SHARPNESS OF NEGOI’S INEQUALITY FOR THE EULER-MASCHERONI CONSTANT

(COMMUNICATED BY ARMEND SHABANI)

CHAO-PING CHEN

Abstract. We present new estimates for the Euler-Mascheroni constant, which improve a result of Negoi.

1. Introduction

The Euler-Mascheroni constant \( \gamma = 0.577215664\ldots \) is defined as the limit of the sequence

\[
D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) .
\]

Several bounds for \( D_n - \gamma \) have been given in the literature [3, 4, 19, 22, 23, 24, 27] (see also [6, 20, 21]). For example, the following bounds for \( D_n - \gamma \) were established in [19, 27]:

\[
\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (n \in \mathbb{N}) .
\]

The convergence of the sequence \( D_n \) to \( \gamma \) is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [5, 6, 7, 8, 10, 13, 16, 18, 20, 21, 25, 26]. For example, Negoi [18] proved that the sequence

\[
T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln \left( n + \frac{1}{2} + \frac{1}{24n} \right)
\]

is strictly increasing and convergent to \( \gamma \). Moreover, the author proved that

\[
\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3} .
\]

The main objective of this work is to establish closer bounds for \( \gamma - T_n \).
2. Lemmas

Before stating and proving the main theorems, we first include here some preliminary results.

The constant \( \gamma \) is deeply related to the gamma function \( \Gamma(x) \) thanks to the Weierstrass formula [1, p. 255]:

\[
\Gamma(x) = e^{-\gamma x} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{k} \right)^{-1} e^{x/k}
\]

for any real number \( x \), except on the negative integers \( \{0, -1, -2, \ldots\} \). The logarithmic derivative of the gamma function:

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}
\]

is known as the psi (or digamma) function.

The following recurrence and asymptotic formulas are well known for the psi function:

\[
\psi(x + 1) = \psi(x) + \frac{1}{x} \quad (2.1)
\]

(see [1, p.258]), and

\[
\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \quad (x \to \infty) \quad (2.2)
\]

(see [1, p.259]). From (2.1) and (2.2), we get

\[
\psi(x + 1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \quad (x \to \infty) \quad (2.3)
\]

It is also known [1, p.258] that

\[
\psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} . \quad (2.4)
\]

The following lemmas are also needed in our present investigation.

**Lemma 2.1.** If the sequence \((\lambda_n)_{n \in \mathbb{N}}\) converges to zero and if there exists the following limit:

\[
\lim_{n \to \infty} n^k (\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1),
\]

then

\[
\lim_{n \to \infty} n^{k-1} \lambda_n = \frac{l}{k-1} \quad (k > 1).
\]

This lemma is suitable for accelerating some convergences, or in constructing some asymptotic expansions. For proofs and other details, see, e.g. [11, 12, 13, 14, 15, 16, 17].

**Lemma 2.2** ([2, Theorem 9]). Let \( k \geq 1 \) and \( n \geq 0 \) be integers. Then for all real numbers \( x > 0 \):

\[
S_k(2n; x) < (-1)^{k+1} \psi^{(k)}(x) < S_k(2n + 1; x), \quad (2.5)
\]

where

\[
S_k(p; x) = \frac{(k - 1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} \left[ B_{2i} \prod_{j=1}^{k-1} (2i + j) \right] \frac{1}{x^{2i+k}},
\]

\( B_{2i} \) being the Bernoulli numbers.
and $B_i$ ($i = 0, 1, 2, \ldots$) are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$$

(see [1] p. 804).

In particular, it follows from (2.5) that

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x)$$

$$< \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}, \quad x > 0 . \quad (2.6)$$

From (2.1) and (2.6), we obtain

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x + 1)$$

$$< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \quad x > 0 . \quad (2.7)$$

3. Main results

3.1. We define the sequence $(u_n)_{n \in \mathbb{N}}$ by

$$u_n = \ln \left( n \frac{1}{2} + \frac{1}{24n} \right) - \psi(n + 1) - \frac{a}{\left( n + b + \frac{c}{n + d} \right)^3} . \quad (3.1)$$

We are interested in finding the values of the parameters $a$, $b$, $c$ and $d$ such that $(u_n)_{n \in \mathbb{N}}$ is the \textit{fastest} sequence which would converge to zero. This provides the best approximations of the form:

$$\psi(n + 1) \approx \ln \left( n \frac{1}{2} + \frac{1}{24n} \right) - \frac{a}{\left( n + b + \frac{c}{n + d} \right)^3} . \quad (3.2)$$

Our study is based on the above Lemma 2.1

\textbf{Theorem 3.1.} Let the sequence $(u_n)_{n \in \mathbb{N}}$ be defined by (3.1). Then for

$$a = \frac{1}{48}, \quad b = \frac{83}{360}, \quad c = \frac{4909}{64800}, \quad d = \frac{11976997}{37112040} , \quad (3.3)$$

we have

$$\lim_{n \to \infty} n^8 (u_n - u_{n+1}) = \frac{1763157528883853}{8311196823552000} . \quad (3.4)$$

and

$$\lim_{n \to \infty} n^7 u_n = \frac{1763157528883853}{581783777648640000} . \quad (3.5)$$

