ON THE CLASS OF $n$-POWER QUASI-NORMAL OPERATORS ON HILBERT SPACE

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Abstract. Let $T$ be a bounded linear operator on a complex Hilbert space $H$. In this paper we investigate some properties of the class of $n$-power quasi-normal operators, denoted $[nQN]$, satisfying $T^n|T|^2 - |T|^2T^n = 0$ and some relations between $n$-normal operators and $n$-quasinormal operators.

1. Introduction and Terminologies

A bounded linear operator on a complex Hilbert space, is quasi-normal if $T$ and $T^*T$ commute. The class of quasi-normal operators was first introduced and studied by A. Brown [5] in 1953. From the definition, it is easily seen that this class contains normal operators and isometries. In [9] the author introduce the class of $n$-power normal operators as a generalization of the class of normal operators and study sum properties of such class for different values of the parameter $n$. In particular for $n = 2$ and $n = 3$ (see for instance [9,10]). In this paper, we study the bounded linear transformations $T$ of complex Hilbert space $H$ that satisfy an identity of the form

$$T^nT^* = T^*TT^n,$$

(1.1)

for some integer $n$. Operators $T$ satisfying (1.1) are said to be $n$-power quasi-normal.

Let $\mathcal{L}(H) = \mathcal{L}(H,H)$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $H$. For $T \in \mathcal{L}(H)$, we use symbols $R(T)$, $N(T)$ and $T^*$ the range, the kernel and the adjoint of $T$ respectively.

Let $W(T) = \{ \langle Tx | x \rangle : x \in H, \|x\| = 1 \}$ the numerical range of $T$. A subspace $M \subset H$ is said to be invariant for an operator $T \in \mathcal{L}(H)$ if $TM \subset M$, and in this situation we denote by $T|M$ the restriction of $T$ to $M$. Let $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$, respectively denote the spectrum, the approximate point spectrum and point spectrum of the operator $T$.

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For any arbitrary operator $T \in \mathcal{L}(H)$, $|T| = (T^*T)^{\frac{1}{2}}$ and

$$[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$$

(the self-commutator of $T$).

An operator $T$ is normal if $T^*T = TT^*$, positive-normal (posinormal) if there exists a positive operator $P \in \mathcal{L}(H)$ such that $TT^* = T^*PT$, hyponormal if $[T^*, T]$ is nonnegative (i.e. $|T^*|^2 \leq |T|^2$, equivalently $\|T^*x\| \leq \|Tx\|$ for all $x \in H$), quasi-hyponormal if $T^*[T^*, T]T$ is nonnegative, paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for all $x \in H$, $n$-isometry if

$$T^*T^n - \binom{n}{1}T^{n-1}T + \binom{n}{2}T^{n-2}T^2 - \cdots + (-1)^n I = 0,$$

$m$-hyponormal if there exists a positive number $m$, such that

$$m^2(T - \lambda I)^*(T - \lambda I) - (T - \lambda I)(T - \lambda I)^* \leq 0; \text{ for all } \lambda \in \mathbb{C}.$$

Let $[N]; [QN]; [H]$; and $(m-H)$ denote the classes constituting of normal, quasi-normal, hyponormal, and $m$-hyponormal operators. Then

$$[N] \subset [QN] \subset [H] \subset [m-H].$$

For more details see [1, 2, 3, 11, 14, 15].

**Definition 1.1.** ([7]) An operator $T \in \mathcal{L}(H)$ is called $(\alpha, \beta)$-normal ($0 \leq \alpha \leq 1 \leq \beta$) if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

or equivalently

$$\alpha \|Tx\| \leq \|T^*x\| \leq \beta \|Tx\| \text{ for all } x \in H.$$

**Definition 1.2.** ([9]) Let $T \in \mathcal{L}(H)$. $T$ is said $n$-power normal operator for a positive integer $n$ if

$$T^nT^* = T^*T^n.$$

The class of all $n$-normal operators is denoted by $[nN]$.

**Proposition 1.3.** ([9]) Let $T \in \mathcal{L}(H)$, then $T$ is of class $[nN]$ if and only if $T^n$ is normal for any positive integer $n$.

**Remark.** $T$ is $n$-power normal if and only if $T^n$ is $(1,1)$-normal.

The outline of the paper is as follows: Introduction and terminologies are described in the first section. In the second section we introduce the class of $n$-power quasi-normal operators in Hilbert spaces and we develop some basic properties of this class. In section three we investigate some properties of a class of operators denoted by $([\mathbb{Z}^n])$ contained the class $[nQN]$.

2. **BASIC PROPERTIES OF THE CLASS $[nQN]$**

In this section, we will study some property which are applied for the $n$-power quasi-normal operators.

**Definition 2.1.** For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(H)$ is said to be $n$-power quasi-normal operator if

$$T^nT^* = T^*T^{n+1}.$$
We denote the set of $n$-power quasi-normal operators by $[nQN]$. It is obvious that the class of all $n$-power quasi-normal operators properly contained classes of $n$-normal operators and quasi-normal operators, i.e., the following inclusions holds 

$$ [nN] \subset [nQN] \quad \text{and} \quad [QN] \subset [nQN]. $$

**Remark.**

(1) A 1-power quasi-normal operator is quasi-normal.

(2) Every quasi-normal operator is $n$-power quasi-normal for each $n$.

(3) It is clear that a $n$-power normal operator is also $n$-power quasi-normal.

