A QUADRATIC TYPE FUNCTIONAL EQUATION

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, the solution and the Hyers–Ulam stability of the following quadratic type functional equation

\[ \sum_{i=1}^{k} \sum_{\varepsilon_i \in \{-1,1\}} f(x_1 + \varepsilon_i x_i) = 2(k-1)f(x_1) + 2 \sum_{i=2}^{k} f(x_i) \]

is investigated.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: “When is it true that a function which approximately satisfies a functional equation \( E \) must be close to an exact solution of \( E \)’?” If there exists an affirmative answer, we say that the equation \( E \) is stable [9]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [10, 9, 21] and monographs [11, 12, 8] and references therein.

Let \( X \) and \( Y \) be normed spaces. A function \( f: X \to Y \) satisfying the functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (x, y \in X) \quad (1.1) \]

is called the quadratic functional equation. It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function \( B \) such that \( f(x) = B(x, x) \) for all \( x \in X \); see [9]. The bi-additive function \( B \) is given by

\[ B(x, x) = \frac{1}{4} (f(x + y) - f(x - y)). \]

The Hyers–Ulam stability of the quadratic equation (1.1) was proved by Skof [22]. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an abelian group. Furthermore, Czerwik [7] deal with stability problem of the quadratic functional equation (1.1) in the spirit of Hyers–Ulam–
Rassias. Also, Jung [13] proved the stability of (1.1) on a restricted domain. For more information on the stability of the quadratic equation, we refer the reader to [2, 3, 16, 4, 14].

**Theorem 1.1.** (Czerwik) Let \( \varepsilon \geq 0 \) be fixed. If a mapping \( f : \mathcal{X} \rightarrow \mathcal{Y} \) satisfies the inequality
\[
\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon \quad (x \in \mathcal{X}) \tag{1.2}
\]
then there exists a unique quadratic mapping \( Q : \mathcal{X} \rightarrow \mathcal{Y} \) such that
\[
\|f(x) - Q(x)\| \leq \frac{1}{2} \varepsilon \quad (x \in \mathcal{X}).
\]
Moreover, if \( f \) is measurable or if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in \mathcal{X} \), then \( Q(tx) = t^2Q(x) \) for all \( x \in \mathcal{X} \) and \( t \in \mathbb{R} \).

The Hyers–Ulam stability of equation (1.1) on a certain restricted domain was investigated by Jung [13] in the following theorem,

**Theorem 1.2.** (Jung) Let \( d > 0 \) and \( \varepsilon \geq 0 \) be given. Assume that a mapping \( f : \mathcal{X} \rightarrow \mathcal{Y} \) satisfies the inequality (1.2) for all \( x, y \in \mathcal{X} \) with \( \|x\| + \|y\| \geq d \). Then there exists a unique quadratic mapping \( Q : \mathcal{X} \rightarrow \mathcal{Y} \) such that
\[
\|f(x) - Q(x)\| \leq \frac{7}{2} \varepsilon \quad (x \in \mathcal{X}). \tag{1.3}
\]
If, moreover, \( f \) is measurable or \( f(tx) \) is continuous in \( t \) for each fixed \( x \in \mathcal{X} \) then \( Q(tx) = t^2Q(x) \) for all \( x \in \mathcal{X} \) and \( t \in \mathbb{R} \).

The quadratic functional equation was used to characterize the inner product spaces [1]. A square norm on an inner product space satisfies the important parallelogram equality
\[
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).
\]
It was shown by Moslehian and Rassias [19] that a normed space \((\mathcal{X}, \|\cdot\|)\) is an inner product space if and only if for any finite set of vectors \(x_1, x_2, \ldots, x_k \in \mathcal{X}\),
\[
\sum_{\varepsilon_j \in \{-1,1\}} \left\| x_1 + \sum_{i=2}^{k} \varepsilon_j x_i \right\|^2 = \sum_{\varepsilon_j \in \{-1,1\}} \left( \|x_1\| + \sum_{i=2}^{k} \varepsilon_j \|x_i\| \right)^2. \tag{1.4}
\]
Motivated by (1.4), we introduce the following functional equation deriving from the quadratic function
\[
\sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1,1\}} f(x_1 + \varepsilon_j x_i) = 2(k-1)f(x_1) + 2\sum_{i=2}^{k} f(x_i), \tag{1.5}
\]
where \( k \geq 2 \) is a fixed integer. It is easy to see that the function \( f(x) = x^2 \) is a solution of functional equation (1.5).

