GROWTH OF A CLASS OF ITERATED ENTIRE FUNCTIONS

(COMMUNICATED BY VICENTIU RADULESCU)

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Abstract. In this paper we generalise a result of J. Sun to n-th iterations of $f(z)$ with respect to $g(z)$.

1. Introduction and Notation

We first consider two entire functions $f(z)$ and $g(z)$ and following Lahiri and Banerjee [5] form the iterations of $f(z)$ with respect to $g(z)$ as follows:

- $f_1(z) = f(z)$
- $f_2(z) = f(g(z)) = f(g_1(z))$
- $f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$
- $\ldots$ $\ldots$ $\ldots$
- $f_n(z) = f(g(f_{n-1}(z)))$ according as $n$ is odd or even

and so

- $g_1(z) = g(z)$
- $g_2(z) = g(f(z)) = g(f_1(z))$
- $\ldots$ $\ldots$ $\ldots$
- $g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z)))$.

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Notation 1.1. Let $f(z)$ and $g(z)$ be two entire functions. Throughout the paper we use the notations $M_{f_1}(r), M_{f_2}(r), M_{f_3}(r)$ etc., to mean $M(r, f), M(M(r, f), g)$,
$M(M(r, f), g)$, $F(r) = O^*(G(r))$ to mean that there exist two positive constants $K_1$ and $K_2$ such that $K_1 \leq \frac{F(r)}{G(r)} \leq K_2$ for any $r$ big enough.

In [2], C. Chuang and C. C. Yang posed the question: For four entire functions $f_1, f_2$ and $g_1, g_2$, when is $T(r, f_1 \log f_1) \sim T(r, f_2 \log g_2)$ as $r \to \infty$, provided $T(r, f_1) \sim T(r, f_2)$ and $T(r, g_1) \sim T(r, g_2)$?

In 2003, Sun [7] showed that in general there is no positive answer and he gave a condition under which there is a positive answer by proving the following theorem.

**Theorem A.** Let $f_1, f_2$ and $g_1, g_2$ be four transcendental entire functions with $T(r, f_1) = O^*((\log r)^e(\log r)^{2\nu})$ and $T(r, g_1) = O^*((\log r)^{\beta})$.

If $T(r, f_1) \sim T(r, f_2)$ and $T(r, g_1) \sim T(r, g_2)$ as $r \to \infty$, then

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty, r \notin E),$$

where $\nu > 0, 0 < \alpha < 1, \beta > 1$ and $\alpha \beta < 1$ and $E$ is a set of finite logarithmic measure.

We extend Theorem A to iterated entire functions.

**Theorem 1.2.** Let $f, g$ and $u, v$ be four transcendental entire functions with $T(r, f) \sim T(r, u), T(r, g) \sim T(r, v), T(r, f) = O^*((\log r)^{e(\log r)^{\alpha}})$ $(0 < \alpha < 1, \nu > 0)$ and $T(r, g) = O^*((\log r)^{\beta})$ where $\beta > 1$ is a constant, then $T(r, f_n) \sim T(r, u_n)$ for $n \geq 2$, where $u_n(z) = u(v(u(v(\ldots (u(z) or v(z))\ldots))))$ according as $n$ is odd or even.

We do not explain the standard notations and definitions of the theory of meromorphic functions because they are available in [4].

2. Lemmas

The following lemmas will be needed in the sequel.

**Lemma 2.1.** [4] Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f).$$

**Lemma 2.2.** [4] Let $f(z)$ be an entire function of order $\rho (\rho < \infty)$. If $k > \rho - 1$, then

$$\log M(r, f) \sim \log M(r - r^{-k}, f) \quad (r \to \infty).$$

**Lemma 2.3.** [4] Let $g(z)$ and $f(z)$ be two entire functions. Suppose that $|g(z)| > R > |g(0)|$ on the circumference $\{|z| = r\}$ for some $r > 0$. Then we have

$$T(r, f(g)) \geq \frac{R - |g(0)|}{R + |g(0)|} T(R, f).$$

**Lemma 2.4.** [4] Let $f$ be an entire function of order zero and $z = re^{i\theta}$. Then for any $\zeta > 0$ and $\eta > 0$, there exist $R_0 = R_0(\zeta, \eta)$ and $k = k(\zeta, \eta)$ such that for all $R > R_0$ it holds

$$\log |f(re^{i\theta})| - N(2R) - \log |z| > -kQ(2R), \quad \zeta R \leq r \leq R,$$
except in a set of circles enclosing the zeros of \( f \), the sum of whose radii is at most \( \eta R \). Here

\[
Q(r) = r \int_r^\infty \frac{n(t,1/f)}{t^2} dt \quad \text{and} \quad N(r) = \int_0^r \frac{n(t,1/f)}{t} dt.
\]

