SUBORDINATION AND SUPERORDINATION FOR FUNCTIONS BASED ON DZIOK-SRIVASTAVA LINEAR OPERATOR

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Abstract. In this article, we obtain some subordination and superordination results involving Dziok-Srivastava linear operator and fractional integral operator for certain normalized analytic functions in the open unit disk.

1. Introduction and Preliminaries.

Let $\mathcal{H}(U)$ denote the class of analytic functions in the unit disk
$$U := \{ z \in \mathbb{C}, |z| < 1 \}.$$ For $n$ positive integer and $a \in \mathbb{C}$, let
$$\mathcal{H}[a,n] := \{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \},$$ and $A_n = \{ f \in H(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \}$ with $A_1 = \mathcal{A}$. A function $f \in \mathcal{H}[a,n]$ is convex in $U$ if it is univalent and $f(U)$ is convex. It is well-known that $f$ is convex if and only if $f(0) = 0$ and
$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ z \in U.$$ Definition 1.1. [1] Denote by $\mathcal{Q}$ the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where
$$E(f) := \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \}$$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Given two functions $F$ and $G$ in the unit disk $U$, the function $F$ is subordinated to $G$, written $F \prec G$, if $G$ is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Alternatively, given two functions $F$ and $G$, which are analytic in $U$, the function $F$ is said to subordinated to $G$ in $U$ if there exists a function $h$, analytic in $U$ with $h(0) = 0$ and $|h(z)| < 1$ for all $z \in U$ such that
$$F(z) = G(h(z)) \text{ for all } z \in U.$$
Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$, then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, if $p \prec q$. If $p$ and $\phi(p(z)), zp'(z))$ are univalent in $U$ and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$, then $p$ is called a solution of the differential superordination. An analytic function $q$ is called subordinant of the solution of the differential superordination if $q \prec p$.

We shall need the following results:

**Lemma 1.1.** [2] Let $q$ be univalent in the unit disk $U$, and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z)), h(z) := \theta(q(z)) + Q(z)$. Suppose that
1. $Q(z)$ is starlike univalent in $U$, and
2. $\Re\frac{zq'(z)}{Q(z)} > 0$ for $z \in U$.
If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and $q$ is the best dominant.

**Lemma 1.2.** [3] Let $q$ be convex univalent in the unit disk $U$ and $\psi$ and $\gamma \in \mathbb{C}$ with $\Re\{1 + \frac{zq'(z)}{Q(z)} + \frac{\gamma}{\psi}\} > 0$. If $p(z)$ is analytic in $U$ and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$, then $p(z) \prec q(z)$ and $q$ is the best dominant.

**Lemma 1.3.** [4] Let $q$ be convex univalent in the unit disk $U$ and $\varphi$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$. Suppose that
1. $zq'(z)\varphi(q(z))$ is starlike univalent in $U$, and
2. $\Re\{\frac{\varphi'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.
If $\varphi(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $U$ and $\varphi(q(z)) + zq'(z)\varphi(q(z)) \prec \varphi(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and $q$ is the best subordinant.

**Lemma 1.4.** [1] Let $q$ be convex univalent in the unit disk $U$ and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\gamma\} > 0$. If $p(z) \in \mathcal{H}(q(0), 1] \cap \mathcal{Q}$, with $p(z) + \gamma zp'(z)$ is univalent in $U$ then $q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$ implies $q(z) \prec p(z)$ and $q$ is the best subordinant.

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of $f$ and $g$ defined by
\[
(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n := (g * f)(z).
\]

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \ldots, l$) and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ ($j = 1, 2, \ldots, m$), the generalized hypergeometric function $\,_{l}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series
\[
\,_{l}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n} \cdots (\alpha_l)_{n}}{(\beta_1)_{n} \cdots (\beta_m)_{n}} \frac{z^n}{n!}
\]
for $l \leq m + 1$, $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, where $(a)_n$ is the Pochhammer symbol defined by
\[
(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 
1, & (n = 0); \\
(a + 1)(a + 2) \cdots (a + n - 1), & (n \in \mathbb{N}).
\end{cases}
\]
Corresponding to the function
\[ h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z_1^m F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z), \]
the Dziok-Srivastava operator (see [5-7]) \( H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \) is defined by the Hadamard product
\[
H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)
= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} a_n z^n
\]
\[
:= H_m^l[\alpha_1] f(z).
\]
We can verify that
\[
z(H_m^l[\alpha_1] f(z))' = H_m^l[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H_m^l[\alpha_1] f(z).
\]
Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator \([8]\), the Carlson-Shaffer linear operator \([9]\), the Ruscheweyh derivative operator \([10]\), the generalized Bernardi-Libera-Livingston linear integral operator \([11]\) and the Srivastava-Owa fractional derivative operator \([12]\):