2. **Solution of the equation (1.5)**

**Theorem 2.1.** A mapping \( f : \mathcal{X} \rightarrow \mathcal{Y} \) satisfies the equation (1.5) for all \( x_1, x_2, \ldots, x_k \in \mathcal{X} \) if and only if \( f \) is quadratic.
Proof. If we replace \( x_1, x_2, \ldots, x_k \) in \([1.5]\) by 0, then we get \( f(0) = 0 \). Putting \( x_3 = x_4 = \cdots = x_k = 0 \) in the equation \((1.5)\) we see that
\[
f(x_1 - x_2) + f(x_1 + x_2) + 2(k - 2)f(x_1) = 2(k - 1)f(x_1) + 2f(x_2).
\]
Hence \( f(x_1 - x_2) + f(x_1 + x_2) = 2f(x_1) + 2f(x_2) \). The converse is trivial. 

Remark. We can prove the theorem above on the punching space \( X - \{0\} \). If we consider \( x_2 = x_3 = \cdots = x_k \), then we observe that
\[
\sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1, 1\}} f(x_1 + \varepsilon_j x_2) = 2(k - 1)f(x_1) + 2 \sum_{i=2}^{k} f(x_i),
\]
whence
\[
(k - 1)(f(x_1 - x_2) + f(x_1 + x_2)) = 2(k - 1)f(x_1) + 2(k - 1)f(x_2).
\]
Hence \( f \) is quadratic.

3. Stability Results

Throughout this section, let \( X \) and \( Y \) be normed and Banach spaces also, we prove the Hyers–Ulam stability of equation \((1.5)\). From now on, we use the following abbreviation
\[
\mathcal{D} f(x_1, x_2, \ldots, x_k) = \sum_{i=2}^{k} \sum_{\varepsilon_j \in \{-1, 1\}} f(x_1 + \varepsilon_j x_i) - 2(k-1)f(x_1) - 2 \sum_{i=2}^{k} f(x_i).
\] (3.1)

Theorem 3.1. Let \( \varepsilon \geq 0 \) be fixed. If a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies
\[
\|\mathcal{D} f(x_1, x_2, \ldots, x_k)\| \leq \varepsilon
\] (3.2)
for all \( x_1, x_2, \ldots, x_k \in X \), then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \frac{1}{2} \varepsilon.
\]
Moreover, if \( f \) is measurable or if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in X \), then \( Q(tx) = t^2 Q(tx) \) for all \( x \in X \) and \( t \in \mathbb{R} \).

Proof. It is enough to put \( x_3 = x_4 = \cdots = x_k = 0 \) in \((3.2)\) and use Theorem 1.1. 

By using an idea from the paper [13], we will prove the Hyers–Ulam stability of equation \((1.5)\) on a restricted domain.

Theorem 3.2. Let \( d > 0 \) and \( \varepsilon \geq 0 \) be given. Suppose that a mapping \( f : X \to Y \) satisfies the inequality \((3.2)\) for all \( x_1, x_2, \ldots, x_k \in X \) with \( \|x_1\| + \|x_2\| + \cdots + \|x_k\| \geq d \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \frac{3 + 2k}{2} \varepsilon
\] (3.3)
for all \( x \in X \). Moreover, if \( f \) is measurable or if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in X \), then \( Q(tx) = t^2 Q(tx) \) for all \( x \in X \) and \( t \in \mathbb{R} \).
Proof. Assume \( \|x_1\| + \|x_2\| + \cdots + \|x_k\| < d \). If \( x_1 = x_2 = \cdots = x_k = 0 \), then we chose a \( t \in \mathcal{X} \) with \( \|t\| = d \). Otherwise, let \( t = (1 + \frac{d}{\|x_{i_0}\|})x_{i_0} \), where \( \|x_{i_0}\| = \max\{\|x_j\| : 1 \leq j \leq k\} \). Clearly, we see that