**Lemma 2.5.** Let \( f \) be a transcendental entire function with

\[
T(r,f) = O^*((\log r)^\beta e^{(\log r)\alpha}) \quad (0 < \alpha < 1, \beta > 0).
\]

Then

1. \( T(r,f) \sim \log M(r,f) \quad (r \to \infty, r \notin E) \),
2. \( T(\sigma r,f) \sim T(r,f) \quad (r \to \infty, \sigma \geq 2, r \notin E) \),

where \( E \) is a set of finite logarithmic measure.

**Lemma 2.6.** Let \( f \) be a transcendental entire function with \( T(r,f) = O^*((\log r)^\beta) \) where \( \beta > 1 \). Then

1. \( T(r,f) \sim \log M(r,f) \quad (r \to \infty, r \notin E) \),
2. \( T(\sigma r,f) \sim T(r,f) \quad (r \to \infty, \sigma \geq 2, r \notin E) \),

where \( E \) is a set of finite logarithmic measure.

**Proof.** Without loss of generality we may assume that \( f(0) = 1 \), otherwise we set \( F(z) = f(z) - f(0) + 1 \).

By Jensen’s theorem,

\[
N(r,1/f) = \int_0^r \frac{n(t,1/f)}{t} dt = 1 \int_0^{2\pi} \log |f(re^{i\theta})|d\theta \leq \log M(r,f)
\]

and so,

\[
n(r,1/f) \log A \leq \int_r^{Ar} \frac{n(t,1/f)}{t} dt \leq \int_0^{Ar} \frac{n(t,1/f)}{t} dt \leq \log M(Ar,f),
\]

for \( r > 1 \) and \( A > 1 \).

Therefore

\[
n(r,1/f) \leq \frac{\log M(Ar,f)}{\log A}. \quad (2.1)
\]

Since \( T(r,f) = O^*((\log r)^\beta), \beta > 1 \), by Lemma 2.1 we have

\[
\log M(r,f) = O^*((\log r)^\beta). \quad (2.2)
\]

Take \( A = r^{\sigma(r)} \) and \( \sigma(r) = \frac{1}{(\log r)^\beta} \). Then by (2.1) we have

\[
n(r,1/f) \leq \frac{\log M(r^{1+\sigma(r)},f)}{\sigma(r) \log r}. \quad (2.3)
\]
Therefore, putting \( r = e^u \) we have

\[
\frac{(\log r^{1+\sigma(r)})^\beta}{r^{1/2}\sigma(r)\log r} = \frac{(1 + \sigma(r))^\beta \log r}{r^{1/2}\sigma(r)\log r}
\]

\[
= \frac{(1 + \frac{1}{u^\beta})^\beta u^\beta}{e^{u/2}u^{1-1/2}u^\beta}
\]

\[
= \frac{(1 + \frac{1}{u^\beta})^\beta}{e^{u/2}e(1/2-\beta)\log u}
\]

\[
= \frac{(1 + \frac{1}{u^\beta})^\beta}{e^{u/2}(\beta-1/2)\log u}.
\]

(2.4)

Since \( \beta > 1 \), for sufficiently large values of \( u \) we have \( \frac{u}{2} - (\beta - 1/2)\log u > 0 \) and \( \frac{u}{2} - (\beta - 1/2)\log u \) increases. By (2.4) for sufficiently large value of \( r \), \( \frac{(\log r^{1+\sigma(r)})^\beta}{r^{1/2}\sigma(r)\log r} \) decreases.

