_1 + E_2 + E_3, \text{ say.} \]

Since \( f_{n, 2k+2}^{(r)} = (f^{(r)})_{n, 2k+2} \), by property (c) of the Steklov mean we get

\[ E_3 \leq C \omega_{2k+2}(f^{(r)}; \eta, I_1). \]

Next, applying Theorem 3.1 and the interpolation property [7], for each \( m = r, r + 1, \ldots, 2k + 2 + r \), it follows that
\[ E_2 \leq C n^{-(k+1)} \sum_{m=r}^{2k+2+r} \left\| f_{\eta,2k+2}^{(m)} \right\|_{C(I_2)} \]
\[ \leq C n^{-(k+1)} \left( \left\| f_{\eta,2k+2} \right\|_{C(I_2)} + \left\| f_{\eta,2k+2}^{(2k+2+r)} \right\|_{C(I_2)} \right) \]
\[ \leq C n^{-(k+1)} \left( \left\| f_{\eta,2k+2} \right\|_{C(I_2)} + \left\| (f^{(r)})^{2k+2} \right\|_{C(I_2)} \right). \]

Hence, by property (b) and (d) of the Steklov mean, we have
\[ E_2 \leq C n^{-(k+1)} \left\{ \|f\|_{C(I_{11})} + \eta^{-(2k+2)} \omega_{2k+2}(f^{(r)}, \eta, I_1) \right\}. \]

Let \( f - f_{\eta,2k} = F \). By our hypothesis, we can write
\[ F(t) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!} (t-x)^m + \frac{F^{(r)}(x) - F^{(r)}(x)}{r!} \psi(t) \]
\[ + h(t,x)(1-\psi(t)), \]
where \( \xi \) lies between \( t \) and \( x \), and \( \psi \) is the characteristic function of the interval \( I_1 \).

For \( t \in I_1 \) and \( x \in I_2 \), we get
\[ F(t) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!} (t-x)^m + \frac{F^{(r)}(x) - F^{(r)}(x)}{r!} (t-x)^r, \]
and for \( t \in [0,1] \setminus [a_1,b_1], x \in I_2 \) we define
\[ h(t,x) = F(t) - \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!} (t-x)^m. \]

Now,
\[ P_n^{(r)}(F(t),k,x) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!} P_n^{(r)}((t-x)^m,k,x) \]
\[ + P_n^{(r)}(\frac{F^{(r)}(x) - F^{(r)}(x)}{r!} (t-x)^r \psi(t),k,x) \]
\[ + P_n^{(r)}(h(t,x)(1-\psi(t)),k,x) := J_1 + J_2 + J_3, \text{ say}. \]

In order to estimate \( J_1 \), in view of Lemma 2.2 we note that
\[ \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!} P_n^{(r)}((t-x)^m,x) = \frac{F^{(r)}(x)}{r!} P_n^{(r)}(t^r,x) \]
\[ = \frac{F^{(r)}(x)}{r!} \left[ r! \prod_{j=1}^{n} (n+j) \right]. \]

By using Lemma 2.2, we get
\[ J_1 = \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} P_n^{(r)}((t-x)^m, k, x) \]
\[ = \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} \sum_{l=0}^{m} \binom{m}{l} (-x)^{m-l} P_n^{(l)}(t^l, k, x) \]
\[ \to \frac{(n+r-1)!}{n!(n-1)!} \mathcal{F}^{(r)}(x). \]

Hence, for sufficiently large \( n \), we have
\[ |J_1| \leq C \| f^{(r)} - f^{(r)}_{\eta,2k+2} \| \mathcal{C}(t_2) \]