\[
\begin{align*}
\|x_1 - t\| + \|x_2 + t\| + \cdots + \|x_k + t\| & \geq d \\
\|x_1 + t\| + \|x_2 + t\| + \cdots + \|x_k + t\| & \geq d \\
\|x_1\| + \|x_2 + 2t\| + \cdots + \|x_k + 2t\| & \geq d \\
\|x_2 + t\| + \|x_3 + t\| + \cdots + \|x_k + t\| + \|t\| & \geq d \\
\|x_1\| + \|t\| & \geq d,
\end{align*}
\]

since \( \|x_j + t\| \geq d \) and \( \|x_j + 2t\| \geq d \), for \( 1 \leq j \leq k \).

From (3.2) and (3.4) and the relations

\[
f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2) = f(x_1 + x_2) + f(x_1 - x_2 - 2t) - 2f(x_1 - t) - 2f(x_2 + t) + f(x_1 + x_2 + 2t) + f(x_1 - x_2 - 2t) - 2f(x_1 - t) - 2f(x_2 + t) - 2f(x_2 + 2t) - 2f(x_2 + 4(t + 2f(x_2 + t) + 4f(t)) + 2f(x_1 + x_2 + 2t) - 2f(x_1 - x_2 - 2t) - 2f(x_1 - t) - 4f(x_1) - 4f(t)
\]

we get

\[
\|Df(x_1, x_2, \cdots, x_k)\| \leq \left\| \sum_{i=2}^{k} \sum_{\epsilon_j \in \{-1, 1\}} f(\alpha_1 + \epsilon_j \alpha_i) - 2(k - 1)f(\alpha_1) - 2 \sum_{i=2}^{k} f(\alpha_i) \right\| + \left\| \sum_{i=2}^{k} \sum_{\epsilon_j \in \{-1, 1\}} f(\beta_1 + \epsilon_j \beta_i) - 2(k - 1)f(\beta_1) - f(\beta_i) \right\| + \left\| \sum_{i=2}^{k} \sum_{\epsilon_j \in \{-1, 1\}} f(\gamma_1 + \epsilon_j \gamma_i) - 2(k - 1)f(\gamma_1) - 2 \sum_{i=2}^{k} f(\gamma_i) \right\| + \left\| \sum_{i=2}^{k} \sum_{\epsilon_j \in \{-1, 1\}} f(\theta_1 + \epsilon_j \theta_i) - 2(k - 1)f(\theta_1) - 2 \sum_{i=2}^{k} f(\theta_i) \right\| + \left\| \sum_{i=2}^{k} \sum_{\epsilon_j \in \{-1, 1\}} f(\eta_1 + \epsilon_j \eta_i) - 2(k - 1)f(\eta_1) - 2 \sum_{i=2}^{k} f(\eta_i) \right\|,
\]

where

\[
\begin{align*}
\alpha_1 &= x_1 - t, & \alpha_i &= x_i + t, & 2 \leq i \leq k \\
\beta_1 &= x_1 + t, & \beta_i &= x_i + t, & 2 \leq i \leq k \\
\gamma_1 &= t, & \gamma_i &= x_i + t, & 2 \leq i \leq k \\
\theta_1 &= x_1, & \theta_i &= x_i + 2t, & 2 \leq i \leq k \\
\eta_1 &= x_1, & \eta_i &= x_i + t, & 2 \leq i \leq k.
\end{align*}
\]
Theorem 4.1. Let $\text{Moslehian et al.}$ [18].

Hence we have
\[
\|Df(x_1, x_2, \cdots, x_k)\| \leq \|Df(\alpha_1, \alpha_2, \cdots, \alpha_k)\| + \|Df(\beta_1, \beta_2, \cdots, \beta_k)\| + 2\|Df(\gamma_1, \gamma_2, \cdots, \gamma_k)\| + \|Df(\theta_1, \theta_2, \cdots, \theta_k)\| + 2(k - 1)\|Df(\eta_1, \eta_2, \cdots, \eta_k)\|
\]
\[
\leq (3 + 2k)\varepsilon. \tag{3.5}
\]

Obviously, inequality (3.2) holds for all $x, y \in X$. According to (3.5) and Theorem 3.1, there exists a unique quadratic mapping $Q : X \rightarrow Y$ which satisfies the inequality (3.3) for all $x_1, x_2, \cdots, x_k \in X$.

