COMPACT MULTIPLICATION OPERATORS ON NONLOCALLY
CONVEX WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Abstract. Let V be a system of weights on a completely regular Hausdorff
space and let $B(E)$ be the topological vector space of all continuous linear
operators on a Hausdorff topological vector space $E$. Let $CV^0_0(X,E)$ and
$CV^b(X,E)$ be the nonlocally convex weighted spaces of continuous functions.
In the present paper, we characterize compact multiplication operators $M_{\psi}$
on $CV^0_0(X,E)$ (or $CV^b(X,E)$) induced by the operator-valued mappings
$\psi : X \to B(E)$ (or the vector-valued mappings $\psi : X \to E$, where $E$ is a
topological algebra).

1. Introduction

The theory of multiplication operators has extensively been studied during the
last three decades on different function spaces. Many authors like Abrahamse [1],
Axler [6], Halmos [12], Singh and Kumar [35], Takagi and Yokouchi [45] have stud-
ied these operators on $L^p$–spaces, whereas Arazy [4], Axler [5], Bonet, Domanski
and Lindström [9], Shields and Williams [34], Feldman [10], Ghatage and Sun
[11], Stegenga [42], and Vukotic [46] have explored these operators on spaces of
analytic functions. Also, a study of these operators on weighted spaces of continu-
ous functions has been made by Singh and Manhas [36, 37, 38, 39, 40], Manhas and
Singh [24], Manhas [21, 22, 23], Khan and Thaheem [17, 18], Alsulami and Khan
[2, 3], and Oubbi [29]. In this paper, we have made efforts to characterize compact
multiplication operators on the nonlocally convex weighted spaces of continuous
functions generalizing some of the results of the author [22, 23] and Alsulami and
Khan [2].
2. PRELIMINARIES

Throughout this paper, we shall assume, unless stated otherwise, that $X$ is a completely regular Hausdorff space and $E$ is a non-trivial Hausdorff topological vector space with a base $\mathcal{N}$ of closed balanced shrinkable neighbourhoods of zero. A neighbourhood $G$ of zero in $E$ is called shrinkable [19] if $tG \subseteq \text{int}G$, for $0 \leq t < 1$. It is proved by Klee [19, Theorem 4 and Theorem 5] that every Hausdorff topological vector space has a base of shrinkable neighbourhoods of zero, and also the Minkowski functional $\rho_G$ of any such neighbourhood $G$ is continuous and satisfies

$$\quad \bar{G} = \{ y \in E : \rho_G(y) \leq 1 \}, \quad \text{int}G = \{ y \in E : \rho_G(y) < 1 \}.$$ 

Let $CV(X, E)$ be the vector space of all continuous $E$-valued functions on $X$. Let $V$ be a set of non-negative upper semicontinuous functions on $X$. Then $V$ is said to be directed upward if for given $u, v \in V$ and $\alpha \geq 0$, there exists $w \in V$ such that $\alpha u, \alpha v \leq w$ (pointwise). A directed upward set $V$ is called a system of weights if for each $x \in X$, there exists $v \in V$ such that $v(x) > 0$. Let $U$ and $V$ be two systems of weights on $X$. Then we say that $U \leq V$ if for every $u \in U$, there exists $v \in V$ such that $u \leq v$. Now, for a given system of weights $V$, we define

$$\quad CV_0(X, E) = \{ f \in CV(X, E) : vf \text{ vanishes at infinity on } X \text{ for each } v \in V \},$$

and

$$\quad CV_b(X, E) = \{ f \in CV(X, E) : vf(X) \text{ is bounded in } E \text{ for each } v \in V \}.$$ 

Clearly $CV_0(X, E) \subseteq CV_b(X, E)$. When $E (= \mathbb{R}$ or $\mathbb{C})$, the above spaces are denoted by $CV_0(X)$ and $CV_b(X)$. The weighted topology on $CV_b(X, E)$ (resp. $CV_0(X, E)$) is defined as the linear topology which has a base of neighbourhoods of zero consisting of all sets of the form

$$\quad N(v, G) = \{ f \in CV_b(X, E) : vf(X) \subseteq G \},$$

where $v \in V$ and $G \in \mathcal{N}$.