The speed of convergence of the sequence $(u_n)_{n \in \mathbb{N}}$ is given by the order estimate $O \left( n^{-7} \right)$. 
Proof. First of all, we write the difference $u_n - u_{n+1}$ as the following power series in $n^{-1}$:

$$u_n - u_{n+1} = \frac{1}{16n^4} - \frac{263 + 8640a + 17280ab}{1440n^5}$$

$$+ \frac{139 - 3840a - 11520ab - 11520ab^2 + 5760ac}{384n^6}$$

$$+ \left( 90720a + 362880ab + 362880ab^3 + 544320ab^2 - 272160ac ight)$$

$$- \frac{435456ac - 108864ac + 362880}{16048n^7}$$

$$+ \left( -193536a - 967680ab + 774144ac + 193560ac + 193560ac + 967680ab^3 ight.$$

$$+ 8663 + 232243ac - 387072ac^2 - 193560ac + 967680ab^4$$

$$+ 580608acd + 967680ac - 193560ac + 967680ab^4 \right) \frac{1}{6048n^8} + O \left( \frac{1}{n^9} \right).$$

(3.6)

The fastest sequence $(u_n)_{n \in \mathbb{N}}$ is obtained when the first four coefficients of this power series vanish. In this case

$$a = \frac{1}{48}, \quad b = \frac{83}{360}, \quad c = \frac{4909}{64800}, \quad d = \frac{11976997}{37112040},$$

we have

$$u_n - u_{n+1} = \frac{176315752883853}{831119682355200n^8} + O \left( \frac{1}{n^9} \right).$$

(3.7)

Finally, by using Lemma 2.1, we obtain assertions (3.4) and (3.5) of Theorem 3.1.

Solution (3.1) provides the best approximation of type (3.2):

$$\approx \ln \left( n + \frac{1}{2} + \frac{1}{24n} \right) - \frac{1}{16n^3} - \frac{1}{16n^5}.$$  

(3.8)

Motivated by approximation (3.8), we establish Theorem 3.2 below, which provides closer bounds for $\gamma - T_n$.

**Theorem 3.2.** For $n \geq 1$, then

$$\frac{1}{48} \left( n + \frac{83}{360} + \frac{4909}{n + 11976997} \right)^2 < \gamma - T_n < \frac{1}{16} \left( n + \frac{83}{360} \right)^2.$$  

(3.9)

**Proof.** We only prove the right-hand inequality in (3.9). The proof of the left-hand inequality in (3.9) is similar. The inequality (3.9) can be written for $n \geq 1$ as

$$\frac{1}{16n^3} < \ln \left( n + \frac{1}{2} + \frac{1}{24n} \right) - \psi(n + 1) < \frac{1}{16} \left( n + \frac{83}{360} \right)^2.$$  

(3.10)
The upper bound of (3.9) is obtained by considering the function \( f(x) \) which is defined, for \( x > 0 \), by
\[
f(x) = \ln \left( x + \frac{1}{2} + \frac{1}{24x} \right) - \psi(x + 1) - \frac{1}{19} \left( \frac{x + \frac{83}{360}}{x + \frac{83}{360}} \right)^3.
\]

We conclude from the asymptotic formula (2.3) that
\[
\lim_{x \to \infty} f(x) = 0.
\]

Differentiating \( f(x) \) and applying the second inequality in (2.7) yields,
\[
f'(x) = \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \psi'(x + 1) + \frac{1049760000}{(360x + 83)^4} > \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \frac{1}{x} \left( \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \right) + \frac{104976000}{(360x + 83)^4}
\]
\[
= \frac{p(x)}{210x^7(24x^2 + 12x + 1)(360x + 83)^4},
\]
where
\[
p(x) = 147550579398783 + 637562673548352(x - 2) + 1095096221221183(x - 2)^2 + 997896029835428(x - 2)^3 + 528831825356263(x - 2)^4 + 164401992148725(x - 2)^5 + 27912981996000(x - 2)^6 + 2004050160000(x - 2)^7 > 0 \quad \text{for} \quad x \geq 2.
\]

Therefore, \( f'(x) > 0 \) for \( x \geq 2 \).