That the converse need not hold can be seen by choosing $T$ to be the unilateral shift, that is, if $H = l^2$, the matrix $T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. It is easily verified that $T^2T^* - T^*T^2 \neq 0$ and $(T^2T^* - T^*T^2)T = 0$. So that $T$ is not 2-power normal but is a 2-power quasi-normal.

**Remark.** An operator $T$ is $n$-power quasi-normal if and only if $[T^n, T^*T] = [T^n, T^*]T = 0$.

**Remark.** An operator $T$ is $n$-power quasi-normal if and only if $T^n|T|^2 = |T|^2T^n$.

First we record some elementary properties of $[nQN]$:

**Theorem 2.2.** If $T \in [nQN]$, then

(1) $T$ is of class $[2nQN]$.

(2) If $T$ has a dense range in $H$, $T$ is of class $[nN]$. In particular, if $T$ is invertible, then $T^{-1}$ is of class $[nQN]$.

(3) If $T$ and $S$ are of class $[nQN]$ such that $[T, S] = [T, S^*] = 0$, then $TS$ is of class $[nQN]$.

(4) If $S$ and $T$ are of class $[nQN]$ such that $ST = TS = T^*S = ST^* = 0$, then $S + T$ is of class $[nQN]$.

**Proof.**

(1) Since $T$ is of $[nQN]$, then

$$ T^nT^*T = T^*TT^n. \quad (2.1) $$

Multiplying (2.1) to the left by $T^n$, we obtain

$$ T^{2n}T^*T = T^*TT^{2n}. $$

Thus $T$ is of class $[2nQN]$. 

(2) Since $T$ is of class $[nQN]$, we have for $y \in R(T)$, $y = Tx$, $x \in H$, and

$$ \|(T^*T^n - T^nT^*)y\| \leq \|(T^nT^* - T^*T^n)Tx\| = \|(T^{n+1}T^* - T^nT^{n+1})x\| = 0. $$

Thus, $T$ is $n$-power normal on $R(T)$ and hence $T$ is of class $[nN]$. In case $T$ invertible, then it is an invertible operator of class $[nN]$ and so

$$ T^nT^* = T^*T^n. $$

This in turn shows that

$$ T^{-n}(T^*T^{-1})^{-1} = [(TT^*)^{-n}]^{-1} = [T^{n+1}T^*]^{-1} = [T^{-1}T^{-1}]T^{-n}, $$
which prove the result.

\[(3)\]
\[(TS)^n(TS)^*TS = T^nS^nT^*S^*TS = T^nT^*TS^nS^*S\]
\[= T^*T^{n+1}S^*S^{n+1} = (TS)^*(TS)^{n+1}.\]

Hence, \(TS\) is of class \([nQN]\).

\[(4)\]
\[(T + S)^n(T + S)^*(T + S) = (T^n + S^n)(T^*T + S^*S)\]
\[= T^nT^*T + S^nS^*S\]
\[= T^*T^{n+1} + S^*S^{n+1}\]
\[= (T + S)^*(T + S)^{n+1}.\]

Which implies that \(T + S\) is of class \([nQN]\).

**Proposition 2.3.** If \(T\) is of class \([nQN]\) such that \(T\) is a partial isometry, then \(T\) is of class \([(n+1)QN]\).

**Proof.** Since \(T\) is a partial isometry, therefore
\[TT^*T = T \quad [4, p.153]. \quad (2.2)\]
Multiplying (2.2) to the left by \(T^*T^{n+1}\) and using the fact that \(T\) is of class \([nNQ]\), we get
\[T^*T^{n+2} = T^*T^{n+2}T^*\]
\[= T^nT^*T.TT^*T\]
\[= T^{n+1}T^*T,\]
which implies that \(T\) is of class \([(n+1)QN]\).

The following examples show that the two classes \([2NQ]\) and \([3NQ]\) are not the same.

**Example 2.4.** Let \(H = \mathbb{C}^3\) and let \(T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3).\) Then by simple calculations we see that \(T\) is not of class \([3QN]\) but of class \([2QN]\).

**Example 2.5.** Let \(H = \mathbb{C}^3\) and let \(S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3).\) Then by simple calculations we see that \(S\) is not of class \([2QN]\) but of class \([3QN]\).

**Proposition 2.6.** Let \(T \in \mathcal{L}(H)\) such that \(T\) is of class \([2QN]\) \(\cap [3QN]\), then \(T\) is of class \([nQN]\) for all positive integer \(n \geq 4\).

**Proof.** We proof the assertion by using the mathematical induction. For \(n = 4\) it is a consequence of Theorem 2.2. 1.

We prove this for \(n = 5\). Since \(T \in [2QN]\),
\[T^2T^*T = T^*T^3, \quad (2.3)\]

multiplying (2.3) to the left by \(T^3\) we get
\[T^5T^*T = T^3T^*T^3.\]
Thus we have
\[ T^5T^*T = T^3T^*T^3 = T^2T^*T^2 = T^*T^6. \]

Now assume that the result is true for \( n \geq 5 \) i.e.
\[ T^nT^*T = T^*TT^n, \]
then
\[ T^{n+1}T^*T = TT^*T^{n+1} = TT^*T^3T^{n-2} = T^3T^*TT^{n-2} = T^*T^4T^{(n-2)} = T^*T^{n+2}. \]

Thus \( T \) is of class \([n+1]QN\).