With this topology, the vector space $CV_b(X, E)$ (resp. $CV_0(X, E)$) is called the weighted space of vector-valued continuous functions which is not necessarily locally convex. For more details on these weighted spaces, we refer to [13, 14, 15, 16, 19, 27]. In case $E$ is a locally convex space, a detailed information can be found in [7, 8, 25, 26, 30, 31, 32, 33, 43, 44]. Let $B(E)$ be the vector space of all continuous linear operators on $E$. We denote by $\mathcal{B}$, the family of all bounded subsets of $E$. For each $B \in \mathcal{B}$ and $G \in \mathcal{N}$, we define the set

$$\quad W(B, G) = \{ T \in B(E) : T(B) \subseteq G \}.$$ 

Then clearly $B(E)$ is a topological vector space with a linear topology which has a base of neighbourhoods of zero consisting of all sets of the form $W(B, G)$. This topology is known as the topology of uniform convergence on bounded subsets of $E$.

By a topological algebra $E$ we mean a topological vector space which is also an algebra such that multiplication in $E$ is separately continuous. Multiplication in $E$ is said to be left (right) hypocontinuous if for each $G \in \mathcal{N}$ and $B \in \mathcal{B}$, there exists $H \in \mathcal{N}$ such that $BH \subseteq (\text{resp. } HB \subseteq G)$. In case $E$ is equipped with both left and right hypocontinuous multiplication, we call $E$ as a topological
algebra with hypocontinuous multiplication. Clearly every topological algebra with joint continuous multiplication is always a topological algebra with hypocontinuous multiplication. For more details on these algebras, we refer to Mallios [20].

For the mapping \( \psi : X \to B(E) \) (or \( \psi : X \to E, \) \( E \) as a topological algebra), we define the linear map \( M_\psi : CV_0(X,E) \to F(X,E) \) by \( M_\psi(f) = \psi f, \) for every \( f \in CV_0(X,E), \) where \( F(X,E) \) denotes the vector space of all functions from \( X \) into \( E \) and the product \( \psi f \) is defined pointwise on \( X \) as \( (\psi f)(x) = \psi_x(f(x)) \) (or \( (\psi f)(x) = \psi(x)(f(x)) \), for every \( x \in X \). In case \( M_\psi \) takes \( CV_0(X,E) \) into itself and is continuous, we call \( M_\psi \), the multiplication operator on \( CV_0(X,E) \) induced by the mapping \( \psi \).

3. **Compact Multiplication Operators**

Throughout this section, we shall assume that for each \( x \in X \), there exists \( f \in CV_0(X) \) such \( f(x) \neq 0 \). In case \( X \) is locally compact Hausdorff space this condition is automatically satisfied.

In order to present the desired results, we need to record some definitions and results as follows.

Let \( T \in B(E) \). Then \( T \) is said to be compact if it maps bounded subsets of \( E \) into relatively compact subsets of \( E \). A completely regular Hausdorff space \( X \) is called a \( K_\mathbb{R} \) \(-\) space if a function \( f : X \to \mathbb{R} \) is continuous if and only if \( f \mid K \) is continuous for each compact subset \( K \) of \( X \). Clearly all locally compact or metrizable spaces are \( K_\mathbb{R} \) \(-\) spaces. A completely regular Hausdorff space \( X \) is said to be a \( V_\mathbb{R} \) \(-\) space with respect to a given system of weights \( V \) on \( X \) if a function \( f : X \to \mathbb{R} \) is necessarily continuous whenever, for each \( v \in V \), the restriction of \( f \) to \( \{x \in X : v(x) \geq 1\} \) is continuous. Also, if \( V_1 \leq V_2 \) for two systems of weights on \( X \), then of course any \( (V_1)_\mathbb{R} \) \(-\) space is again a \( (V_2)_\mathbb{R} \) \(-\) space. For more details on \( V_\mathbb{R} \) \(-\) spaces, we refer to Bierstedt [8].

