**ON THE GENERALIZED ABSOLUTE CESÀRO SUMMABILITY**

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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**Abstract.** In this paper, a known theorem dealing with $|C, \alpha|$ summability factors, has been generalized for $|C, \alpha, \beta|_k$ summability factors. Some new results have also been obtained.

1. **Introduction**

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $u_n^{\alpha, \beta}$ and $t_n^{\alpha, \beta}$ the n-th Cesàro means of order $(\alpha, \beta)$, with $\alpha + \beta > -1$, of the sequence $(s_n)$ and $(na_n)$, respectively, i.e., (see [2])

$$u_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_n^{-v} A_v^\beta s_v$$

$$t_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_n^{-v} A_v^\beta v a_v,$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_n^{-\alpha+\beta} = 0 \quad \text{for} \quad n > 0. \quad (1.3)$$

The series $\sum a_n$ is said to be summable $|C, \alpha, \beta|_k$, $k \geq 1$ and $\alpha + \beta > -1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} | u_n^{\alpha, \beta} - u_n^{\alpha, \beta} | < \infty. \quad (1.4)$$

Since $t_n^{\alpha, \beta} = n(u_n^{\alpha, \beta} - u_{n-1}^{\alpha, \beta})$ (see [4]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} | t_n^{\alpha, \beta} |^k < \infty. \quad (1.5)$$

If we take $\beta = 0$, then $|C, \alpha, \beta|_k$ summability reduces to $|C, \alpha|_k$ summability (see [5]). It should be noted that obviously $(C, \alpha, 0)$ mean is the same as $(C, \alpha)$

2000 **Mathematics Subject Classification.** 40D15, 40F05, 40G05, 40G99.

**Key words and phrases.** Absolute Cesàro summability, infinite series, summability factors.

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mean. A sequence \((\lambda_n)\) is said to be convex if \(\Delta^2 \lambda_n \geq 0\), where \(\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}\).

Pati [6] has proved the following theorem dealing with \(|C, \alpha|\) summability factors.

**Theorem 1.1.** If \((\lambda_n)\) is a convex sequence such that \(\sum n^{-1} \lambda_n\) is convergent and the sequence \((\theta_n^\alpha)\) defined by

\[
\theta_n^\alpha = |t_n^\alpha|, \quad \alpha = 1, \tag{1.6}
\]

\[
\theta_n^\alpha = \max_{1 \leq v \leq n} |t_v^\alpha|, \quad 0 < \alpha < 1 \tag{1.7}
\]
satisfies the condition

\[
\theta_n^\alpha = O(1)(C, 1), \tag{1.8}
\]
then the series \(\sum a_n \lambda_n\) is summable \(|C, \alpha|\) for \(0 < \alpha \leq 1\).

2. The Main Result

The aim of this paper is to generalize Theorem 1.1 for \(|C, \alpha, \beta|_k\) summability.

We shall prove the following theorem.

**Theorem 2.1.** If \((\lambda_n)\) is a convex sequence such that \(\sum n^{-1} \lambda_n\) is convergent and the sequence \((\theta_n^{\alpha, \beta})\) defined by

\[
\theta_n^{\alpha, \beta} = |t_n^{\alpha, \beta}|, \quad \alpha = 1, \beta > -1 \tag{2.1}
\]

\[
\theta_n^{\alpha, \beta} = \max_{1 \leq v \leq n} |t_v^{\alpha, \beta}|, \quad 0 < \alpha < 1, \beta > -1 \tag{2.2}
\]
satisfies the condition

\[
(\theta_n^{\alpha, \beta})^k = O(1)(C, 1), \tag{2.3}
\]
then the series \(\sum a_n \lambda_n\) is summable \(|C, \alpha, \beta|_k\) for \(0 < \alpha \leq 1, \beta > -1\) and \(k \geq 1\).

It should be noted that if we take \(k = 1\) and \(\beta = 0\), then we get Theorem 1.1.

We need the following lemmas for the proof of our theorem.

**Lemma 2.2.** ([3]) If \((\lambda_n)\) is a convex sequence such that \(\sum n^{-1} \lambda_n\) is convergent, then

\[
n \Delta \lambda_n \to 0,
\]

\[
\sum_{n=1}^{\infty} (n + 1) \Delta^2 \lambda_n
\]
is convergent.