**Proposition 2.7.** If \( T \) is of class \([nQN]\) such that \( N(T^*) \subset N(T) \), then \( T \) is of class \([nN]\).

**Proof.** In view of the inclusion \( N(T^*) \subset N(T) \), it is not difficult to verify the normality of \( T^n \).

Next couple of results shows that \([nQN]\) is not translation invariant

**Theorem 2.8.** If \( T \) and \( T-I \) are of class \([2QN]\), then \( T \) is normal.

**Proof.** First we see that the condition on \( T-I \) implies
\[ T^2(T^*T) - T^2T^* - 2T(T^*T) + 2TT^* = (T^*T)T^2 - T^*T^2 - 2(T^*T)T + 2T^*T. \]

Since \( T \) is of class \([2QN]\), we have
\[ -T^2T^* - 2T(T^*T) + 2TT^* = -T^*T^2 - 2(T^*T)T + 2T^*T, \]
or
\[ -TT^* - 2(T^*T)T^* + 2TT^* = -T^{*2}T - 2T^*(T^*T) + 2T^*T \]  \hspace{1cm} (2.4)

We first show that (2.4) implies
\[ N(T^*) \subset N(T) \]  \hspace{1cm} (2.5)

Suppose \( T^*x = 0 \). From (2.4), we get
\[ -3T^{*2}Tx + 2T^*Tx = 0. \]  \hspace{1cm} (2.6)

Then
\[ -3T^{*3}Tx + 2T^{*2}Tx = 0. \]

Therefore, as \( T \) is of class \([2QN]\),
\[ -3T^{*2}TT^*x + 2T^{*2}Tx = 0 \]
and hence
\[ 2T^{*2}Tx = 0. \]

Consequently, (2.6) gives \( 2T^*Tx = 0 \) or \( Tx = 0 \). This proves (2.5). As observe in Proposition 2.7 and Proposition 1.3 \( T^2 \) is normal. This along with (2.4) gives
\[ -(T^*T) + TT^* = -(T^*T)T + T^*T, \]
or
\[ T^* (T^* T - TT^*) = T^* T - TT^*. \] (2.7)

If \( N(T^* - I) = \{0\} \), then (2.7) implies \( T \) is normal.

Now assume that \( N(T^* - I) \) is non trivial. Let \( T^* x = x \). Then (2.6) gives
\[ T^{*2} T x - T^* T x = T^* T x - T x. \]

Since \( T^{*2} T = TT^* \), we have
\[ T^* T x = T x. \]

Therefore
\[ ||T x||^2 = < T^* T x \mid x >= < T x \mid x >= < x \mid T^* x >= ||x||^2. \]

Hence
\[
||T x - x||^2 = ||T x||^2 + ||x||^2 - 2 Re < T x \mid x > \\
= ||T x||^2 - ||x||^2 \\
= 0.
\]

Or \( T x = x \). Thus \( N(T^* - I) \subset N(T - I) \). This along with (2.7), yields
\[ T(T^* T - TT^*) = T^* T - TT^* \]
and so
\[ T(T^* T - TT^*) T = (T^* T - TT^*) \]
or
\[ TT^* T^2 - T^2 T^* T = T^* T^2 - TT^* T. \]

Since \( T^2 T^* = T^* T^2 \) and \( T^3 T^* = T^* T^3 \) we deduce that \( T^* T^2 = TT^* T \). Thus \( T \) is quasinormal. From (2.5), the normality of \( T \) follows.

In attempt to extend the above result for operators of class \([nQN]\), we prove

**Theorem 2.9.** If \( T \) is of class \([2QN] \cap [3QN]\) such that \( T - I \) is of class \([nQN]\), then \( T \) is normal.

**Proof.** Since \( T - I \) is of class \([nQN]\), we have
\[
\sum_{k=1}^{n} a_k T^k T^* T - \sum_{k=1}^{n} a_k T^k T^* = T^* T \sum_{k=1}^{n} a_k T^k - T^* \sum_{k=1}^{n} a_k T^k, \quad a_k = (-1)^{n-k}{n \choose k}.
\]

Under the condition on \( T \), we have by Proposition 2.6
\[
a_1 T (T^* T) - \left( \sum_{k=1}^{n} a_k T^k \right) T^* = a_1 (T^* T) T - T^* \left( \sum_{k=1}^{n} a_k T^k \right)
\]
or
\[
a_1 (T^* T) T^* - T \left( \sum_{k=1}^{n} a_k T^* T^k \right) = a_1 T^* (T^* T) - \left( \sum_{k=1}^{n} a_k T^* T^k \right) T. \] (2.8)

(2.8) implies that \( N(T^*) \subset N(T) \). In fact, let \( T^* x = 0 \). From (2.8), we have
\[
a_1 T^{*2} T x - \left( \sum_{k=1}^{n} a_k T^* T^k \right) T x = 0.
\]

\( T \) is of class \([2QN]\) and of class \([3QN]\), we deduce that
\[
a_1 T^{*2} T x - a_1 T^* T x - a_2 T^{*2} T x = 0 \] (2.9)
and hence
\[ a_1 T^{*3}Tx - a_1 T^{*2}T - a_2 T^{*3}Tx = 0 \]
Hence
\[ a_1 T^{*2}Tx. \]
Consequently (2.9) gives \( T^*Tx = 0 \), which implies that \( Tx = 0 \).
It follows by Proposition 2.7 that \( T^k \) is normal for \( k = 2, 3, ..., n \) and hence
\( T(T^*T) - TT^* = (T^*T)T - T^*T \)
or
\( T^*(TT^* - T^*T) = TT^* - T^*T. \)
Hence,
\( (T^* - I)(TT^* - T^*T) = 0. \)
A similar argument given in as in the proof of Theorem 2.8 gives the desired result.

