ON CERTAIN SUBCLASS OF MEROMORPHIC HARMONIC FUNCTIONS WITH FIXED RESIDUE $\alpha$

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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ABSTRACT. In this paper, we consider some properties such as growth and distortion theorem, coefficient problems, linear combinations for certain subclass of meromorphic harmonic functions with positive coefficients.

1. Introduction

Let $A(p)$ denote the set of function analytic in $D\setminus\{p\}$, Where $D = \{z : |z| < 1\}$. In the annulus $\{z : p < |z| < 1\}$ every function $h$ in $S_p$ has an expansion of the form

$$h(z) = \frac{\alpha}{z - p} + \sum_{n=1}^{\infty} a_n z^n,$$

where $\alpha = \text{Res}(f, p)$, with $0 < \alpha \leq 1$, $z \in D\setminus\{p\}$.

The function $h$ given in (1.1) was studied by Jinxi Ma [8] and Ghanim and Darus [1].

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (see [5]). In [7], there is a more comprehensive study on harmonic univalent functions.

Denote by $SH_p$ the class of the functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the punctured unit disk $D\setminus\{p\}$.

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Then for $f = h + g$ we may express the analytic function $h$ as the form (1.1) and $g$ as

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

then, we have

$$f(z) = h(z) + g(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad (1.2)$$

where $\alpha = \text{Res}(f, p)$, with $0 < \alpha \leq 1$, $z \in D \setminus \{p\}$.

Let $\mathcal{H}_p$ be subclass of $Sh_p$ consisting of function of the form

$$f(z) = h(z) + g(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad (a_n, b_n \geq 0) \quad (1.3)$$

where $\alpha = \text{Res}(f, p)$, with $0 < \alpha \leq 1$, $z \in D \setminus \{p\}$, which are univalent harmonic in the punctured unit disc $D \setminus \{p\}$. $h(z)$ and $g(z)$ are analytic in $D \setminus \{p\}$ and $D$, respectively and $h(z)$ has a simple pole at the point $p$ with residue $\alpha$.

For $\alpha = 1$ and $p = 0$ the function $f$ studied by Bostanci, Yalçın and Öztürk [4].

A function $f \in SH_p$ is said to be in the subclass $SH^*_p$ of meromorphically harmonic starlike in $D \setminus \{p\}$ if it satisfies the condition

$$\Re \left\{ -\frac{zh'(z) + zg'(z)}{h(z) + g(z)} \right\} > 0, \quad (z : p < |z| < 1). \quad (1.4)$$

Also, a function $f \in SH_p$ is said to be in the subclass $CH_p$ of meromorphically harmonic convex in $D \setminus \{p\}$ if it satisfies the condition

$$\Re \left\{ -\frac{z^2 h''(z) + zh'(z) + z^2 g''(z) + zg'(z)}{zh'(z) + zg'(z)} \right\} > 0, \quad (z : p < |z| < 1). \quad (1.5)$$

This classification (1.4) for univalent functions was studied by Ghanim and Daus [1], [2], and the classification (1.5) with $\alpha = 1$ and $p = 0$ was first used by Jahangiri [6].

Next, we define the operator $I^k$ on the class $SH_p$ as follows:

$$I^0 f(z) = f(z),$$

$$I^k f(z) = I^k h(z) + I^k g(z), \quad k = 1, 2, 3, \ldots, \quad (1.6)$$

where

$$I^k h(z) = z \left( I^{k-1} h(z) \right)' + \frac{\alpha (2z - p)}{(z-p)^2} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n.$$ 

and

$$I^k g(z) = z \left( I^{k-1} g(z) \right)' = \sum_{n=1}^{\infty} n^k b_n z^n.$$
With the help of the differential operator $I^k$, we define the class $SH_p^*(k, \alpha, \beta)$

**Definition 1.1.** The function $f \in SH_p$ is said to be a member of the class $SH_p^*(k, \alpha, \beta)$ if it satisfies

$$\left| \frac{z (I^k h(z))^\prime + z (I^k g(z))^\prime}{I^k f(z)} + 1 \right| \leq \left| \frac{z (I^k h(z))^\prime + z (I^k g(z))^\prime}{I^k f(z)} + 2\beta - 1 \right|,$$  \hspace{1cm} (1.7)

$(k \in N_0 = N \cup 0)$ for some $\beta(0 \leq \beta < 1)$ and for all $z \in D \backslash \{p\}$.