Direct computation would yield
\[
f(1) = \gamma + \ln \left( \frac{37}{24} \right) - \frac{87910307}{86938307} = -0.00110059\ldots,
\]
\[
f(2) = \gamma + \ln \left( \frac{121}{48} \right) - \frac{155288881}{103563254} = -0.000072039\ldots.
\]

Consequently, the sequence \( (f(n))_{n \in \mathbb{N}} \) is strictly increasing. This leads us to
\[
f(n) < \lim_{n \to \infty} f(n) = 0, \quad n \geq 1,
\]
which means that the upper bound in assertion (3.9) of Theorem 3.2 holds true for all \( n \in \mathbb{N} \). The proof of Theorem 3.2 is thus completed. \( \square \)

**Remark 1.** In fact, the following inequality holds true:
\[
\gamma - T_n < \left( n + \frac{83}{360} + \frac{1}{48} \right)^3 - \left( n + \frac{83}{360} + \frac{4909}{13086847471578024} \right)^3 - \left( n + \frac{83}{360} + \frac{11976997}{37412049} + \frac{4909}{13086847471578024} \right)^3 - \left( n + \frac{83}{360} + \frac{11976997}{37412049} + \frac{111976997}{37412049} + \frac{4909}{13086847471578024} \right)^3,
\]
for \( n \in \mathbb{N} \).
3.2. We now define the sequence \((v_n)_{n \in \mathbb{N}}\) by

\[ v_n = \ln \left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n + 1) - \frac{1}{a_1 n^3 + b_1 n^2 + c_1 n + d_1}. \]  

(3.12)

where \(a_1, b_1, c_1, d_1 \in \mathbb{R}\). Following the same method used in the proof of Theorem 3.1, we find that for

\[ a_1 = 48, \quad b_1 = \frac{166}{5}, \quad c_1 = \frac{5569}{300}, \quad d_1 = \frac{58741}{28000}, \]  

(3.13)

we have

\[ \lim_{n \to \infty} n^8(v_n - v_{n+1}) = \frac{183358033}{9953280000} \]  

and

\[ \lim_{n \to \infty} n^7v_n = \frac{183358033}{69672960000}. \]  

(3.14)

The speed of convergence of the sequence \((v_n)_{n \in \mathbb{N}}\) is given by the order estimate \(O(n^{-7})\).

**Theorem 3.3.** For \(n \geq 1\), then

\[ \frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \gamma - T_n. \]  

(3.15)

**Proof.** The inequality (3.15) can be written for \(n \geq 1\) as

\[ \frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \ln \left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n + 1). \]  

(3.16)

We consider the function \(F(x)\) defined for \(x > 0\) by

\[ F(x) = \ln \left(x + \frac{1}{2} + \frac{1}{24x}\right) - \psi(x + 1) - \frac{1}{48x^3 + \frac{166}{5}x^2 + \frac{5569}{300}x + \frac{58741}{28000}}. \]

We conclude from the asymptotic formula (2.3) that

\[ \lim_{x \to \infty} F(x) = 0. \]

Differentiating \(F(x)\) and applying the first inequality in (2.7) yields,

\[ F'(x) = \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \psi'(x + 1) \]

\[ = \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\right) \]

\[ = \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\right) \]

\[ + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \]

\[ < \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\right) \]

\[ + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \]

\[ = -\frac{q(x)}{210x^8(24x^2 + 12x + 1)(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2}, \]

where

\[ q(x) = 18130487257947165687 + 6355299367883953745(x - 3) \]

\[ + 94471229612034347921(x - 3)^2 + 79408865830190450709(x - 3)^3 \]

\[ + 41975644888778012717(x - 3)^4 + 14553520724815257633(x - 3)^5 \]

\[ + 332239327176291138(x - 3)^6 + 482867798807968875(x - 3)^7 \]

\[ + 40622141576265200(x - 3)^8 + 1509403327656000(x - 3)^9 > 0 \quad \text{for} \quad x \geq 3. \]
Therefore, $F'(x) < 0$ for $x \geq 3$.

Direct computation would yield

$$F(1) = -\frac{8640343}{8556343} + \gamma + \ln 37 - 3 \ln 2 - \ln 3 = 0.000262469 \ldots,$$

$$F(2) = -\frac{140286189}{93412126} + \gamma + 2 \ln 11 - 4 \ln 2 - \ln 3 = 0.000006718 \ldots,$$

$$F(3) = -\frac{509165071}{277634766} + \gamma + \ln 23 - 3 \ln 2 - 2 \ln 3 = 0.000000589 \ldots.$$

Consequently, the sequence $(F(n))_{n \in \mathbb{N}}$ is strictly decreasing. This leads us to

$$F(n) > \lim_{n \to \infty} F(n) = 0, \quad n \geq 1,$$

which means that inequality (3.15) holds true for all $n \in \mathbb{N}$. \hfill \Box

**Remark 2.** The lower bound in (3.15) is sharper than one in (3.9).

**Remark 3.** In fact, the following inequality holds true:

$$\gamma - T_n < \frac{1}{48n^3 + \frac{196}{3}n^2 + \frac{569}{200}n + \frac{2871}{28000} - \frac{133358033}{30240000n}}.$$

(3.17)

for $n \in \mathbb{N}$.

**Remark 4.** The numerical calculations presented in this work were performed by using the Maple software for symbolic computations.

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**References**


CHAO-PING CHEN
School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454003, Henan Province, People’s Republic of China
E-mail address: chenchaoping@sohu.com