**Theorem 2.10.** If \( T \) and \( T^* \) are of class \( [nQN] \), then \( T^n \) is normal.

First we establish

**Lemma 2.11.** If \( T \) is of class \( [nQN] \), then \( N(T^n) \subset N(T^{*n}) \) for \( n \geq 2 \).

**Proof.** Suppose \( T^nx = 0 \). Then
\[ T^n(T^*T)T^{n-1}x = 0. \]
By hypotheses,
\[ T^*TT^nT^{n-1}x = 0, \]
which implies
\[ TT^nT^{n-1}x = 0. \]
Hence
\[ T^nT^{n-1}x = 0. \]
Under the condition on \( T \), we have
\[ T^*TT^nT^{n-2}x = 0 \]
Hence
\[ T^nT^{n-2}x = 0. \]
By repeating this process we can find
\[ T^nT^{n-1}x = 0. \]

**Proof of Theorem 2.10.** By hypotheses and Lemma 2.11
\[ N(T^{*n}) = N(T^n). \]
Since \( T \) is of \( [nQN] \), \( [T^nT^* - T^*T^n]T^n = 0 \), i.e. \( [T^nT^* - T^*T^n] = 0 \) on \( clR(T) \).
Also the fact that \( N(T^*) \) is a subset of \( N(T^n) \) gives \( [T^nT^* - T^*T^n] = 0 \) on \( N(T^*) \).
Hence the result follows.

**Theorem 2.12.** If \( T \) and \( T^2 \) are of class \( [2QN] \), and \( T \) is of class \( [3QN] \), then \( T^2 \) is quasinormal.
**Proof.** The condition that $T^2$ is of class $[2QN]$ gives

$$T^*4(T^*2T^2) = (T^*2T^2)T^*4$$

Implies

$$T^*5(T^*T)T = (T^*2T^2)T^*4$$

Since $T$ if of class $[3QN]$, we have

$$T^*2(T^*T)T^*3T = (T^*2T^2)T^*4$$

And hence

$$T^*2(T^*T)^2T^*2 = (T^*2T^2)^2T^*4 \quad [T \text{ is of class } [2QN]].$$

Implies

$$(T^*T)^2T^*4 = (T^*2T^2)T^*4 \quad [T \text{ is of class } [2QN]]$$

or

$$T^4((T^*T)^2 - T^*2T^2) = 0.$$  

By Lemma 2.11,

$$T^*2T^2((T^*T)^2 - T^*2T^2) = 0$$

or

$$T^2[(T^*T)^2 - T^*2T^2)] = 0. \tag{2.10}$$

Hence

$$T^*2[((T^*T)^2 - T^*2T^2)] = 0, \quad [N(T^2) \text{ is a subset of } N(T^*2)].$$

Or

$$[((T^*T)^2 - T^*2T^2)]T^2 = 0. \tag{2.11}$$

Since $T$ is of class $[2QN]$, $T^2$ commutes with $(T^*T)^2$. Hence from (2.10) and (2.11), we get the desired conclusion.

**Theorem 2.13.** If $T$ and $T^2$ are of class $[2QN]$ and $N(T) \subset N(T^*)$, then $T^2$ is quasinormal.

**Proof.** By the condition that $T^2$ is of class $[2QN]$, we have

$$(T^*2T^2)^*4 = T^*4(T^*2T^2)$$

$$= T^*4(T^*2T^2)$$

$$= T^*4(T^*T)T^*2$$

$$= T^*(T^*T)T^*4T \quad [T \text{ is of class } [2QN]]$$

Thus we have

$$\{(T^*2T^2)^*4 - (T^*(T^*T))^2T^*2\} = 0$$

or

$$T^2\{T^2(T^*2T^2) - (T^*(T^*T))^2\} = 0.$$ 

Then under the kernel condition

$$T\{T^2(T^*2T^2) - (T^*(T^*T))^2\} = 0$$

or

$$\{(T^*2T^2)^*4 - (T^*(T^*T))^2\} = 0 \quad \text{for } x \in clR(T^*).$$

Since $N(T) \subset N(T^*)$,

$$\{(T^*2T^2)^*4 - (T^*(T^*T))^2\}y = 0 \quad \text{for } y \in N(T).$$
Thus
\[
\{(T^*T^2)T^*T^2 - [T^*(T^*T)]^2\} = 0
\]
or
\[
T^2(T^*T^2) = [(T^*T)T]^2
= T^*T^2 T^*T^2
= T^*T^2(T^*T)T
= T^*(T^*T)T^3
\text{[Tis of class [2QN]}
= (T^*T^2)T^2.
\]
This proves the result.

**Theorem 2.14.** Let \( T \) be an operator of class \([2QN]\) with polar decomposition \( T = U|T| \). If \( N(T^*) \subset N(T) \), then the operator \( S \) with polar decomposition \( U^2|T| \) is normal.