A subset \( H \subseteq CV_0(X,E) \) is called equicontinuous at \( x_0 \in X \) if for every neighbourhood \( G \) of zero in \( E \), there exists a neighbourhood \( N \) of \( x_0 \) in \( X \) such that \( f(x) - f(x_0) \in G \), for every \( x \in N \) and \( f \in H \). If \( H \) is equicontinuous at every point of \( X \), then we say that \( H \) is equicontinuous on \( X \). Moreover, using nets, we say that a subset \( H \subseteq CV_0(X,E) \) is equicontinuous on \( X \) if and only if for every \( x \in X \) and for every net \( x_\alpha \to x \) in \( X \),

\[
\sup \{ \rho_G(f(x_\alpha) - f(x)) : f \in H \} \to 0, \quad \text{for every} \; G \in \mathcal{N}.
\]

The following generalized Arzela-Ascoli type theorem and related results can be found in Khan and Oubbi [16].

**Theorem 3.1.** Let \( X \) be a completely regular Hausdorff \( V_\mathbb{R} \) \(-\) space and let \( E \) be a quasi-complete Hausdorff topological vector space. Then a subset \( M \subseteq CV_0(X,E) \) is relatively compact if and only if

(i) \( M \) is equicontinuous;

(ii) \( M(x) = \{ f(x) : f \in M \} \) is relatively compact in \( E \), for each \( x \in X \);

(iii) \( vM \) vanishes at infinity on \( X \) for each \( v \in V \) (i.e., for each \( v \in V \) and \( G \in \mathcal{N} \), there exists a compact set \( K \subseteq X \) such that \( v(x)f(x) \in G \), for all \( f \in M \) and \( x \in X \setminus K \)).
Corollary 3.2. Let $X$ be a locally compact Hausdorff space and let $E$ be a quasi-complete Hausdorff topological vector space. Let $V$ be a system of constant weights on $X$. Then a subset $M \subseteq CV_0(X,E)$ is relatively compact if and only if

(i) $M$ is equicontinuous;

(ii) $M(x) = \{f(x) : f \in M\}$ is relatively compact in $E$, for each $x \in X$;

(iii) $M$ uniformly vanishes at infinity on $X$ (i.e., for every $G \in N$, there exists a compact set $K \subseteq X$ such that $f(x) \in G$, for all $f \in M$ and $x \in X \setminus K$).

Remark. Theorem 3.2 and Corollary 3.7 of [24] are proved for a completely regular Hausdorff $V_R$-space $X$. But with slight modification in the proofs both the results are still valid if we take $X$ as a completely regular Hausdorff $V_R$-space.

Now we are ready to present the characterization of compact multiplication operators on $CV_0(X,E)$.

Theorem 3.3. Let $X$ be a completely regular Hausdorff $V_R$-space and let $E$ be a non-zero quasi-complete Hausdorff topological vector space. Let $\psi : X \rightarrow B(E)$ be an operator-valued mapping. Then $M_\psi : CV_0(X,E) \rightarrow CV_0(X,E)$ is a compact multiplication operator if the following conditions are satisfied:

(i) $\psi : X \rightarrow B(E)$ is continuous in the topology of uniform convergence on bounded subsets of $E$;

(ii) for every $v \in V$ and $G \in N$, there exist $u \in V$ and $H \in N$, such that $u(x)y \in H$ implies that $v(x)\psi_x(y) \in G$, for every $x \in X$ and $y \in E$;

(iii) for every $x \in X$, $\psi(x)$ is a compact operator on $E$;

(iv) $\psi : X \rightarrow B(E)$ vanishes at infinity uniformly on $X$, i.e., for each $G \in N$ and $B \in B$, there exists a compact set $K \subseteq X$ such that $\psi_x(B) \subseteq G$, for every $x \in X \setminus K$;

(v) for every bounded set $F \subseteq CV_0(X,E)$, the set $\{\psi_x : f \in F\}$ is equicontinuous for every $x \in X$.

