RIEMANNIAN MANIFOLDS WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION SATISFYING SOME SEMISYMMETRY CONDITIONS

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Abstract. We study on a Riemannian manifold \((M, g)\) with a semi-symmetric non-metric connection. We obtain some characterizations for \((M, g)\) satisfying some semisymmetry conditions.

1. Introduction

Let \(\tilde{\nabla}\) be a linear connection in an \(n\)-dimensional differentiable manifold \(M\). The torsion tensor \(T\) and the curvature tensor \(\tilde{R}\) of \(\tilde{\nabla}\) are given respectively by

\[
T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],
\]

\[
\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z.
\]

The connection \(\tilde{\nabla}\) is symmetric if its torsion tensor \(T\) vanishes, otherwise it is not symmetric. The connection \(\tilde{\nabla}\) is a metric connection if there is a Riemannian metric \(g\) in \(M\) such that \(\tilde{\nabla}g = 0\), otherwise it is non-metric [15]. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In 1932, H. A. Hayden [8] introduced a metric connection \(\hat{\nabla}\) with a non-zero torsion on a Riemannian manifold. Such a connection is called Hayden connection. In [7, 11], Friedmann and Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection \(\hat{\nabla}\) is said to be a semi-symmetric connection if its torsion tensor \(T\) is of the form

\[
T(X, Y) = \omega(Y)X - \omega(X)Y,
\]

where the 1-form \(\omega\) is defined by

\[
\omega(X) = g(X, U),
\]

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and $U$ is a vector field. Hayden connection with the torsion tensor of the form (1.1) is a semi-symmetric metric connection [10].

In [1], Agashe and Chafle introduced the idea of a semi-symmetric non-metric connection on a Riemannian manifold. This was further developed by Agashe and Chafle [2], De and Kamila [5], De, Sengupta and Binh [12].

In [13, 14], Szabó studied semisymmetric Riemannian manifolds, that is Riemannian manifolds satisfying the condition $R \cdot R = 0$. It is well known that locally symmetric manifolds (i.e., Riemannian manifolds satisfying the condition $\nabla R = 0$) are trivially semisymmetric. But the converse statement is not true. If $R \cdot S = 0$ then the manifold is called Ricci-semisymmetric. It is trivial that every semisymmetric manifold is Ricci-semisymmetric but the converse statement is not true.

In this paper, we consider Riemannian manifolds admitting a semi-symmetric non-metric connection such that $U$ is a unit parallel vector field with respect to the Levi-Civita connection $\nabla$. We investigate the conditions $R \cdot R = 0$, $R \cdot R = 0$, $R \cdot R - R \cdot R = 0$, $R \cdot R = 0\cdot R - R \cdot S = 0$ and $R \cdot S = 0$ on $M$, where $R$ and $\tilde{R}$ (resp. $S$ and $\tilde{S}$) denote the curvature tensors (resp. Ricci tensors) of $\nabla$ and $\tilde{\nabla}$.

The paper is organized as follows. In Section 2 and Section 3, we give the necessary notions and results which will be used in the next sections. In Section 4, we prove that $R \cdot \tilde{R} = 0$ holds on $M$ if and only if $M$ is semisymmetric. Furthermore, we show that $M$ is a quasi-Einstein manifold under certain conditions.

2. Preliminaries

An $n$-dimensional Riemannian manifold $(M, g)$, $(n \geq 3)$, is said to be an Einstein manifold if its Ricci tensor $S$ satisfies the condition $S = \frac{k}{n} g$, where $r$ denotes the scalar curvature of $M$. If the Ricci tensor $S$ is of the form

$$S(X, Y) = ag(X, Y) + bD(X)D(Y), \quad (2.1)$$

where $a, b$ are scalars of which $b \neq 0$ and $D$ is a non zero 1-form, then $M$ is called a quasi-Einstein manifold [4].

For a $(0, k)$-tensor field $T$, $k \geq 1$, on $(M, g)$ we define the tensor $R \cdot T$ (see [6]) by

$$(R(X, Y) \cdot T)(X_1, ..., X_k) = -T(R(X, Y)X_1, X_2, ..., X_k)$$

$$- ... - T(X_1, ..., X_{k-1}, R(X, Y)X_k). \quad (2.2)$$

If $R \cdot R = 0$ then $M$ is called semisymmetric [13]. In addition, if $E$ is a symmetric $(0, 2)$-tensor field then we define the $(0, k + 2)$-tensor $Q(E, T)$ (see [6]) by

$$Q(E, T)(X_1, ..., X_k; X, Y) = -T((X \wedge _E Y)X_1, X_2, ..., X_k)$$

$$- ... - T(X_1, ..., X_{k-1}, (X \wedge _E Y)X_k). \quad (2.3)$$

where $X \wedge _E Y$ is defined by

$$(X \wedge _E Y)Z = E(Y, Z)X - E(X, Z)Y.$$

3. Semi-symmetric non-metric connection

Let $\nabla$ be the Levi-Civita connection of a Riemannian manifold $M$. The semi-symmetric non-metric connection $\tilde{\nabla}$ is defined by