**Definition 1.2.** The fractional derivative of order \( \alpha \) is defined, for a function \( f \) by
\[
D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \leq \alpha < 1,
\]
where the function \( f \) is analytic in simply-connected region of the complex \( z \)-plane \( \mathbb{C} \) containing the origin and the multiplicity of \( (z-\zeta)^{-\alpha} \) is removed by requiring
\[
\log(z - \zeta) \text{ to be real when } (z - \zeta) > 0.
\]

**Definition 1.3.** The fractional integral of order \( \alpha \) is defined, for a function \( f \), by
\[
I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,
\]
where the function \( f \) is analytic in simply-connected region of the complex \( z \)-plane \( \mathbb{C} \) containing the origin and the multiplicity of \( (z-\zeta)^{\alpha-1} \) is removed by requiring
\[
\log(z - \zeta) \text{ to be real when } (z - \zeta) > 0.
\]

**Remark 1.1.** \([12]\)
\[
D_z^\alpha \{z^\mu\} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \alpha + 1)} \{z^{\mu-\alpha}\}, \quad \mu > -1; \quad 0 \leq \alpha < 1
\]
and
\[
I_z^\alpha \{z^\mu\} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} \{z^{\mu+\alpha}\}, \quad \mu > -1; \quad \alpha > 0.
\]
The main object of the present paper is to find the sufficient conditions for certain normalized analytic functions \( f, g \) to satisfy
\[
\left[ \frac{I_z^\alpha H_m^l[\alpha_1] g_1(z)}{\rho_\alpha(z)} \right]^\mu < \left[ \frac{I_z^\alpha H_m^l[\alpha_1] f(z)}{\rho_\alpha(z)} \right]^\mu < \left[ \frac{I_z^\alpha H_m^l[\alpha_1] g_2(z)}{\rho_\alpha(z)} \right]^\mu
\]
and
\[
q_1(z) < \left[ \frac{I_z^\alpha H_m^l[\alpha_1] f(z)}{\rho_\alpha(z)} \right]^\mu < q_2(z), \quad \rho_\alpha(z) \neq 0, \quad z \in U
\]
where $\mu \geq 1$, $q_1$ and $q_2$ are given univalent functions in $U$. Also, we obtain the results as special cases. Further, in this paper, we study the existence of univalent solution for the fractional differential equation

$$D^\alpha_\rho \rho_\alpha(z)u(z) = H^I_m[\alpha_1]f(z),$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$, $\rho : U \to \mathbb{C}\setminus\{0\}$ is an analytic functions in $z \in U$ and $f : U \to \mathbb{C}$ is a univalent function in $U$. The existence is obtained by applying Schauder fixed point theorem. Moreover, we discuss some properties of this solution involving fractional differential subordination. The following results are used in the sequel.

**Theorem 1.1.** (Arzela-Ascoli) (see [13]) Let $E$ be a compact metric space and $\mathcal{C}(E)$ be the Banach space of real or complex valued continuous functions normed by

$$\|f\| := \sup_{t \in E}|f(t)|.$$ 

If $A = \{f_n\}$ is a sequence in $\mathcal{C}(E)$ such that $f_n$ is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Let $M$ be a subset of Banach space $X$ and $A : M \to M$ an operator. The operator $A$ is called compact on the set $M$ if it carries every bounded subset of $M$ into a compact set. If $A$ is continuous on $M$ (that is, it maps bounded sets into bounded sets ) then it is said to be completely continuous on $M$.

**Theorem 1.2.** (Schauder) (see [14]) Let $X$ be a Banach space, $M \subset X$ a nonempty closed bounded convex subset and $P : M \to M$ is compact. Then $P$ has a fixed point.

Recently, the subordination and superordination containing the Dziok-Srivastava linear operator are studied by many authors [15].

2. Subordination and superordination.

In this section, we study some important properties of the fractional differential and integral operators $D^\alpha_\rho$, $I^\alpha_\rho$, given by the authors [16] which are useful in the next results of the subordination and superordination.