From Lemma 2.4 using (2.2) and (2.3), we have

\[
Q(r) = r \int_r^\infty n(t,1/f)dt
\]

\[
\leq r \int_r^\infty \frac{\log M(t^{1+\sigma(t)},f)}{t^2\sigma(t)\log t}dt
\]

\[
= r \int_r^\infty \frac{\log (t^{1+\sigma(t)})^\beta}{t^2\sigma(t)\log t}dt
\]

\[
\leq O^* \left( r \int_r^\infty \frac{\log t^{1+\sigma(t)}\beta}{t^2\sigma(t)\log t}dt \right)
\]

\[
\leq \frac{r^{1/2}O^* \left( \log r^{1+\sigma(r)}\beta \right)}{\sigma(r)\log r} \int_r^\infty t^{-3/2}dt
\]

\[
= \frac{2O^* \left( \log r^{1+\sigma(r)}\beta \right)}{\sigma(r)\log r}
\]

\[
= \frac{2 \log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}.
\]

Therefore

\[
\frac{Q(r)}{\log M(r,f)} \leq \frac{2 \log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r} \frac{\log M(r,f)}{\log M(r,f)}
\]

\[
\leq \frac{2K_2(\log r^{1+\sigma(r)})^\beta}{\sigma(r)\log r} K_1(\log r)^\beta,
\]

for some suitable constants \( K_1 \) and \( K_2 \)

\[
= \frac{2K_2(1+\sigma(r))^\beta \log r^\beta}{K_1 \sigma(r)\log r (\log r)^\beta}
\]

\[
= \frac{2K_2(1+\sigma(r))^\beta}{K_1 \sigma(r)\log r}
\]

\[
\to 0 \text{ as } r \to \infty.
\]
So
\[ Q(r) = o(\log M(r, f)) \] (2.5)
Since \( T(r, f) = O^*(\log r)^\delta \), \( n(r, 1/f) = o(r) \).
The concluding part of the proof of the lemma is similar to that of Lemma 5 of J. Sun [7]. But still for the sake of completeness and for convenience of readers, we outline the proof.

\[
\log M(r, f) \leq \log \prod_{n=1}^{\infty} (1 + r/t)dn(t, 1/f) \\
\leq \int_0^\infty rdn(t, 1/f) \\
= r \int_0^\infty \frac{n(t, 1/f)}{t(t+r)} dt \\
= r \left( \int_0^r + \int_r^\infty \right) \frac{n(t, 1/f)}{t(t+r)} dt \\
\leq \frac{1}{r} r \int_0^r \frac{n(t, 1/f)}{t} dt + r \int_r^\infty \frac{n(t, 1/f)}{t^2} dt \\
= N(r) + Q(r) 
\] (2.6)

So, from Lemma 2.4 and (2.5), (2.6) we have

\[
\log |f(re^{i\theta})| > N(2R) - kQ(2R) \quad (\zeta R \leq r \leq R, \ r \notin E) \\
= N(2R) + Q(2R) - (k+1)Q(2R) \\
\geq \log M(2R, f) + (k+1)o(\log M(2R, f)) \\
= \log M(2R, f)(1 - o(1)) \\
\geq \log M(r, f)(1 - o(1)) 
\] (2.7)

where \( E \) is a set of finite logarithmic measure.

On the other hand

\[
\log |f(z)| \leq \log M(r, f) \leq \log M(\sigma r, f) \quad (|z| = r, \sigma \geq 2, r) 
\] (2.9)

Let \( 2R = \sigma r, \sigma \geq 2 \) then from (2.7), (2.8) and (2.9) we have,

\[
\log |f(z)| \sim \log M(\sigma r, f) \quad (r \to \infty, \sigma \geq 2, r \notin E) 
\] (2.10)

and

\[
\log |f(z)| \sim \log M(r, f) \quad (r \to \infty, r \notin E). 
\] (2.11)

From (2.11) for sufficiently large value of \( r \), we have,

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log M(r, f)(1 + o(1))d\theta \\
= \log M(r, f)(1 + o(1)) \quad (r \to \infty, r \notin E).
\]

So,

\[
\lim_{r \to \infty} \frac{T(r, f)}{\log M(r, f)} = 1, \quad (r \notin E) 
\]
i.e.

\[
T(r, f) \sim \log M(r, f), \quad (r \notin E). 
\] (2.12)
From (2.10) and (2.11) we have
\[ \log M(r, f) \sim \log M(\sigma r, f) \quad (r \to \infty, \sigma \geq 2, r \notin E). \] (2.13)
From (2.12) and (2.13) we have
\[ T(\sigma r, f) \sim T(r, f) \quad (r \to \infty, \sigma \geq 2, r \notin E). \] (2.14)
From (2.12) and (2.14) we have the required result.
This proves the lemma.