$$\tilde{\nabla}_XY = \nabla_XY + \omega(Y)X, \quad (3.1)$$
where
\[ \omega(X) = g(X, U), \]
and \( X, Y, U \) are vector fields on \( M \) [1]. Let \( R \) and \( \tilde{R} \) denote the Riemannian curvature tensors of \( \nabla \) and \( \tilde{\nabla} \), respectively. Then we know that [1]
\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - \theta(Y, Z)g(X, W) + \theta(X, Z)g(Y, W),
\]
where
\[
\theta(X, Y) = g(AX, Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y). \tag{3.3}
\]
Here \( A \) is a \((1,1)\)-tensor field which is metrically equivalent to \( \theta \). Now assume that \( U \) is a parallel unit vector field with respect to the Levi-Civita connection, i.e., \( \nabla U = 0 \) and \( \|U\| = 1 \). Then
\[
(\nabla_X \omega)Y = \nabla_X \omega(Y) - \omega(X)\omega(Y) = 0. \tag{3.4}
\]
So \( \theta \) is a symmetric \((0,2)\)-tensor field. Since \( U \) is a parallel unit vector field, it is easy to see that \( \tilde{R} \) is a generalized curvature tensor and it is trivial that \( R(X, Y)U = 0 \).

Hence by a contraction, we find
\[
\tilde{S}(Y, U) = \omega(QY) = 0,
\]
where \( \tilde{S} \) denotes the Ricci tensor of \( \tilde{\nabla} \) and \( Q \) is the Ricci operator defined by \( g(QX, Y) = S(X, Y) \). It is easy to see that we have also the following relations:
\[
\tilde{\nabla}_X U = X, \tag{3.5}
\]
\[
\tilde{R}(X, Y)U = \omega(Y)X - \omega(X)Y, \quad \tilde{R} \cdot \theta = 0, \tag{3.6}
\]
\[
\tilde{S} = S + (n - 1)(\omega \otimes \omega), \tag{3.7}
\]
and
\[
\tilde{r} = r + (n - 1), \tag{3.8}
\]
where \( \tilde{S} \) and \( \tilde{r} \) denote the Ricci tensor and the scalar curvature of \( M \) with respect to semi-symmetric non-metric connection \( \tilde{\nabla} \).

4. Main Results

The tensors \( \tilde{R} \cdot R \) and \( Q(\theta, T) \) are defined in the same way as in (2.2) and (2.3). Let \( (R \cdot R)_{hijklm} \) and \( (\tilde{R} \cdot R)_{hijklm} \) denote the local components of the tensors \( R \cdot \tilde{R} \) and \( \tilde{R} \cdot R \), respectively. Hence, we have the following Proposition:

**Proposition 4.1.** Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If \( U \) is a parallel unit vector field with respect to the Levi-Civita connection \( \nabla \) then
\[
(R \cdot \tilde{R})_{hijklm} = (R \cdot R)_{hijklm} \tag{4.1}
\]
and
\[
(\tilde{R} \cdot R)_{hijklm} = (R \cdot R)_{hijklm} - Q(-\omega \otimes \omega, R)_{hijklm}. \tag{4.2}
\]

**Proof.** Applying (3.2) in (2.2) and using (2.3), we obtain
\[
R \cdot \tilde{R} = R \cdot R \tag{4.3}
\]
and
\[
(\tilde{R} \cdot R)_{hijklm} = (R \cdot R)_{hijklm} - Q(\theta, R)_{hijklm} = (R \cdot R)_{hijklm} - Q(-\omega \otimes \omega, R)_{hijklm}. \]

This completes the proof of the proposition. \( \square \)
As an immediate consequence of Proposition 4.1, we have the following theorem:

**Theorem 4.2.** Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection and \(U\) a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\). Then \(R \cdot \tilde{R} = 0\) if and only if \(M\) is semisymmetric.

**Theorem 4.3.** Let \((M, g)\) be an \((n \geq 3)\)-dimensional semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\) and \(\tilde{R} \cdot R = 0\) then \(M\) is a quasi-Einstein manifold.

**Proof.** Since \(M\) is semisymmetric and the condition \(\tilde{R} \cdot R = 0\) holds on \(M\), from Proposition 4.1, we have

\[
Q(\omega \otimes \omega, R)_{hijklm} = 0. \tag{4.4}
\]

Contracting (4.4) with \(g^{ij}\) we get

\[
Q(\omega \otimes \omega, S)_{hklm} = 0,
\]

which gives us

\[
S = r(\omega \otimes \omega),
\]

where \(r : M \to \mathbb{R}\) is a function. So by virtue of (2.1), \(M\) is a quasi-Einstein manifold. Thus the proof of the theorem is completed.

**Theorem 4.4.** Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\) and \(R \cdot \tilde{R} - \tilde{R} \cdot R = 0\), then \(M\) is a quasi-Einstein manifold.