Proof. According to [24, Corollary 3.7] and Remark 1, conditions (i) and (ii) imply that $M_\psi$ is a multiplication operator on $CV_0(X,E)$. Let $S \subseteq CV_0(X,E)$ be a bounded set. To prove that $M_\psi$ is a compact operator, it is enough to show that the set $M_\psi(S)$ satisfies all the conditions of Theorem 1. Fix $x_0 \in X$. We shall verify that the set $M_\psi(S)$ is equicontinuous at $x_0$. Let $G \in \mathcal{N}$. Then there exists $H \in \mathcal{N}$ such that $H + H \subseteq G$. Choose $v \in V$ such that $v(x_0) \geq 1$. Let $F_v = \{x \in X : v(x) > 1\}$. Consider the set $B = \{f(x) : x \in F_v, f \in S\}$. Clearly the set $B$ is bounded in $E$. By condition (i), there exists a neighbourhood $K_1$ of $x_0$ such that $\psi_x - \psi_{x_0} \in W(B,H)$, for every $x \in K_1$. Further, it implies that $\psi_x(f(x) - f(x_0)) \in H$, for every $x \in K_1 \cap F_v$ and $f \in S$. Again, by condition (v), there exists a neighbourhood $K_2$ of $x_0$ such that $\psi_{x_0}(f(x) - f(x_0)) \in H$, for every $x \in K_2$ and $f \in S$. Let $N = K_2 \cap K_1 \cap F_v$. Then for every $x \in N$ and $f \in S$, we have

$$\psi_x(f(x)) - \psi_{x_0}(f(x_0)) = \psi_x(f(x)) - \psi_{x_0}(f(x)) + \psi_{x_0}(f(x)) - \psi_{x_0}(f(x_0)) \in H + H \subseteq G.$$ 

This proves the equicontinuity of the set $M_\psi(S)$ at $x_0$ and hence it is equicontinuous on $X$. This established the condition (i) of Theorem 1. To prove condition (ii) of Theorem 1, we shall show that the set $M_\psi(S)(x_0)$ is relatively compact in $E$ for each $x_0 \in X$. Since the set $B = \{f(x_0) : f \in S\}$ is bounded in $E$ and $\psi_{x_0}$ is compact
operator on $E$, by condition (iii), it follows that the set $\psi_{x_0}(B) = M_\psi(S)(x_0)$ is relatively compact in $E$. Finally we shall establish condition (iii) of Theorem 1 by showing that the set $vM_\psi(S)$ vanishes at infinity on $X$ for each $v \in V$. Fix $v \in V$ and $G \in \mathcal{N}$. Since the set $B = \{v(x)f(x) : x \in X, f \in S\}$ is bounded in $E$, according to Condition (iv), there exists a compact set $K \subseteq X$ such that $\psi_x(B) \subseteq G$, for every $x \in X \setminus K$. That is, $v(x)\psi_x(f(x)) \in G$, for every $x \in X \setminus K$ and $f \in S$. This proves that the set $vM_\psi(S)$ vanishes at infinity on $X$. With this the proof of the theorem is complete.

\begin{theorem} Let $X$ be a completely regular Hausdorff $V_R$-space and let $E$ be a non-zero quasi-complete Hausdorff topological vector space. Let $U$ be a system of constant weights on $X$ such that $\cup \subseteq V$. Let $\psi : X \to B(E)$ be an operator-valued mapping. Then conditions (i) through (v) in Theorem 3 are necessary and sufficient for $M_\psi$ to be a compact multiplication operator on $CV_0(X,E)$.
\end{theorem}

\textbf{Proof.} We suppose that $M_\psi$ is a compact multiplication operator on $CV_0(X,E)$.