It is easy to check that $SH_p^*(0, 1, \beta)$ is the class of meromorphically starlike functions of order $\beta$ and $SH_p^*(0, 1, 0)$ gives the meromorphically starlike functions for all $z \in D \backslash \{p\}$.

Let us write

$$SH_p^*[k, \alpha, \beta] = SH_p^*(k, \alpha, \beta) \cap \mathbb{H}_p$$ \hspace{1cm} (1.8)

where $\mathbb{H}_p$ is the class of functions of the form (1.3) that are analytic and harmonic in $D \backslash \{p\}$.

Next, our first results will concern on the coefficient estimates for the classes $SH_p^*(k, \alpha, \beta)$ and $SH_p^*[k, \alpha, \beta]$.

2. Main Results

Here we provide a sufficient condition for a function, analytic in $D \backslash \{p\}$ to be in $SH_p^*(k, \alpha, \beta)$.

**Theorem 2.1.** If $f(z) = h(z) + \overline{g(z)}$ is of the form (1.2) and satisfies the condition

$$\sum_{n=1}^{\infty} n^k (n + \beta) (1 - p) (|a_n| + |b_n|) \leq \alpha (1 - \beta) \quad (k \in N_0),$$ \hspace{1cm} (2.1)

where $(0 \leq \beta < 1)$, then $f$ is harmonic univalent sense preserving in $D \backslash \{p\}$ and $f \in SH_p^*(k, \alpha, \beta)$.

**Proof:** Suppose that (2.1) holds true for $0 \leq \beta < 1$. Consider the expression

$$M(z) = \left| z (I^k h(z))^\prime + z (I^k g(z))^\prime + I^k f(z) \right|$$

and

$$\left| z (I^k h(z))^\prime + z (I^k g(z))^\prime + (2\beta - 1) I^k f(z) \right|,$$

then for $|z| = r$, and since $|z - p| \geq |z| - p = r - p$, we have

$$M(z) = \left| \frac{-\alpha z}{(z - p)^2} + \frac{\alpha}{z - p} + \sum_{n=1}^{\infty} n^k (n + 1) (a_n z^n + \overline{b_n z^n}) \right|$$

$$- \left| \frac{-\alpha z + \alpha (z - p)(2\beta - 1)}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 2\beta - 1) (a_n z^n + \overline{b_n z^n}) \right|$$

$$= \left| \frac{-\alpha p}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 1) (a_n z^n + \overline{b_n z^n}) \right|$$
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\[-\frac{2\alpha z + 2\alpha \beta z - 2\alpha \beta p + \alpha p}{(z-p)^2} + \sum_{n=1}^{\infty} n^k (n+2\beta-1) (a_n z^n + b_n z^n)\]

and

\[M(r) \leq \frac{\alpha p}{(r-p)^2} + \sum_{n=1}^{\infty} n^k (n+1) (|a_n| + |b_n|) r^n\]

\[-\frac{2\alpha [(1-\beta)r + \beta p]}{(r-p)^2} - \frac{\alpha p}{(r-p)^2} + \sum_{n=1}^{\infty} n^k (n+2\beta-1) (|a_n| + |b_n|) r^n\]

\[= \sum_{n=1}^{\infty} 2n^k (n+\beta) (|a_n| + |b_n|) r^n - \frac{2\alpha (1-\beta)}{(r-p)}.\]

That is

\[(r - p) M(r) \leq \sum_{n=1}^{\infty} 2n^k (n+\beta) (|a_n| + |b_n|) (r - p) r^n - 2\alpha (1 - \beta) \quad (2.2)\]

The inequality in (2.2) holds true for all \(r (0 \leq r < 1)\). Therefore, letting \(r \rightarrow 1\) in (2.2), we obtain

\[(1 - p) M(r) \leq \sum_{n=1}^{\infty} 2n^k (n+\beta) (|a_n| + |b_n|) (1 - p) - 2\alpha (1 - \beta).\]