**Proof.** It follows by Proposition 2.7 that \( T^2 \) is normal and \( N(T^*) = N(T^+2) \) and by Lemma 2.11 we have
\[
N(T) = N(T^*). \tag{2.12}
\]
As a consequence, \( U \) turns out to be normal and it is easy to verify that
\[
|T|U|T|^2U^*|T| = |T|U^*|T|^2U|T|.
\]
Since
\[
N(|T|) = N(U) = N(U^*),
\]
\[
|T|U|T|^2U^* = |T|U^*|T|^2U
\]
and hence
\[
U|T|^2U^* = U^*|T|^2U.
\]
Again by the normality of \( U \), we have
\[
U|T|U^* = U^*|T|U \tag{2.13}
\]
Also \( U^+2U^2 = U^*U \), showing \( U^2 \) to be normal partial isometry with \( N(U^2) = N(|T|) \). Thus \( U^2|T| \) is the polar decomposition Note that (2.13) the normality shows that \( U^2 \) and \( |T| \) are commuting. Consequently
\[
(U^2|T|)^*(U^2|T|) = |T|U^*U^2|T|
= |T|U^2U^*2|T|
= (U^2|T|)(U^2|T|)^*.
\]
This completes the proof.

**Corollary 2.15.** If \( T \) is of class \([2QN]\) and \( 0 \notin W(T) \), then \( T \) is normal

**Proof.** Since \( 0 \notin W(T) \) gives \( N(T) = N(T^*) = \{0\} \) and so by our Proposition 2.7, \( T^2 \) is normal. Then \( [T^*T, TT^*] = 0 \). Now the conclusion follows form [8].

**Theorem 2.16.** Let \( T \) is of class \([2QN]\) such that \([T^*T, TT^*] = 0\). Then \( T^2 \) is quasinormal.
Proof.

\[(T^{*2}T^2)T^2 = T^*(T^*T)T^3\]
\[= T^*T^2T^*T^2\]
\[= (T^*T)(TT^*)T^2\]
\[= (TT^*)(T^*T)T^2\]
\[= TT^*T^2T^*\]
\[= T(T^*T)(TT^*)T\]
\[= T(T^*T)(T^*T)T\]
\[= T^2(T^*2T^2).\]

This proves the result.

**Theorem 2.17.** If \(T\) is of class \([2QN]\) and \([3QN]\) with \(N(T) \subset N(T^*)\), then \(T\) is quasinormal.

Proof.

\[T^3(T^*T) = T^*(T^*T)T^{*2}\] [\(T\) is of class \([2QN]\)]
\[= (T^*T)T^*3\]

Hence

\[[T^{*2}T - T^*TT^*]T^{*2} = 0\]

or

\[T^2[T^{*2}T^2 - TT^*T] = 0,\]

Since \(N(T) \subset N(T^*), N(T) = N(T^2)\) and therefore

\[T[T^{*2}T^2 - TT^*T] = 0, \text{ or } [T^{*2}T - T^*TT^*]T^* = 0.\]

Again by \(N(T) \subset N(T^*),\) we get the desired conclusion.

**Theorem 2.18.** If an operator \(T\) of class \([2QN]\) is a 2-isometry, then it is an isometry.

Proof. By the definition of a 2-isometry,

\[(T^{*2}T^2)(T^*T) - 2(T^*T)^2 + T^*T = 0.\]

Since \(T\) is of class \([2QN]\)

\[T^{*2}(T^*T)T^2 - 2(T^*T)^2 + T^*T = 0,\]

that is

\[T^{*3}T^3 - 2(T^*T)^2 + T^*T = 0.\] (2.14)

Also

\[T^*[T^{*2}T^2 - 2T^*T + I]T = 0\]

i.e.

\[T^{*3}T^3 - 2T^{*2}T^2 + T^*T = 0.\] (2.15)

From (2.14) and (2.15) \(T^{*2}T^2 = (T^*T)^2\) and hence

\[(T^*T)^2 - 2(T^*T) + I = T^{*2}T^2 - 2T^*T + I = (T^*T - I)^2 = 0\]

or

\[T^*T = I.\]
Theorem 2.19. If an operator $T$ is of class $[2QN] \cap [3QN]$ is an $n$-isometry, then $T$ is an isometry.

Proof. By the definition of $n$-isometry,
$$T^n T^* T - \binom{n}{1}T^{n-1}T^{*n} + \cdots + (-1)^{n-2} \binom{n}{n-2}T^{*2}T^2 + (-1)^{n-1} \binom{n}{n-1}T^{*n}T + (-1)^n T^*T = 0.$$ 

Since $T$ is of class $[2QN] \cap [3QN]$, we have by Proposition 2.6
$$T^{n+1} T^n + \cdots + (-1)^{n-2} \binom{n}{n-2}T^{*3}T^3 + (-1)^n \binom{n}{n-1}(T^*T)^2 + (-1)^n T^*T = 0. \quad (2.16)$$

Also
$$T^*[T^nT^n - \binom{n}{1}T^*n T^{n-1} + \cdots + (-1)^{n-1} \binom{n}{n-1}T^{*n}T + (-1)^n I]T = 0$$
i.e.
$$T^{n+1} T^n + \cdots + (-1)^{n-1} \binom{n}{n-1}T^{*2}T^2 + (-1)^n T^*T = 0 \quad (2.17)$$

From (2.16) and (2.17) $T^nT^2 = (T^*T)^2.$ Consequently $(T^*)^k T^k = (T^*T)^k$, $\forall \, k \in \mathbb{N}$, and hence
$$(T^*T)^n - \binom{n}{1}(T^*T)^{n-1} + \cdots + (-1)^{n-1} \binom{n}{n-1}(T^*T) + (-1)^n I = 0 = (I - T^*T)^n.$$ 

This completes the proof.