Now we study asymptotic behavior of function equation (1.5).

**Theorem 3.3.** Suppose that $f : X \rightarrow Y$ is a mapping. Then $f$ is quadratic if and only if for $k \in \mathbb{N}$ ($k \geq 2$)
\[
\|Df(x_1, x_2, \cdots, x_k)\| \rightarrow 0 \tag{3.6}
\]
as $\|x_1\| + \|x_2\| + \cdots + \|x_k\| \rightarrow \infty$.

**Proof.** If $f$ is quadratic then (3.6) evidently holds. Conversely, by using the limits (3.6) we can find for every $n \in \mathbb{N}$ a sequence $\varepsilon_n$ such that $\|Df(x_1, x_2, \cdots, x_k)\| \leq \frac{1}{n}$ for all $x_1, x_2, \cdots, x_k \in X$ with $\|x_1\| + \|x_2\| + \cdots + \|x_k\| \geq \varepsilon_n$.

By Theorem 3.2 for every $n \in \mathbb{N}$ there exists a unique quadratic mapping $Q_n$ such that
\[
\|f(x) - Q_n(x)\| \leq \frac{3 + 2k}{2n} \tag{3.7}
\]
for all $x \in X$. Since $\|f(x) - Q_1(x)\| \leq \frac{3 + 2k}{2n}$ and $\|f(x) - Q_n(x)\| \leq \frac{3 + 2k}{2n} \leq \frac{3 + 2k}{2}$, by the uniqueness of $Q_1$ we conclude that $Q_n = Q_1$ for all $n \in \mathbb{N}$. Now, by tending $n$ to the infinity in (3.7) we deduce that $f = Q_1$. Therefore $f$ is quadratic. \qed

4. Stability on bounded domains

Throughout this section, we denote by $B_r(0)$ the closed ball of radius $r$ around the origin and $B_r = B_r(0) - \{0\}$. In this section we used some ideas from the paper’s Moslehian et al. [18].

**Theorem 4.1.** Let $X$ and $Y$ be normed and Banach spaces $p > 2, r > 0, \varphi : X^k \rightarrow [0, \infty) (k \geq 2)$ be a function such that $\varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_k}{2}) \leq \frac{1}{2^p} \varphi(x_1, x_2, \cdots, x_k)$ for all $x_1, x_2, \cdots, x_k \in B_r$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$ and
\[
\|Df(x_1, x_2, \cdots, x_k)\| \leq \varphi(x_1, x_2, \cdots, x_k) \tag{4.1}
\]
for all $x_1, x_2, \cdots, x_k \in B_r$ with $x_i \pm x_j \in B_r$ for $1 \leq i, j \leq k$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that
\[
\|f(x) - Q(x)\| \leq \frac{1}{(2^p - 4)(k - 1)} \varphi(x, x, \cdots, x), \tag{4.2}
\]
where $x \in B_r$.

**Proof.** Let $x_1, x_2, \cdots, x_k \in B_r$. If we consider $x_2 = x_3 = \cdots = x_k$ in (4.1), then we see that
\[
\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2)\| \leq \frac{1}{k - 1} \varphi(x_1, x_2, \cdots, x_2). \tag{4.3}
\]
Replacing \( x_1, x_2 \) in \((4.3)\) by \( \frac{x}{2} \), we get
\[
\| f(x) - 4f\left(\frac{x}{2}\right) \| \leq \frac{1}{k-1} \varphi \left( \frac{x}{2}, \frac{x}{2}, \ldots, \frac{x}{2} \right).
\] (4.4)

Replacing \( x \) by \( \frac{x}{2^r} \) in \( B_r \) and multiplying with \( 4^n \) in \((4.4)\), we obtain
\[
\|4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right)\| \leq \frac{4^n}{k-1} \varphi \left( \frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, \ldots, \frac{x}{2^{n+1}} \right).
\] (4.5)