Next, applying Schwarz inequality for integration and then for summation and using Remarks 1-2, we get
\[ J_2 \leq \frac{2}{r!} \| f^{(r)} - f^{(r)}_{\eta,2k+2} \| \mathcal{C}(t_2) \sum_{j=0}^{k} |C(j, k)| \sum_{i,j \geq 0} \frac{(d_j n)^i}{x^r(1-x)^r} \sum_{i,j \geq 0} \frac{d_j n}{x^r(1-x)^r} \]
\[ \times \sum_{\nu=1}^{d_j n} p_{d_j n, \nu}(x) |\nu - d_j n x| \left( \int_0^1 p_{d_j n-1, \nu-1}(t) \psi(t)|t-x|^{r} dt \right) \]
\[ \leq \frac{2}{r!} \| f^{(r)} - f^{(r)}_{\eta,2k+2} \| \mathcal{C}(t_2) \sum_{j=0}^{k} |C(j, k)| \sum_{i,j \geq 0} \frac{(d_j n)^i+1}{x^r(1-x)^r} \times \]
\[ \times \sum_{\nu=1}^{d_j n} p_{d_j n, \nu}(x) |\nu - d_j n x| \left( \int_0^1 p_{d_j n-1, \nu-1}(t) \left( \int_0^1 p_{d_j n-2, \nu-2}(t) dt \right)^{1/2} \right) \times \]
\[ \times \left( \int_0^1 p_{d_j n-1, \nu-1}(t)(t-x)^r dt \right)^{1/2} \]
\[ \leq C \| f^{(r)} - f^{(r)}_{\eta,2k+2} \| \mathcal{C}(t_2) \sum_{j=0}^{k} |C(j, k)| \times \]
\[ \times \sum_{i,j \geq 0} \frac{(d_j n)^i}{x^r(1-x)^r} \left( \sum_{\nu=1}^{d_j n} p_{d_j n, \nu}(x) (\nu - d_j n x)^2 \right)^{1/2} \times \]
\[ \times \left( d_j n \sum_{\nu=1}^{d_j n} p_{d_j n, \nu}(x) \int_0^1 p_{d_j n-1, \nu-1}(t)(t-x)^2 dt \right)^{1/2} \]
\[ \leq C \| f^{(r)} - f^{(r)}_{\eta,2k+2} \| \mathcal{C}(t_2) \sum_{j=0}^{k} |C(j, k)| \sum_{i,j \geq 0} \frac{(d_j n)^i}{x^r(1-x)^r} O(n^{3/2}) O(n^{-r/2}) \]
or

$$J_2 \leq C^\prime \| f^{(r)} - f_{\eta,2k+2}^{(r)} \|_{C(I_2)}.$$ 

Since \( t \in [0,1] \setminus I_1 \), we can choose a \( \delta > 0 \) in such a way that \( |t - x| \geq \delta \) for all \( x \in I_2 \). Thus, by Lemma 2.2, we obtain

$$|J_3| \leq \sum_{j=0}^{k} |C(j,k)| \sum_{2i+j \geq r} (d_j n)^i \frac{q_{i,j,r}(x)}{x^i (1-x)^r} d_j n \sum_{\nu=1}^{d_j n} \| p_{d_j n,\nu}(x) \nu - d_j nx \| x^i \times$$

$$\int_{|t-x| \geq \delta} p_{d_j n-1,\nu-1}(t) |h(t,x)| \| dt + \frac{d_j n!}{(d_j n - r)!} (1-x)^{d_j n-r} |h(0,x)|$$

For \( |t-x| \geq \delta \), we can find a constant \( C > 0 \) such that \( |h(t,x)| \leq C \). Hence, proceeding as a manner similar to the estimate of \( J_2 \), it follows that \( J_3 = O(n^{-s}) \) for any \( s > 0 \).

Combining the estimates of \( J_1 - J_3 \), we obtain

$$E_1 \leq C \| f^{(r)} - f_{\eta,2k+2}^{(r)} \|_{C(I_2)}$$

$$\leq C \omega_{2k+2} f^{(r)}, \eta, f_1 \| \text{(in view of (c) of Steklov mean)}.$$ 

Therefore, with \( \eta = n^{-1/2} \) the theorem follows. \( \square \)

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