By the hypothesis (2.1) it follows that (1.7) holds, so that \(f \in SH_p^k(\alpha, \beta)\). Note that \(f\) is sense-preserving in \(U \setminus \{p\}\). This is because

\[|f'(z)| \geq \frac{1}{|z-p|} - \sum_{n=1}^{\infty} n |a_n||z|^{n-1}\]

\[\geq \frac{1}{|z-p|} - \sum_{n=1}^{\infty} n |a_n||z|^{n-1} \geq 1 - \sum_{n=1}^{\infty} n |a_n|\]

\[\geq 1 - \sum_{n=1}^{\infty} n (n + \beta)(1 - p)|a_n|\]

\[\geq \sum_{n=1}^{\infty} n (n + \beta)(1 - p)|b_n|\]

\[\geq \sum_{n=1}^{\infty} n |b_n| \geq \sum_{n=1}^{\infty} n |b_n||z|^{n-1} \geq |g'(z)|\]

Hence the theorem.

**Corollary 2.2.** Let \(k = \beta = 0\) and \(p \rightarrow 0\) in the Theorem 2.1, then we have

\[\sum_{n=1}^{\infty} n (|a_n| + |b_n|) \leq \alpha.\]

**Corollary 2.3.** Let \(k = \beta = 0, \alpha = 1\) and \(p \rightarrow 0\) in the Theorem 2.1, then we have

\[\sum_{n=1}^{\infty} n (|a_n| + |b_n|) \leq 1,\]

the result was achieved by Bostanci, Yalçın and Öztürk [4].
Corollary 2.4. Let $k = 1$, $\beta = 0$ and $p \to 0$ in the Theorem 2.1, then we have

$$\sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) \leq \alpha$$

Corollary 2.5. Let $k = 1$, $\beta = 0$, $\alpha = 1$ and $p \to 0$ in Theorem 2.1, then we have

$$\sum_{n=1}^{\infty} n^2 (|a_n| + |b_n|) \leq 1$$

the result was achieved by Bostancı, Yalçın and Öztürk [4].

Next we give a necessary and sufficient condition for a function $f \in S_h^p$ to be in the class $S_{h}^{\infty}[k,\alpha,\beta]$.

Theorem 2.6. Let $f \in S_h^p$ be a function defined by (1.3). Then $f \in S_{h}^{\infty}[k,\alpha,\beta]$ if and only if the inequality

$$\sum_{n=1}^{\infty} n^k (n + \beta) (1 - p) (a_n + b_n) \leq \alpha (1 - \beta) \quad (k \in N_0) \quad (2.3)$$

is satisfied. The result is sharp.

Proof: In view of Theorem 2.1, it sufficies to show that the 'only if' part is true. Assume that $f \in S_{h}^{\infty}[k,\alpha,\beta]$. Then

$$\left| \frac{z (I_k h(z))' + z (I_k g(z))'}{I_k f(z)} + 1 \right| = \left| \frac{- \alpha p}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 1) (a_n z^n + b_n z^n) \, z^n}{- 2 \alpha z + 2 \alpha \beta z - 2 \alpha \beta p + \alpha p}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 2 \beta - 1) (a_n z^n + b_n z^n) \right| \leq 1, \quad (2.4)$$

$z \in D \setminus \{p\}$.

Since $\Re(z) \leq |z|$ for all $z$, it follows from (2.4) that

$$\Re \left\{ \frac{- \alpha p}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 1) (a_n z^n + b_n z^n) \, z^n}{- 2 \alpha [(1 - \beta) z + \beta p] + \alpha p}{(z - p)^2} + \sum_{n=1}^{\infty} n^k (n + 2 \beta - 1) (a_n z^n + b_n z^n) \right\} \leq 1, \quad (2.5)$$

$z \in D \setminus \{p\}$. We now choose the values $z$ on the real axis. Upon clearing the denominator in (2.5) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} n^k (n + 1) (1 - p) (a_n + b_n) \leq \alpha (1 - \beta)$$
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2α (1 − β) − ∞ \sum_{n=1}^{∞} n^k (n + 2β - 1) (1 - p) (a_n + b_n),

which immediately yields the required condition (2.3).