**Lemma 2.3.** ([1]). If \(0 < \alpha \leq 1, \beta > -1\) and \(1 \leq v \leq n\), then

\[
|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A^\beta_v a_p| \leq \max_{1 \leq m \leq v} \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A^\beta_p a_p |. \tag{2.4}
\]

**Proof of the theorem.** Let \(T_n^{\alpha, \beta}\) be the n-th \((C, \alpha, \beta)\) mean of the sequence \((na_n \lambda_n)\). Then, by (1.2), we have

\[
T_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A^\beta_v v a_v \lambda_v.
\]
First, applying Abel’s transformation and then using Lemma 2.3, we have that

\[ T_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha - 1} A_p^\beta a_p + \frac{\lambda_n}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha - 1} A_v^\beta v a_v, \]

thus,

\[ |T_n^{\alpha, \beta}| \leq \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \left| \sum_{p=1}^{v} A_{n-p}^{\alpha - 1} A_p^\beta a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha - 1} A_v^\beta v a_v \]

\[ \leq \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} A_v^\alpha A_v^\beta \theta_v^{\alpha, \beta} |\Delta \lambda_v| + |\lambda_n| \theta_n^{\alpha, \beta} \]

\[ = T_{n,1}^{\alpha, \beta} + T_{n,2}^{\alpha, \beta}, \quad \text{say.} \]

Since

\[ |T_{n,1}^{\alpha, \beta} + T_{n,2}^{\alpha, \beta}|^k \leq 2^k (|T_{n,1}^{\alpha, \beta}|^k + |T_{n,2}^{\alpha, \beta}|^k), \]

in order to complete the proof of the theorem, by (5), it is sufficient to show that

\[ \sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}^{\alpha, \beta}|^k < \infty \quad \text{for} \quad r = 1, 2. \]

Whenever \( k > 1 \), we can apply Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), we get that

\[ \sum_{n=1}^{m+1} \frac{1}{n} \left| T_{n,1}^{\alpha, \beta} \right|^k \leq \sum_{n=2}^{m+1} \frac{1}{n^{1+\alpha+\beta}} \sum_{v=1}^{n-1} A_v^\alpha A_v^\beta \theta_v^{\alpha, \beta} \Delta \lambda_v \left| k \right| \]

\[ = O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+\alpha+\beta}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta \lambda_v \theta_v^{\alpha, \beta} \right\}^k \]

\[ \times \left( \sum_{v=1}^{n-1} \Delta \lambda_v \right)^{-k-1} \]

\[ = O(1) \sum_{v=1}^{m} v^{\alpha k} \Delta \lambda_v \theta_v^{\alpha, \beta} \int_{v}^{\infty} \frac{dx}{x^{1+\alpha+\beta}} \]

\[ = O(1) \sum_{v=1}^{m} v^{\alpha k} \Delta \lambda_v \theta_v^{\alpha, \beta} \]

\[ = O(1) \sum_{v=1}^{m} \Delta \lambda_v \theta_v^{\alpha, \beta} \]

\[ = O(1) \sum_{v=1}^{m} \Delta \lambda_v \theta_v^{\alpha, \beta} + O(1) m \Delta \lambda_m \]

\[ = O(1) \quad \text{as} \quad m \to \infty, \]

in view of hypotheses of the theorem and Lemma 2.2.
Similarly, we have that
\[
\sum_{n=1}^{m} \frac{1}{n} |\lambda_n \theta_n^{\alpha,\beta}|^k = O(1) \sum_{n=1}^{m} \frac{\lambda_n}{n} (\theta_n^{\alpha,\beta})^k \\
= O(1) \sum_{n=1}^{m-1} \Delta(n^{-1}\lambda_n) \sum_{v=1}^{n} (\theta_v^{\alpha,\beta})^k \\
+ O(1) \frac{\lambda_m}{m} \sum_{v=1}^{n} (\theta_v^{\alpha,\beta})^k \\
= O(1) (\lambda_1 - \lambda_m) + O(1) \sum_{n=1}^{m-1} \frac{\lambda_{n+1}}{n+1} + O(1) \lambda_m \\
= O(1) \text{ as } m \to \infty.
\]

Therefore, by (1.5), we get that
\[
\sum_{n=1}^{\infty} \frac{1}{n} |T_n^{\alpha,\beta}|^k < \infty \quad \text{for } r = 1, 2.
\]

This completes the proof of the theorem. If we take \(\beta = 0\), then we get a new result for \(|C, \alpha|_k\) summability factors. Also, if we take \(\beta = 0\) and \(\alpha = 1\), then we get another new result for \(|C, 1|_k\) summability factors.

**References**


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