**Lemma 2.7.** Let \( f_1 \) and \( f_2 \) be two entire functions with \( T(r, f_1) = O^*(\log r)^\beta \) where \( \beta > 1 \) and \( T(r, f_1) \sim T(r, f_2) \) then \( M(r, f_1) \sim M(r, f_2) \).

**Proof.** From Lemma 2.6 we have,
\[ \log M(r, f_1) \sim T(r, f_1) \sim T(r, f_2) \sim \log M(r, f_2) \quad (r \to \infty, r \notin E) \]
where \( E \) is a set of finite logarithmic measure.
Since \( \log M(r, f_1) \sim \log M(r, f_2) \), so for given \( \epsilon > 0 \), there exist \( r_1, r_2 > 0 \) such that
\[ \frac{\log M(r, f_1)}{\log M(r, f_2)} < 1 + \frac{\log(1 + \epsilon)}{\log M(r, f_2)} \quad \text{for } r > r_1 \] (2.15)
and
\[ \frac{\log M(r, f_2)}{\log M(r, f_1)} < 1 + \frac{\log(1 + \epsilon)}{\log M(r, f_1)} \quad \text{for } r > r_2 \] (2.16)
Now from (2.15) we have
\[ \log M(r, f_1) < \log M(r, f_2) + \log(1 + \epsilon). \]
So,
\[ \frac{M(r, f_1)}{M(r, f_2)} < 1 + \epsilon \quad \text{for } r > r_1. \] (2.17)
Similarly from (2.16)
\[ \frac{M(r, f_2)}{M(r, f_1)} < 1 + \epsilon \quad \text{for } r > r_2. \]
i.e.
\[ \frac{M(r, f_1)}{M(r, f_2)} > 1 - \epsilon \quad \text{for } r > r_2. \] (2.18)
From (2.17) and (2.18) we have
\[ 1 - \epsilon < \frac{M(r, f_1)}{M(r, f_2)} < 1 + \epsilon \quad \text{for } r > r_0 = \max \{r_1, r_2\}. \]
So, \( M(r, f_1) \sim M(r, f_2) \).
This proves the lemma.

**Lemma 2.8.** Let \( f_1 \) and \( f_2 \) be two entire functions with \( T(r, f_1) = O^*(\log r)^\nu e^{(\log r)^\alpha} \) where \( \nu > 0 \) and \( 0 < \alpha < 1 \) and \( T(r, f_1) \sim T(r, f_2) \) then \( M(r, f_1) \sim M(r, f_2) \).

**Proof.** From Lemma 2.5 we have,
\[ \log M(r, f_1) \sim T(r, f_1) \sim T(r, f_2) \sim \log M(r, f_2) \quad (r \to \infty, r \notin E) \]
where \( E \) is a set of finite logarithmic measure and concluding part follows from Lemma 2.7.

□
3. Theorems

In [6] K. Niiio and N. Suita proved the following theorem.

**Theorem 3.1.** Let \( f(z) \) and \( g(z) \) be entire functions. If \( M(r, g) > \frac{2+\epsilon}{\epsilon} |g(0)| \) for any \( \epsilon > 0 \), then we have

\[
T(r, f(g)) \leq (1 + \epsilon)T(M(r, g), f).
\]

In particular, if \( g(0) = 0 \), then

\[
T(r, f(g)) \leq T(M(r, g), f)
\]

for all \( r > 0 \).

The following theorem is the generalization of the above.

**Theorem 3.2.** Let \( f(z) \) and \( g(z) \) be two entire functions. Then we have

\[
T(R_2, f) \leq T(r, f_n) \leq T(R_3, f)
\]

where \( |f(z)| > R_1 > \frac{2+\epsilon}{\epsilon} |f(0)| \) and \( |g(z)| > R_2 > \frac{2+\epsilon}{\epsilon} |g(0)| \), \( R_3 = \max \{M_{f_{n-1}}(r), M_{g_{n-1}}(r)\} \) for sufficiently large values of \( r \) and any integer \( n \geq 2 \).