To prove condition (i), we fix $x_0 \in X, B \in \mathcal{B}$ and $G \in \mathcal{N}$. Let $v \in V$ and $f \in CV_0(X)$ such that $v(x_0) \geq 1$ and $f(x_0) = 1$. Let $K_1 = \{x \in X : v(x)f(x) \geq 1\}$. Then $K_1$ is a compact subset of $X$ such that $x_0 \in K_1$. According to [26, Lemma 2, p. 69], there exists $h \in CV_0(X)$ such that $h(K_1) = 1$. For each $y \in B$, we define the function $g_y : X \to E$ as $g_y(x) = h(x)y$, for every $x \in X$. If we put $F = \{g_y : y \in B\}$, then $F$ is clearly bounded in $CV_0(X,E)$ and hence the set $M_\psi(F)$ is relatively compact in $CV_0(X,E)$. According to Theorem 1, the set $M_\psi(F)$ is equicontinuous at $x_0$. This means that there exits a neighbourhood $K_2$ of $x_0$ such that $\psi_x(g_y(x)) - \psi_{x_0}(g_y(x_0)) \in G$, for every $x \in K_2$ and $y \in B$. Let $K = K_1 \cap K_2$.

Then we have $\psi_x(y) - \psi_{x_0}(y) \in G$, for every $x \in K$ and $y \in B$. This shows that $\psi_x - \psi_{x_0} \in W(B,G)$, for every $x \in K$. This proves that $\psi : X \to B(E)$ is continuous at $x_0$ and hence on $X$. In view of Remark 1, the proof of condition (ii) follows from Corollary 3.7 of [24]. To establish condition (iii), let $x_0 \in X$.

We select $f \in CV_0(X)$ such that $f(x_0) = 1$. Let $B \in \mathcal{B}$. Then for each $y \in B$, we define the function $h_y : X \to E$ as $h_y(x) = f(x)y$, for every $x \in X$. Clearly the set $S = \{h_y : y \in B\}$ is bounded in $CV_0(X,E)$ and hence the set $M_\psi(S)$ is relatively compact in $CV_0(X,E)$. Again, according to Theorem 1, it follows that the set $M_\psi(S)(x_0) = \{\psi_{x_0}(y) : y \in B\}$ is relatively compact in $E$. This proves that $\psi_{x_0}$ is a compact operator on $E$.