A distortion property for functions in the class $SH^*_p[k, α, β]$ is contained in the following theorem:

**Theorem 2.7.** If the function $f$ defined by (1.3) is in the class $SH^*_p[k, α, β]$, then, for $|z| = r$, we have

$$|f(z)| \leq \frac{α}{r - p} + \frac{α(1 - β)}{(1 + β)(1 - p)} r.$$

**Proof:** Let $f \in SH^*_p[k, α, β]$. Taking the absolute value of $f$ we obtain

$$|f(z)| \leq \frac{α}{r - p} + \sum_{n=1}^{∞} (a_n + b_n) r^n$$

$$\leq \frac{α}{r - p} + \frac{α(1 - β)}{(1 + β)(1 - p)} \sum_{n=1}^{∞} n^k (n + β) (1 - p) (a_n + b_n) r^n$$

$$\leq \frac{α}{r - p} + \frac{α(1 - β)}{(1 + β)(1 - p)} r.$$

The functions

$$f(z) = \frac{α}{z - p} + \frac{α(1 - β)}{(1 + β)(1 - p)} z \quad \text{and} \quad f(z) = \frac{α}{z - p} + \frac{α(1 - β)}{(1 + β)(1 - p)} \overline{z}$$

for $0 < α \leq 1$ and $0 \leq β < 1$ show that the bound given in Theorem 2.7 are sharp in $D \setminus \{p\}$.

**Theorem 2.8.** Set

$$h_0(z) = \frac{α}{z - p}, \quad g_0(z) = 0,$$

$$h_n(z) = \frac{α}{z - p} + \frac{α(1 - β)}{n^k (n + β)(1 - p)} z^n$$

(2.6)

for $n = 1, 2, 3, ...$, and

$$g_n(z) = \frac{α(1 - β)}{n^k (n + β)(1 - p)} \overline{z}^n$$

(2.7)

for $n = 1, 2, 3, ...$.

Then $f \in SH^*_p[k, α, β]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{∞} (λ_n h_n + γ_n g_n),$$

(2.8)

where $λ_n ≥ 0$, $γ_n ≥ 0$ and $\sum_{n=0}^{∞} (λ_n + γ_n) = 1$.

In particular, the extreme points of $SH^*_p[k, α, β]$ are $\{h_n\}$ and $\{g_n\}$.

**Proof:** From (2.6), (2.7) and (2.8), we have

$$f(z) = \sum_{n=0}^{∞} (λ_n h_n + γ_n g_n)$$
\[ f (z) = \sum_{n=0}^{\infty} (\lambda_n + \gamma_n) \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} \frac{\alpha (1-\beta)}{n^k (n+\beta) (1-p)} \lambda_n z^n + \sum_{n=0}^{\infty} \frac{\alpha (1-\beta)}{n^k (n+\beta) (1-p)} \gamma_n z^n. \]

Then
\[ \sum_{n=1}^{\infty} \left( n^k (n+\beta) (1-p) \right) \frac{\lambda_n}{n^k (n+\beta) (1-p)} + \sum_{n=0}^{\infty} \left( n^k (n+\beta) (1-p) \right) \frac{\gamma_n}{n^k (n+\beta) (1-p)} = \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) - \lambda_0 = 1 - \lambda_0 \leq 1 \]

So \( f \in S H^*_p[k, \alpha, \beta] \).

Conversely, suppose that \( f \in S H^*_p[k, \alpha, \beta] \). Set
\[ \lambda_n = \frac{n^k (n+\beta)(1-p)}{\alpha (1-\beta)} a_n, \quad n \geq 1, \]
and
\[ \gamma_n = \frac{n^k (n+\beta)(1-p)}{\alpha (1-\beta)} b_n, \quad n \geq 0. \]

Then by Theorem 2.6, \( 0 \leq \lambda_n \leq 1 \) \( (n = 1, 2, 3, ...) \) and \( 0 \leq \gamma_n \leq 1 \), \( (n = 0, 1, 2, ...) \).

We define
\[ \lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=0}^{\infty} \gamma_n \]
and note that, by Theorem 2.6, \( \lambda_0 \geq 0 \).

Consequently, we obtain
\[ f (z) = \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n), \]
as required.

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