Definition 2.20. An operator $A \in \mathcal{L}(H)$ is said to be quasi-invertible if $A$ has zero kernel and dense range.

Definition 2.21. ([18]) Two operators $S$ and $T$ in $\mathcal{L}(H)$ are quasi-similar if there are quasi-invertible operators $A$ and $B$ in $\mathcal{L}(H)$ which satisfy the equations
$$AS = TA \quad \text{and} \quad BT = SB.$$ 

If $M$ is a closed subspace of $H, H = M \oplus M^\perp$. If $T$ is in $\mathcal{L}(H)$, then $T$ can be written as a $2 \times 2$ matrix with operators entries,
$$T = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$
where $W \in \mathcal{L}(M), \, X \in \mathcal{L}(M^\perp, M), \, Y \in \mathcal{L}(M, M^\perp), \, \text{and} \, Z \in \mathcal{L}(M^\perp)$ (cf. Conway [6]).

Proposition 2.22. If $S$ and $T$ are quasi-similar $n$-power quasi-normal operators in $\mathcal{L}(H)$ such that $N(S) = N(T), N(T)$ and $N(S)$ are reducing respectively for $T$ and $S$, then $S_1 = S|_{N(S)^\perp}$ and $T_1 = T|_{N(T)^\perp}$ are quasi-similar $n$-power quasi-normal operators.

Proof. Since $S$ and $T$ are quasi-similar, there exists quasi-invertible operators $A$ and $B$ such that $AS = TA$ and $SB = BT$. The $N(S)$ is invariant under both $A$ and $B$. Thus the matrices of $S,T,A$ and $B$ with respect to decomposition $H = N(S) \oplus N(S)^\perp$ are
$$\begin{pmatrix} S_1 & O \\ O & O \end{pmatrix}, \begin{pmatrix} T_1 & O \\ O & O \end{pmatrix}, \begin{pmatrix} A_1 & O \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 & O \\ B_2 & B_3 \end{pmatrix}$$
respectively. It is easy to verify that the ranges of $A_1$ and $B_1$ are dense in $N(S)^\perp$. We now show that $N(A_1) = N(B_1) = \{0\}$.

Suppose that $x \in N(A_1)$. Then $TA(x \oplus 0) = 0$. The equation $AS = TA$ implies that $x \in N(S_1)$. This implies that $x = 0$, and so $N(A_1) = \{0\}$. Likewise $N(B_1) = \{0\}$.
Therefore \( A_1 \) and \( B_1 \) are quasi-invertible operators on \( N(S)\perp \) and equations 
\[ \begin{align*}
A S &= T A \\
S B &= T B
\end{align*} \]
imply that \( A_1 S_1 = T_1 A_1 \) and \( S_1 B_1 = B_1 T_1 \). Hence \( S_1 \) and \( T_1 \) are quasi-similar. By a similar way as in [10, Proposition 2.1.(iv)] we can see that the operators \( S_1 \) and \( T_1 \) are \( n \)-power quasi-normal.

3. THE \((\mathbb{Z}^n)\) -CLASS OPERATORS

In this section we consider the class \((\mathbb{Z}_n^\alpha)\) of operators \( T \) satisfies
\[ |T^nT^*T - T^*TT^n|^\alpha \leq c_\alpha^2 (T - \lambda I)^*n(T - \lambda I)^n, \]
for all \( \lambda \in \mathbb{C} \) and for a positive \( \alpha \). The motivation is due to S. Mecheri [13] who considered the class of operators \( T \) satisfying
\[ |TT^* - T^*T|^\alpha \leq c_\alpha^2 (T - \lambda I)^*(T - \lambda I) \]
and A. Uchiyama and T. Yoshino [19] who discussed the class of operators \( T \) satisfying
\[ |TT^* - T^*T|^\alpha \leq c_\alpha^2 (T - \lambda I)(T - \lambda I)^*. \]

**Definition 3.1.** For \( T \in \mathcal{L}(H) \) we say that \( T \) belongs to the class \((\mathbb{Z}_n^\alpha)\) for some \( \alpha \geq 1 \) if there is a positive number \( c_\alpha \) such that
\[ |T^nT^*T - T^*TT^n|^\alpha \leq c_\alpha^2 (T - \lambda I)^*n(T - \lambda I)^n, \]
for all \( \lambda \in \mathbb{C} \), or equivalently, if there is a positive number \( c_\alpha \) such that
\[ \|T^nT^*T - T^*TT^n\|_{\frac{\alpha}{2}} \leq c_\alpha \|T - \lambda I\|^n_x \]
for all \( x \in H \) and \( \lambda \in \mathbb{C} \). Also, let
\[ \mathbb{Z}_n = \bigcup_{\alpha \geq 1} \mathbb{Z}_n^\alpha. \]

**Remark.** An operator \( T \) of class \([nQN]\), it is of class \((\mathbb{Z}_n)\).

In the following examples we give an example of an operator not in the classes \( \mathbb{Z}_n \), and an operator of these classes, which are not of class \([nQN]\).