Now, to prove condition (iv), we suppose that $\psi : X \to B(E)$ does not vanishes at infinity on $X$. This implies that there exist $G \in \mathcal{N}$ and $B \in \mathcal{B}$ such that for every compact set $K \subseteq X$, there exists $y_k \in X\setminus K$ for which $\psi_{x_k}(B) \subseteq G$. Further, it implies that there exists $y_k \in B$ such that $\psi_{x_k}(y_k) \notin G$. According to [41, Lemma 3.1], there exists an open neighbourhood $N_k$ of $x_k$ such that each $v \in V$ is bounded on $N_k$. Let $O_k = N_k \cap X\setminus K$. Then $O_k$ is an open neighbourhood of $x_k$ for each compact set $K \subseteq X$. Further, according to [26, Lemma 2, p. 69], there exists $f_k \in CV_0(X)$ such that $0 \leq f_k \leq 1$, $f_k(x_k) = 1$ and $f_k(X\setminus O_k) = 0$. For each compact $K \subseteq X$, we define the function $h_k : X \to E$ as $h_k(x) = f_k(x)y_k$, for every $x \in X$. Clearly the set $M = \{h_k : K \subseteq X, K \text{ compact subset}\}$ is bounded in $CV_0(X,E)$ and hence the set $M_\psi(M)$ is relatively compact in $CV_0(X,E)$. Since $U \subseteq V$, we can select $v \in V$ such that $v(x) \geq 1$, for every $x \in X$. Again, Theorem 1 implies that the set $vM_\psi(M)$ vanishes at infinity on $X$. This implies that there exists a compact set $K_0 \subseteq X$ such that $v(x)\psi_x(h_k(x)) \in G$, for all $h_k \in M$ and for every $x \in X\setminus K_0$. Since $v(x) \geq 1$, for all $x$, it follows that $\psi_x(f_k(x)y_k) \in G$, for every $x \in X\setminus K_0$.
For \( x = x_{k_0} \), we have \( \psi_{x_{k_0}} (y_{k_0}) \in G \), which is a contradiction. This proves that 
\( \psi : X \to B(E) \) vanishes at infinity on \( X \).
Finally, we shall prove condition (v). Let \( F \subseteq CV_0(X,E) \) be a bounded set.
Fix \( x_0 \in X \) and \( G \in \mathcal{N} \). Then there exists \( H \in \mathcal{N} \) such that 
\( H + H \subseteq G \). Clearly the set \( B = \{ f(x) : x \in X, f \in F \} \) is bounded in \( E \).
Since \( \psi : X \to B(E) \) is continuous at \( x_0 \), there exists a neighbourhood \( N_1 \) of \( x_0 \) in \( X \) such that \( \psi_x (f(x)) - \psi_{x_0} (f(x)) \in H \), for every \( x \in N_1 \) and \( f \in F \). Again, since 
the set \( M_\psi (F) \) is relatively compact in \( CV_0(X,E) \), according to Theorem 1, the set \( M_\psi (F) \) is 
equicontinuous at \( x_0 \). This implies that there exists a neighbourhood \( N_2 \) of \( x_0 \) in \( X \) such that 
\( \psi_x (f(x)) - \psi_{x_0} (f(x_0)) \in H \), for every \( x \in N_2 \) and \( f \in F \). Let \( N = N_1 \cap N_2 \). Then for every \( x \in N \) and \( f \in F \), we have 
\( \psi_{x_0} (f(x) - f(x_0)) = \psi_{x_0} (f(x)) - \psi_{x_0} (f(x_0)) \in H + H \subseteq G \).
This proves condition (v). This completes the proof of the theorem as the sufficient
part is already proved in Theorem 3. □

**Theorem 3.5.** Let \( X \) be a completely regular Hausdorff \( V_R \) - space and let \( E \) be a quasi-complete Hausdorff topological algebra with hypocontinuous multiplication containing the unit element \( e \). Let \( U \) be a system of constant weights on \( X \) such that \( U \leq V \). Then the vector-valued mapping \( \psi : X \to E \) induces a compact
multiplication operator \( M_\psi \) on \( CV_0(X,E) \) if and only if

(i) \( \psi : X \to E \) is continuous;
(ii) for every \( v \in V \) and \( G \in \mathcal{N} \), there exist \( u \in V \) and \( H \in \mathcal{N} \) such that 
\( u(x) y \in H \) implies that \( v(x) \psi(x) y \in G \), for every \( x \in X \) and \( y \in E \);
(iii) for every \( x \in X \), the operator \( L_{\psi(x)} : E \to E \), defined by \( L_{\psi(x)}(y) = \psi(x)y \),
for every \( y \in E \), is compact;
(iv) \( \psi : X \to E \) vanishes at infinity on \( X \);
(v) for every bounded set \( F \subseteq CV_0(X,E) \), the set \( \{ L_{\psi(x)}af : f \in F \} \) is equicontinuous for every \( x \in X \).

**Proof.** In [24, Theorem 3.2], Manhas and Singh have characterized the weighted composition operators \( W_{\psi,\phi} \) on \( CV_0(X,E) \) induced by the mappings \( \phi : X \to X \) and \( \psi : X \to E \). If we take \( \phi : X \to X \) as the identity map, then Theorem 3.2 of 
Manhas and Singh [24] and Remark 1 implies that \( M_\psi \) is a multiplication operator on \( CV_0(X,E) \) if and only if condition (i)-(ii) of Theorem 5 hold. Also, using
similar arguments of Theorem 4, it can be shown that \( M_\psi \) is a compact operator on \( CV_0(X,E) \). □