**Proof.** Using (4.1) and (4.2) we get

\[
Q(\omega \otimes \omega, R)_{hijklm} = 0.
\]

Using the same method as in the proof of Theorem 4.3, we obtain that \(M\) is a quasi-Einstein manifold. So we get the result as required.

**Proposition 4.5.** Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\) then

\[
(R \cdot \tilde{S})_{hklm} = (R \cdot S)_{hklm}, \tag{4.5}
\]

\[
(\tilde{R} \cdot S)_{hklm} = (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}. \tag{4.6}
\]

**Proof.** Applying (3.7) and (3.2) in (2.2) and using (2.3), we obtain

\[
R \cdot \tilde{S} = R \cdot S
\]

and

\[
(\tilde{R} \cdot S)_{hklm} = (R \cdot S)_{hklm} - Q(\theta, S)_{hklm}
= (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}.
\]

This completes the proof of the proposition.

As an immediate consequence of Proposition 4.5, we have the following theorem:
Theorem 4.6. Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection and \(U\) a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\). Then \(\mathcal{R} \cdot \tilde{S} = 0\) if and only if \(M\) is Ricci-semisymmetric.

Theorem 4.7. Let \((M, g)\) be an \((n \geq 3)\)-dimensional Ricci-semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\) and \(\mathcal{R} \cdot \tilde{S} = 0\), then \(M\) is a quasi-Einstein manifold.

Proof. Since the condition \(\mathcal{R} \cdot \tilde{S} = 0\) holds on \(M\), from Proposition 4.5, we have

\[ Q(\omega \otimes \omega, S)_{hklm} = 0. \]

So by the same reason as in the proof of Theorem 4.3, \(M\) is a quasi-Einstein manifold. Thus the proof of the theorem is completed. □

Theorem 4.8. Let \((M, g)\) be an \((n \geq 3)\)-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to Levi-Civita connection \(\nabla\) and \(\mathcal{R} \cdot \tilde{S} = \mathcal{R} \cdot S = 0\), then \(M\) is a quasi-Einstein manifold.

Proof. Using (4.5) and(4.6) we get

\[ Q(\omega \otimes \omega, S)_{hklm} = 0. \]

Using the same method as in the proof of Theorem 4.3, we obtain that \(M\) is a quasi-Einstein manifold. This proves the theorem. □

Theorem 4.9. Let \((M, g)\) be an \((n \geq 3)\)-dimensional Ricci-semisymmetric Riemannian manifold admitting a semi-symmetric non-metric connection. If \(U\) is a parallel unit vector field with respect to the Levi-Civita connection \(\nabla\) and \(\mathcal{R} \cdot \tilde{S} = 0\), then \(M\) is a quasi-Einstein manifold.

Proof. Applying (3.7) and (3.2) in (2.2) and using (2.3) we obtain,

\[ (\mathcal{R} \cdot \tilde{S})_{hklm} = (R \cdot S)_{hklm} - Q(-\omega \otimes \omega, S)_{hklm}. \]

We suppose that \(\mathcal{R} \cdot \tilde{S} = 0\) and \(R \cdot S = 0\). So using the same method as in the proof of Theorem 4.3, we obtain that \(M\) is a quasi Einstein manifold. Thus the proof of the theorem is completed.

The following example shows that there is a Riemannian manifold with a semi-symmetric non-metric connection having a parallel vector field associated to the 1-form satisfying \(R \cdot \tilde{R} = R \cdot R\). □

Example. Let \(M^{2m+1}\) be a \((2m + 1)\)-dimensional almost contact manifold endowed with an almost contact structure \((\varphi, \xi, \eta)\), that is, \(\varphi\) is a \((1,1)\)-tensor field, \(\xi\) is a vector field and \(\eta\) is a 1-form such that

\[ \varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1. \]

Then

\[ \varphi(\xi) = 0 \quad \text{and} \quad \eta \circ \varphi = 0. \]

Let \(g\) be a compatible Riemannian metric with \((\varphi, \xi, \eta)\), that is,

\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \]
or equivalently,

\[ g(X, \varphi Y) = -g(\varphi X; Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \]

for all \( X, Y \in \mathfrak{X}(M) \). Then, \( M^{2m+1} \) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\varphi, \xi, \eta, g)\). An almost contact metric manifold is cosymplectic \([3]\) if \( \nabla_X \varphi = 0 \). From the formula \( \nabla_X \varphi = 0 \), it follows that

\[ \nabla X \xi = 0, \quad \nabla X \eta = 0, \quad \text{and} \quad R(X, Y) \xi = 0. \]

If we define a connection

\[ \tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) X \]

on the above manifold, then we obtain

\[ T(X, Y) = \eta(Y) X - \eta(X) Y \]

and

\[ \theta = -\eta \otimes \eta, \]

which shows that \( \tilde{\nabla} \) is a semi-symmetric non-metric connection and by virtue of Proposition 4.1, we have \( R \cdot \tilde{R} = R \cdot R \).

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**References**