**Theorem 2.1**[16] For $\alpha, \in (0, 1]$ and $f$ is a continuous function, then

$$1 - D^\alpha_\rho f(z) = \frac{(z)^{\alpha-1}}{\Gamma(\alpha)}f(0) + I^\alpha_\rho Df(z); \quad D = \frac{d}{dz}$$

$$2 - I^\alpha_\rho D^\alpha_\rho f(z) = D^\alpha_\rho I^\alpha_\rho f(z) = f(z).$$

But, first we consider the subordination results involving Dziok-Srivastava linear operator and fractional integral operator as the following:

**Theorem 2.2.** Let $f, g$ be analytic in $U$, \[ \frac{I^\alpha_\rho H^I_m[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0 \] be univalent in $U$ such that \[ \frac{I^\alpha_\rho H^I_m[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0 \] and \[ z\left[I^\alpha_\rho H^I_m[\alpha_1]g(z)\right]'' \] be starlike univalent in $U$. If the subordination

\[ \frac{I^\alpha_\rho H^I_m[\alpha_1]f(z)}{\rho_\alpha(z)} = 1 + \mu \left(\frac{zI^\alpha_\rho H^I_m[\alpha_1]f(z)'}{I^\alpha_\rho H^I_m[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}\right) \]
Assume that \( \Re\{Q(z)\} > 0, \ z \in U \), where
\[
Q(z) := \frac{\rho(z)}{\rho_\alpha(z)} \left( 1 + \frac{zG(z)}{G(z)} \right)
\]
Then
\[
\left[ \frac{I_\nu H_{m_0}[\alpha_1][f(z)]}{\rho_\alpha(z)} \right]_\mu \prec \left[ \frac{I_\nu H_{m_0}[\alpha_1]g(z)}{\rho_\alpha(z)} \right]_\mu
\]
and \( \left[ \frac{I_\nu H_{m_0}[\alpha_1]g}{\rho_\alpha} \right]_\mu \) is the best dominant.

**Proof.** Setting
\[
p(z) := \left[ \frac{I_\nu H_{m_0}[\alpha_1][f(z)]}{\rho_\alpha(z)} \right]_\mu, \quad q(z) := \left[ \frac{I_\nu H_{m_0}[\alpha_1]g(z)}{\rho_\alpha(z)} \right]_\mu.
\]
Our aim is to apply Lemma 1.1. First we show that \( \Re\{1 + \frac{zq''(z)}{q'(z)}\} > 0 \).

\[
\Re\{1 + \frac{zq''(z)}{q'(z)}\} = \Re\{1 + \frac{zG'(z)}{G(z)} + (\mu - 1)\frac{G(z)\rho_\alpha(z)}{I_\nu H_{m_0}[\alpha_1]g(z)} \} > 0.
\]
Assume that
\[
\theta(\omega) := \omega \text{ and } \phi(\omega) := 1,
\]
it can easily be observed that \( \theta, \phi \) are analytic in \( \mathbb{C} \). Also, we let
\[
Q(z) := zq'(z)\phi(z) = zq'(z),
\]
\[
h(z) := \theta(q(z)) + Q(z) = q(z) + zq'(z).
\]
By the assumptions of the theorem we find that \( Q \) is starlike univalent in \( U \) and that
\[
\Re\{\frac{zq''(z)}{Q(z)}\} = \Re\{2 + \frac{zq''(z)}{q'(z)}\} > 0.
\]
By using Theorem 2.1, a computation shows
\[
p(z) + zq'(z) = \left[ \frac{I_\nu H_{m_0}[\alpha_1][f(z)]}{\rho_\alpha(z)} \right]_\mu \left\{ 1 + \mu \left( \frac{zI_\nu H_{m_0}[\alpha_1][f(z)]'}{I_\nu H_{m_0}[\alpha_1][f(z)]} - \frac{z\rho'(z)}{\rho(z)} \right) \right\}
\]
\[
\prec \left[ \frac{I_\nu H_{m_0}[\alpha_1][g(z)]}{\rho_\alpha(z)} \right]_\mu \left\{ 1 + \mu \left( \frac{zI_\nu H_{m_0}[\alpha_1][g(z)]'}{I_\nu H_{m_0}[\alpha_1][g(z)]} - \frac{z\rho'(z)}{\rho(z)} \right) \right\}
\]
\[
= q(z) + zq'(z).
\]
Thus in view of Lemma 1.1, \( p(z) \prec q(z) \) and \( q \) is the best dominant.