It follows from \((4.5)\) that
\[
\|4^n f\left(\frac{x}{2^n}\right) - 4^{n+m} f\left(\frac{x}{2^{n+m}}\right)\| \leq \frac{1}{k-1} \sum_{i=1}^{m} 4^{n+i-1} \varphi \left( \frac{x}{2^{n+i}}, \frac{x}{2^{n+i}}, \ldots, \frac{x}{2^{n+i}} \right)
\]
\[
\leq \frac{2^{2(n-1)}}{2^{m(k-1)}} \varphi(x, x, \ldots, x) \sum_{i=1}^{m} \frac{1}{2^{(p-1)i}}.
\] (4.6)

It follows that \( \{4^n f\left(\frac{x}{2^n}\right)\} \) is Cauchy and so is convergent. Therefore we see that a mapping
\[
\hat{Q}(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \quad (x \in B_r),
\]
satisfies
\[
\| f(x) - \hat{Q}(x) \| \leq \frac{1}{(2^p - 4)(k-1)} \varphi(x, x, \ldots, x),
\]
and \( \hat{Q}(0) = 0 \), when taking the limit \( m \to \infty \) in \((4.6)\) with \( n = 0 \).

Next fix \( x \in B_r \). Because of \( \frac{x}{2} \in B_r \), we have
\[
4\hat{Q}\left(\frac{x}{2}\right) = \lim_{n \to \infty} 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) = \hat{Q}(x).
\]

Therefore \( 4^{n+m} \hat{Q}\left(\frac{x}{2^{n+m}}\right) = \hat{Q}(x) \) and so the mapping \( Q : X \to Y \) given by \( Q(x) := 4^n \hat{Q}\left(\frac{x}{2^n}\right) \), where \( n \) is least non-negative integer such that \( \frac{x}{2^n} \in B_r \) is well-defined.

It is easy to see that \( Q(x) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right) \quad (x \in X) \) and \( Q|_{B_r(0)} = \hat{Q} \).

Now let \( x, y \in X \). There is a large enough \( n \) such that \( \frac{x}{2^n}, \frac{y}{2^n}, \frac{x+y}{2^n}, \frac{x-y}{2^n} \in B_r(0) \). Put \( x_1 = \frac{x}{2^n} \) and \( x_2 = \frac{y}{2^n} \) in \((4.3)\) and multiplying both sides with \( 4^n \) to obtain
\[
\|4^n f\left(\frac{x+y}{n}\right) + 4^n f\left(\frac{x-y}{2^n}\right) - 4^n 2f\left(\frac{x}{2^n}\right) - 4^n 2f\left(\frac{y}{2^n}\right)\| \leq \frac{4^n}{k-1} \varphi \left( \frac{x}{2^n}, \frac{y}{2^n}, \ldots, \frac{y}{2^n} \right)
\]
\[
\leq \frac{4^n}{2^{np}(k-1)} \varphi(x, y, y, \ldots, y).
\]

whence, by taking the limit as \( n \to \infty \), we get \( Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \). Hence \( Q \) is quadratic. Uniqueness of \( Q \) can be proved by using the strategy used in the proof of Theorem \( 3.2 \).

\[\square\]

**Corollary 4.2.** Let \( X \) and \( Y \) be normed and Banach spaces \( p > 2, r > 0, \theta > 0 \). Suppose that \( f : X \to Y \) is a mapping satisfying \( f(0) = 0 \) and
\[
\|D f(x_1, x_2, \ldots, x_k)\| \leq \theta \|x_1\|^\frac{p}{2} \|x_2\|^\frac{p}{2} \cdots \|x_k\|^\frac{p}{2}
\] (4.7)
for all \( x_1, x_2, \ldots, x_k \in B_r \) with \( x_i \pm x_j \in B_r \) for \( 2 \leq i, j \leq k \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\| f(x) - Q(x) \| \leq \frac{\theta p}{(2^p - 4)(k-1)},
\] (4.8)

where \( x \in B_r \).
Proof. Apply Theorem 4.1 with \( \varphi(x_1, x_2, \ldots, x_k) = \theta \|x_1\|^p \|x_2\|^p \cdots \|x_k\|^p. \)

References


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