**Proof.** By Theorem 3.1 we have for odd \( n \) and any \( \epsilon > 0 \) arbitrary small

\[
T(r, f_n) = T(r, f_{n-1}(f)) \leq (1 + \epsilon)T(M(r, f), f_{n-1}) \leq (1 + \epsilon)T(M_{f_1}(r), f_{n-2}(g)) \leq (1 + \epsilon)^2T(M_{f_2}(r), f_{n-3}(f)) \leq (1 + \epsilon)^nT(M_{f_n}(r), f).
\]

Similarly when \( n \) is even, we have

\[
T(r, f_n) = T(r, f_{n-1}(g)) \leq (1 + \epsilon)T(M(r, g), f_{n-1}) \leq (1 + \epsilon)T(M_{g_1}(r), f_{n-2}(f)) \leq (1 + \epsilon)^2T(M_{g_2}(r), f_{n-3}(f)) \leq (1 + \epsilon)^nT(M_{g_{n-1}}(r), f).
\]

Therefore

\[
T(r, f_n) \leq (1 + \epsilon)^nT(R_3, f) \text{ for any integer } n \geq 2.
\]

Since \( \epsilon > 0 \) was arbitrary, we have for sufficiently large values of \( r \)

\[
T(r, f_n) \leq T(R_3, f). \tag{3.2}
\]
Also using Lemma 2.3 we have for odd \( n \)
\[
T(r, f_n) = T(r, f_{n-1}(f)) \\
\geq \left( \frac{R_1 - |f(0)|}{R_1 + |f(0)|} \right) T(R_1, f_{n-1}) \\
> (1 - \epsilon)T(R_1, f_{n-2}(g)) \\
\geq (1 - \epsilon) \left( \frac{R_2 - |g(0)|}{R_2 + |g(0)|} \right) T(R_2, f_{n-2}) \\
> (1 - \epsilon)^2T(R_2, f_{n-2}) \\
\geq (1 - \epsilon)^3T(R_1, f_{n-3}) \\
\vdots \\
\geq (1 - \epsilon)^n T(R_1, f(g)) \\
\geq (1 - \epsilon)^{n-1}T(R_2, f).
\]

Similarly when \( n \) is even we obtain
\[
T(r, f_n) = T(r, f_{n-1}(g)) \\
\geq \left( \frac{R_2 - |g(0)|}{R_2 + |g(0)|} \right) T(R_2, f_{n-1}) \\
> (1 - \epsilon)T(R_2, f_{n-2}(f)) \\
\vdots \\
\geq (1 - \epsilon)^n T(R_1, f(g)) \\
\geq (1 - \epsilon)^{n-1}T(R_2, f).
\]

So,
\[
T(r, f_n) \geq (1 - \epsilon)^{n-1}T(R_2, f).
\]

Since \( \epsilon > 0 \) was arbitrary, we have for sufficiently large values of \( r \)
\[
T(r, f_n) \geq T(R_2, f). \tag{3.3}
\]

Hence from (3.2) and (3.3) we obtain (3.1).

This proves the theorem. \( \square \)

4. PROOF OF THE THEOREM 1.2

Proof. From Theorem 3.2 we have
\[
T(R_1, f) \leq T(r, f_n) \leq T(R_2, f) \tag{4.1}
\]
\[
T(R_1', u) \leq T(r, u_n) \leq T(R_2', u) \tag{4.2}
\]
and choose \( R_1 \) and \( R_1' \) in such way that \( |g(z)| > R_1 > \frac{2 + \epsilon}{\epsilon} |g(0)|, \ |v(z)| > R_1' > \frac{2 + \epsilon}{\epsilon} |v(0)| \) and \( T(R_1, f) \sim T(R_1', f) \), where \( R_2 = \max\{M_{f_n-1}(r), \ M_{g_n-1}(r)\} \) and \( R_2' = \max\{M_{u_n-1}(r), \ M_{v_n-1}(r)\} \) for sufficiently large value of \( r \) and arbitrary small \( \epsilon > 0 \).

Since \( T(r, f) \sim T(r, u) \), so
\[
T(R_1, f) \sim T(R_1', f) \sim T(R_1', u) \\
i.e. \ T(R_1, f) \sim T(R_1', u) \ (r \to \infty, r \notin E). \tag{4.3}
\]
Also from Lemma 2.8 we have $M(r, f) \sim M(r, u)$.
So,
$$M(M(r, f), g) \sim M(M(r, u), v) \quad (r \to \infty), \text{ using Lemma 2.2}$$
i.e. $M(M(M(r, f), g), f) \sim M(M(M(r, u), v), u) \quad (r \to \infty)$. 