**Example 3.2.** If \( f \) is a sequence of complex numbers, \( f = (f(0), f(1), f(2), \ldots)^T \).

The \( p \)-Cesàro operators \( C_p \) acting on the Hilbert space \( l^2 \) of square-summable complex sequences \( f \) is defined by
\[ (C_p f)(k) = \frac{1}{(k + 1)^p} \sum_{i=1}^{k} f(i) \quad \text{for fixed real } p > 1 \quad \text{and } k = 0, 1, 2, \ldots. \]

These operators was studied extensively in [16] where it was shown, that these operators are bounded and
\[ (C_p^* f)(k) = \sum_{i=k}^{\infty} \frac{1}{(i + 1)^p} f(i). \]

In matrix form, we have
\[ \begin{pmatrix}
\frac{1}{2^p} & 0 & 0 & \ldots \\
\frac{1}{3^p} & \frac{1}{2^p} & 0 & \ldots \\
\frac{1}{4^p} & \frac{1}{3^p} & \frac{1}{2^p} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]
We consider the sequence \( f \) defined as follows
\[
f(0) = 1 \quad \text{and} \quad f(k) = \prod_{j=1}^{k} \frac{j^p}{(1 + j)^p - 1} \quad \text{for} \quad k \geq 1.
\]
In [16] it is verified that \( f \in l^2 \), is eigenvector for \( C_p \) associated with eigenvalue 1, so \( f \in N(C_p - I) \), but \( f \notin N(C_p^* - I) \). It follows that \( \| (C_p - I^p) f \| = 0 \).

On the other hand, we have
\[
(C_p^*C_p - C_p^*C_p) f = (C_p^* - I)C_p f \neq 0.
\]
Hence, \( C_p \) is a bounded operator but not of classes \( \mathbb{Z}^n \).

**Example 3.3.** Let \( T \) be a weighted shift operator on \( l^2 \) with weights \( \alpha_1 = 2, \alpha_k = 1 \) for all \( k \geq 2 \). That is
\[
T_{\alpha}(x_1, x_2, x_3, \ldots) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots) \quad \text{and} \quad T^*(x_1, x_2, \ldots) = (\alpha_1 x_2, \alpha_2 x_3, \ldots).
\]
A simple computation shows that
\[
(T^nT^n T - T^*TT^n)(x) = (0, 0, \ldots, 0, 6x_1, 0, 0, \ldots)
\]
with \( 6x_1 \) at the \( (n+1) \)th place.

Moreover
\[
(T^nT^n T - T^*TT^n)(x) = (-6x_{n+1}, 0, 0, \ldots) \quad \text{and} \quad |T^nT^n T - T^*TT^n|^2 x = (-36x_1, 0, 0, \ldots).
\]
Therefore \( T \) is not of class \( [nQN] \) and however \( T \) is of class \( \mathbb{Z}_3^n \subseteq \mathbb{Z}^n \).

**Lemma 3.4.** For each \( \alpha, \beta \) such as \( 1 \leq \alpha \leq \beta \), we have \( \mathbb{Z}_3^n \subseteq \mathbb{Z}_\beta^n \).

**Proof.**
\[
|T^nT^*T - T^*T^n+1|^{\beta} = |T^nT^n T - T^*T^n+1|^{\beta} \leq |T^nT^n T - T^*T^n+1|^{\beta-\alpha} |T^nT^*T - T^*T^n+1|^{\alpha}
\]
\[
\leq (2|T||n+2|^{\beta-\alpha} c_\alpha^n) (T - \lambda I)^{\alpha} (T - \lambda I)^{\alpha}
\]
where
\[
C_\beta^2 = (2|T||n+2|^{\beta-\alpha} c_\alpha^n).
\]
There exists an Hilbert space \( H^* \), \( H \subseteq H^* \), and an isometric *-homomorphism preserving order, i.e., for all \( T, S \in \mathcal{L}(H) \) and \( \lambda, \mu \in \mathbb{C} \), we have

**Proposition 3.5.** ([6],[13] Berberian technique) Let \( H \) be a complex Hilbert space.
Then there exists a Hilbert space \( H^* \supseteq H \) and a map
\[
\Phi : \mathcal{L}(H) \to \mathcal{L}(H^*) : T \mapsto T^*
\]
satisfying: \( \Phi \) is an *-isometric isomorphism preserving the order such that

1. \( \Phi(T^*) = \Phi(T)^* \).
2. \( \Phi(\lambda T + \mu S) = \lambda \Phi(T) + \mu \Phi(S) \).
3. \( \Phi(I_H) = I_{H^*} \).
4. \( \Phi(TS) = \Phi(T) \Phi(S) \).
5. \( \| \Phi(T) \| = \| T \| \).
6. \( \Phi(T) \leq \Phi(S) \) if \( T \leq S \).
7. \( \sigma(\Phi(T)) = \sigma(T), \quad \sigma_{\alpha}(\Phi(T)) = \sigma_{\alpha}(\Phi(T)) \).
8. If \( T \) is a positive operator, then \( \Phi(T^\alpha) = |\Phi(T)|^\alpha \) for all \( \alpha > 0 \).
Lemma 3.6. If an operator $T^2$ is of class $[nQI]$, then $\Phi(T^2)$ is of class $[nQI]$.

Lemma 3.7. If an operator $T$ is of class $[nQI]$, then $\Phi(T)$ is of class $[nQI]$.

Proof. Since $T$ is of class $Z^n$, there exists $\alpha \geq 1$ and $c_{\alpha} > 0$ such that

$$|T^nT^*T - T^*T^{n+1}|^\alpha \leq c_{\alpha}^2(T - \lambda)^n(T - \lambda)^{n+1}$$

for all $\lambda \in \mathbb{C}$.