**Corollary 3.6.** Let \( X \) be a locally compact Hausdorff space and let \( E \) be a non-zero
quasi-complete Hausdorff topological vector space. Let \( V \) be a system of constant
weights on \( X \). Let \( \psi : X \to B(E) \) be an operator-valued mapping. Then \( M_\psi : CV_0(X,E) \to CV_0(X,E) \) is a compact multiplication operator if and only if the
following conditions are satisfied:

(i) \( \psi : X \to B(E) \) is continuous in the topology of uniform convergence on
bounded subsets of \( E \);
(ii) for every \( G \in \mathcal{N} \), there exists \( H \in \mathcal{N} \), such that \( y \in H \) implies that \( \psi_x (y) \in G \), for every \( x \in X \) and \( y \in E \);
(iii) for every \( x \in X \), \( \psi(x) \) is a compact operator on \( E \);
Remark. (i) In case $E$ is a quasi-complete locally convex Hausdorff space and $X$ is a locally compact Hausdorff space, Theorem 4 reduces to [23, Theorem 3.4].

(ii) In Corollary 6, if $E$ is a quasi-complete locally convex Hausdorff space, it reduces to Theorem 2.4 of Manhas [22].

(iii) If $X$ is as $V_{\mathbb{R}}$-space without isolated points, then it is proved in [2, Corollary 4] that there is no non-zero compact multiplication operator $M_\psi$ on $CV_0(X,E)$. But if $X$ is a $V_{\mathbb{R}}$-space with isolated points, then Theorem 4 provide (e.g. see Example 1 below) non-zero compact multiplication operators $M_\psi$ on some weighted spaces $CV_0(X,E)$ whereas it is not the case with some of the $L^p$-spaces and spaces of analytic functions. In [35], Singh and Kumar have shown that the zero operator is the only compact multiplication operator on $L^p$-spaces (with non-atomic measure). In [9], Bonet Domanski and Lindstrom have shown that there is no non-zero compact multiplication operator on Weighted Banach Spaces of analytic functions. Also, recently, Ohno and Zhao [28] have proved that the zero operator is the only compact multiplication operator on Bloch Spaces.

Example 3.1. Let $X = \mathbb{Z}$, the set of integers with the discrete topology and let $V = K^{+}(\mathbb{Z})$, the set of positive constant functions on $\mathbb{Z}$. Let $E = C_b(\mathbb{R})$ be the Banach space of bounded continuous complex valued functions on $\mathbb{R}$. For each $t \in \mathbb{Z}$, we define an operator $A_t : C_b(\mathbb{R}) \rightarrow C_b(\mathbb{R})$ as $A_t f(s) = f(t+s)$, for every $f \in C_b(\mathbb{R})$ and for every $s \in \mathbb{R}$. Clearly, for each $t \in \mathbb{Z}$, $A_t$ is a compact operator. Let $\psi : \mathbb{Z} \rightarrow B(E)$ be defined as $\psi(t) = e^{-|t|}A_t$, for $t \in \mathbb{Z}$. Then all the conditions of Corollary 6 are satisfied by the mapping $\psi$ and hence $M_\psi$ is a compact multiplication operator on $C_0(\mathbb{Z},E)$. In case we take $E = C(\mathbb{R})$ with compact-open topology, then the mapping $\psi : \mathbb{Z} \rightarrow B(E)$ defined as above does not induce the compact multiplication operator $M_\psi$ on $C_0(\mathbb{Z},E)$. But, if $E = C(\mathbb{R})$ with compact-open topology and we define $\psi : \mathbb{Z} \rightarrow B(E)$ as $\psi(t) = e^{-|t|}A_{t_0}$, for every $t \in \mathbb{Z}$, where $A_{t_0}$ is a fixed compact operator on $C(\mathbb{R})$ defined as $A_{t_0}f(s) = f(t_0)$, for every $f \in C(\mathbb{R})$ and for every $s \in \mathbb{R}$, then it turns out that $M_\psi$ is a compact multiplication operator on $C_0(\mathbb{Z},E)$.

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References


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