Finally, for odd $n$,
$$M_{f_{n-1}}(r) \sim M_{u_{n-1}}(r) \quad (r \to \infty). \quad (4.4)$$
Similarly, for even $n$,
$$M_{g_{n-1}}(r) \sim M_{v_{n-1}}(r) \quad (r \to \infty). \quad (4.5)$$

From (4.4) and (4.5) for any integer $n \geq 2$, we have $R_2 \sim R_2'$ for large $r$.

So from $T(r, f) \sim T(r, u)$ and $R_2 \sim R_2'$ we have
$$T(R_2, u) \sim T(R'_2, f) \quad (r \to \infty) \quad (4.6)$$

So from (4.1), (4.2), (4.3) and (4.6) we have $T(r, f_n) \sim T(r, u_n)$.
This proves the theorem. \hfill \Box

**Theorem 4.1.** Let $f, g$ and $u, v$ be four transcendental entire functions with $T(r, f) \sim T(r, u)$, $T(r, g) \sim T(r, v)$, $T(r, f) = O^*((\log r)^\beta)$ and $T(r, g) = O^*((\log r)^\beta)$ where $\beta > 1$ is a constant, then $T(r, f_n) \sim T(r, u_n)$.

**Note 4.2.** The conditions of Theorem 1.2 and Theorem 4.1 are not strictly sharp. Which are illustrated by the following examples.

**Example 4.3.** Let $f(z) = e^z, g(z) = z$ and $u(z) = 2e^z, v(z) = 2z$. Then we have $f_2 = f(g) = e^z, u_2 = u(v) = 2e^{2z}$ and $f_4 = f(g(f(g))) = e^{e^z}, u_4 = u(v(u(v))) = 2e^{4e^{2z}}$.

Also
$$T(r, f) = \frac{r}{\pi}, \quad T(r, u) = \frac{r}{\pi} + \log 2,$$
$$T(r, g) = \log r, \quad T(r, v) = \log r + \log 2,$$
$$T(r, f_2) = \frac{r}{\pi}, \quad T(r, u_2) = \frac{2r}{\pi} + \log 2.$$ 

Thus
$$T(r, f) \sim T(r, u), T(r, g) \sim T(r, v) \quad (r \to \infty).$$

But
$$\frac{T(r, f_2)}{T(r, u_2)} = 2 \quad \text{as} \quad r \to \infty,$$
so
$$T(r, f_2) \sim T(r, u_2).$$

Also
$$T(r, f_4) \leq \log M(r, f_4) = e^r$$
and
$$3T(2r, u_4) \geq \log M(r, u_4) = \log 2 + 4e^{2r}$$
i.e. $T(r, u_4) \geq \frac{1}{3} \log 2 + \frac{4}{3} e^r$
i.e. $\frac{1}{T(r, u_4)} \leq \frac{1}{3} \log 2 + \frac{4}{3} e^r$. 

Therefore
\[ \frac{T(r, f_4)}{T(r, u_4)} \leq \frac{e^r}{\frac{1}{3} \log 2 + \frac{4}{3} e^r} = 3/4 \text{ as } r \to \infty, \]
so
\[ T(r, f_4) \sim T(r, u_4). \]
Thus, \( T(r, f_n) \sim T(r, u_n) \) does not hold for all \( n \geq 2 \). Here \( T(r, f) \neq O^*((\log r)^\beta) \) where \( \beta > 1 \) is a constant.

**Example 4.4.** Let \( f(z) = e^z, g(z) = \log z \) and \( u(z) = 2e^z, v(z) = \log 2z \). Then we have
\[
\begin{align*}
  f_2 &= f(g) = z, u_2 = u(v) = 4z, \\
  f_3 &= f(g(f)) = e^z, u_3 = u(v(u)) = 8e^z, \\
  f_4 &= f(g(f(g))) = z, u_4 = u(v(u(v))) = 16z.
\end{align*}
\]
Here
\[ T(r, f) = \frac{r}{\pi}, \quad T(r, u) = \frac{r}{\pi} + \log 2, \]
\[ \therefore T(r, f) \sim T(r, u) \quad (r \to \infty). \]