It follows from the properties of the map $\Phi$ that

$$\Phi([T^nT^*T - T^*T^{n+1}]) \leq \Phi(c_{\alpha}^2(T - \lambda)^n(T - \lambda)^{n+1})$$

for all $\lambda \in \mathbb{C}$.

By the condition 8. above we have

$$\Phi([T^nT^*T - T^*T^{n+1}]) = |\Phi([T^nT^*T - T^*T^{n+1}])|^\alpha$$

Therefore

$$|\Phi(T)|^n|\Phi(T)| - \Phi(T) \Phi(T)^{n+1}| \leq \Phi(c_{\alpha}^2(T - \lambda)^n(T - \lambda)^{n+1})$$

for all $\lambda \in \mathbb{C}$.

Hence $\Phi(T)$ is of class $Z^n$.

Proposition 3.8. Let $T$ be a class $Z^n$ operator and assume that there exists a subspace $M$ that reduces $T$, then $T|\bar{M}$ is of class $Z^n$ operator.

Proof. Since $T$ is of class $Z^n$, there exists an integer $p \geq 1$ and $c_{p} > 0$ such that

$$||T^nT^*T - T^*T^{n+1}||^p \leq c_{2p}||T - \lambda I||^n x||, \text{ for all } x \in H, \text{ for all } \lambda \in \mathbb{C}.$$ 

If $M$ reduces $T$, $T$ can be written respect to the composition $H = M \oplus M^\perp$ as follows:

$$T = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

By a simple calculation we get

$$T^nT^*T - T^*T^{n+1} = \begin{pmatrix} A^nA^*A - A^*A^{n+1} & O \\ O & B^nB^*B - B^*B^{n+1} \end{pmatrix}$$

By the uniqueness of the square root, we obtain

$$|T^nT^*T - T^*T^{n+1}| = \begin{pmatrix} |A^nA^*A - A^*A^{n+1}| & O \\ O & |B^nB^*B - B^*B^{n+1}| \end{pmatrix}.$$ 

Now by iteration to the order $2^p$, it results that

$$|T^nT^*T - T^*T^{n+1}|^{2^p-1} = \begin{pmatrix} |A^nA^*A - A^*A^{n+1}|^{2^p-1} & 0 \\ 0 & |B^nB^*B - B^*B^{n+1}|^{2^p-1} \end{pmatrix}.$$ 

Therefore for all $x \in M$, we have

$$||T^nT^*T - T^*T^{n+1}||^{2^p-1} \leq c_{2p}||T - \lambda I||^n x|| \leq (A - \lambda I)^n x||.$$ 

Hence $A$ is of class $Z^{n}_{2^p} \subset Z^n$.

Theorem 3.9. Let $T$ be of class $Z^n$.

1. If $\lambda \in \sigma_p(T)$, $\lambda \neq 0$, then $\lambda \in \sigma_p(T^*)$, furthermore if $\lambda \neq \mu$, then $E_{\lambda}$ (the proper subspace associated with $\lambda$) is orthogonal to $E_{\mu}$.

2. If $\lambda \in \sigma_q(T)$, then $\lambda \in \sigma_q(T^*)$.

3. $TT^*T - T^*T^2$ is not invertible.

Proof.
ON THE CLASS OF $n$-POWER QUASI-NORMAL OPERATORS ON HILBERT SPACE

(1) If $T \in \mathbb{Z}_1$, then $T \in \mathbb{Z}_1^\alpha$ for some $\alpha \geq 1$ and there exists a positive constant $c_\alpha$ such that

$$|TT^* T - T^* T^2|^\alpha \leq c_\alpha (T - \lambda I)^*(T - \lambda I)$$

for all $\lambda \in \mathbb{C}$.

As $Tx = \lambda x$ implies $|TT^* T - T^* T^2|^2 x = 0$ and $(TT^* - T^* T)x = 0$ and hence

$$\| (T - \lambda)^* x \| = \| (T - \lambda) x \|$$

$$\lambda \langle x | y \rangle = \langle \lambda x | y \rangle = \langle Tx | y \rangle = \langle x | T^* y \rangle = \langle x | \overline{\mu} y \rangle = \mu \langle x | y \rangle.$$  

Hence

$$\langle x | y \rangle = 0.$$

(2) Let $\lambda \in \sigma_a(T)$ from the condition 7. above, we have

$$\sigma_a(T) = \sigma_a(\Phi(T)) = \sigma_p(\phi(T)).$$

Therefore $\lambda \in \sigma_p(\phi(T))$. By applying Lemma 3.7 and the above condition 1., we get

$$\lambda \in \sigma_p(\Phi(T)^*).$$

(3) Let $T \in \mathbb{Z}_1$. then there exists an integer $p \geq 1$ and $c_p > 0$ such that

$$\| TT^* T - T^* T^2 \|^2 \leq c_p^2 \| (T - \lambda I) x \|^2$$

for all $x \in H$ and for all $\lambda \in \mathbb{C}$.

It is know that $\sigma_a(T) \neq \emptyset$. If $\lambda \in \sigma_a(T)$, then there exists a normed sequence $(x_m)$ in $H$ such that $\| (T - \lambda I) x_m \| \rightarrow 0$ as $m \rightarrow \infty$. Then

$$(TT^* T - T^* T^2) x_m \rightarrow 0$$

and so, $(TT^* T - T^* T^2)$ is not invertible.

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References


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