Also
\[ T(r, g) \leq \log \log r, \]
and
\[
\begin{align*}
  3T(2r, v) &\geq \log \log 2r \\
  \text{i.e.} \quad T(r, v) &\geq \frac{\log \log r}{3} \\
  \text{i.e.} \quad \frac{1}{T(r, v)} &\leq \frac{3}{\log \log r}.
\end{align*}
\]
So
\[ \frac{T(r, g)}{T(r, v)} \leq 3. \]
Again
\[ T(r, v) \leq \log \log 2r, \]
and
\[
\begin{align*}
  3T(2r, g) &\geq \log \log r \\
  \text{i.e.} \quad T(r, g) &\geq \frac{\log \log r/2}{3} \\
  \text{i.e.} \quad \frac{1}{T(r, g)} &\leq \frac{3}{\log \log r/2}.
\end{align*}
\]
So
\[ \frac{T(r, v)}{T(r, g)} \leq \frac{3}{\log \log r/2} \leq 3 \quad \text{as } r \to \infty. \]
\[ \therefore \quad \frac{1}{3} \leq \frac{T(r, g)}{T(r, v)} \leq 3 \quad \text{as } r \to \infty. \]
Also

\[ T(r, f_2) = \log r, \quad T(r, u_2) = \log r + \log 4, \]
\[ T(r, f_3) = \frac{r}{\pi}, \quad T(r, u_3) = \frac{r}{\pi} + \log 8, \]
\[ T(r, f_4) = \log r, \quad T(r, u_4) = \log r + \log 16. \]

Here \( T(r, g) \sim T(r, v) \). But still \( T(r, f_n) \sim T(r, u_n) \) for \( n = 2, 3, 4 \).

**Example 4.5.** Let \( f(z) = e^z, g(z) = (\log z)^2 \) and \( u(z) = 2e^z, v(z) = (\log 2z)^2. \)

Then we have

\[
\begin{align*}
  f_2 &= f(g) = e^{(\log z)^2}, u_2 = u(v) = 2e^{(\log 2z)^2}, \\
  f_3 &= f(g(f)) = e^z, u_3 = u(v(u)) = 2e^{(\log 4)^2}e^2z^2, \\
  f_4 &= f(g(f(g))) = e^{(\log z)^4}, u_4 = u(v(u(v))) = 2e^{(\log 4)^2}4^{2(\log 2z)^2}e^{(\log 2z)^4}, \\
  f_5 &= f(g(f(g(f)))) = e^{z^2}, u_5 = u(v(u(v(v)))) = 32e^{(\log 4)^2}4^{2(\log 4e^z)^2}e^{(\log 4e^z)^4}.
\end{align*}
\]

Also

\[ T(r, f) = \frac{r}{\pi}, \quad T(r, u) = \frac{r}{\pi} + \log 2, \]
\[ \therefore \quad T(r, f) \sim T(r, u). \]

and

\[ \frac{1}{3} \leq \frac{T(r, g)}{T(r, v)} \leq 3 \quad \text{as} \quad r \to \infty. \]

Here \( T(r, f) \neq O^*((\log r)^{\beta}) \) where \( \beta > 1 \) is a constant and \( T(r, g) \sim T(r, v) \). But

\[ T(r, f_2) = (\log r)^2 \quad \text{and} \quad T(r, u_2) = (\log r)^2 + 2\log 2\log r + (\log 2)^2 + \log 2 \]

so

\[ T(r, f_2) \sim T(r, u_2) \quad \text{as} \quad r \to \infty, \]

and

\[ T(r, f_3) = \frac{r^2}{\pi} \quad \text{and} \quad T(r, u_3) = \log 2 + (\log 4)^2 + 2r \log 4 + \frac{r^2}{\pi}, \]

so

\[ T(r, f_3) \sim T(r, u_3) \quad \text{as} \quad r \to \infty, \]

and

\[ T(r, f_4) = (\log r)^4 \quad \text{and} \quad T(r, u_4) = \log 2 + (\log 4)^2 + O(\log r)^2 + (\log 2r)^4, \]

so

\[ T(r, f_4) \sim T(r, u_4) \quad \text{as} \quad r \to \infty, \]

and

\[ T(r, f_5) = \frac{r^4}{\pi} \quad \text{and} \quad T(r, u_5) = \log 2 + (\log 4)^2 + O(r^2) + \frac{r^4}{\pi}, \]

so

\[ T(r, f_5) \sim T(r, u_5) \quad \text{as} \quad r \to \infty, \]

and so on.

**Acknowledgments.** The authors are thankful to the referee for several suggestions which considerably improve the presentation of the paper.
References


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