**Corollary 2.1.** Let \( f, g \) be analytic in \( U \). \( \left[ \frac{I_\nu L(a,c)}{\rho_\alpha} \right]_\mu \) be univalent in \( U \) and \( z(\left[ \frac{I_\nu L(a,c)}{\rho_\alpha} \right]_\mu)' \) be starlike univalent in \( U \). If the subordination
\[
\left[ \frac{I_\nu L(a,c)}{\rho_\alpha(z)} \right]_\mu \left\{ 1 + \mu \left( \frac{zI_\nu L(a,c)[f(z)]'}{I_\nu L(a,c)[f(z)]} - \frac{z\rho'(z)}{\rho(z)} \right) \right\}
\]
\[
\prec \left[ \frac{I_\nu L(a,c)[g(z)]}{\rho_\alpha(z)} \right]_\mu \left\{ 1 + \mu \left( \frac{zI_\nu L(a,c)[g(z)]'}{I_\nu L(a,c)[g(z)]} - \frac{z\rho'(z)}{\rho(z)} \right) \right\}
\]
By putting holds and  

\[ \Re \left\{ \frac{zG'(z)}{G(z)} + (\mu - 1) \frac{zG(z)\rho_a(z)}{I^\alpha L(a,c)g(z)} \right\} > 0, \quad z \in U, \]

where 

\[ G(z) := \left[ \frac{I^\alpha L(a,c)g(z)}{\rho_a(z)} \right]' . \]

Then 

\[ \left[ \frac{I^\alpha L(a,c)f(z)}{\rho_a(z)} \right]^\mu \times \left[ \frac{I^\alpha L(a,c)g(z)}{\rho_a(z)} \right]^\mu \]

and \( \left[ \frac{I^\alpha L(a,c)g}{\rho_a} \right]^\mu \) is the best dominant.

**Proof.** By putting \( l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1 \) and \( \beta_1 = c \) in Theorem 2.2.

**Corollary 2.2.** Let \( f, g \) be analytic in \( U, \left[ \frac{I^\alpha g}{\rho_a} \right]^\mu \) be univalent in \( U \) such that \( I^\alpha L_m[\alpha_1][g(z)] \neq 0 \) and \( z(\left[ \frac{I^\alpha g}{\rho_a} \right]^\mu)' \) be starlike univalent in \( U \). If the subordination 

\[ \left[ \frac{I^\alpha f(z)}{\rho_a(z)} \right]^\mu \{ 1 + \mu(z \left[ I^\alpha f(z) \right]^\gamma \frac{z\rho'(z)}{\rho(z)} - z\rho'(z) \} \times \left[ \frac{I^\alpha g(z)}{\rho_a(z)} \right]^\mu \{ 1 + \mu(z \left[ I^\alpha g(z) \right]^\gamma \frac{z\rho'(z)}{\rho(z)} - z\rho'(z) \} \]

holds and 

\[ \Re \left\{ \frac{zG'(z)}{G(z)} + (\mu - 1) \frac{zG(z)\rho_a(z)}{I^\alpha f(z)} \right\} > 0, \quad z \in U, \quad \text{where} \quad G(z) := \left[ \frac{I^\alpha g(z)}{\rho_a(z)} \right]' . \]

Then 

\[ \left[ \frac{I^\alpha f(z)}{\rho_a(z)} \right]^\mu \times \left[ \frac{I^\alpha g(z)}{\rho_a(z)} \right]^\mu \]

and \( \left[ \frac{I^\alpha g}{\rho_a} \right]^\mu \) is the best dominant.

**Proof.** By putting \( l = 1, m = 0, \alpha_1 = 1 \), in Theorem 2.2.

**Theorem 2.3.** Let \( f, g \) be analytic in \( U, q \) be convex univalent in \( U \) with \( \Re\{1 + \frac{zq'(z)}{\rho(z)} + \frac{1}{\gamma} \}, \gamma \in \mathbb{C} \) and \( \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]}{\rho_a(z)} \right]^\mu \) be analytic in \( U \). If the subordination 

\[ \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu \{ 1 + \mu z(\gamma \left[ I^\alpha H^\alpha_m[\alpha_1]f(z) \right]^\gamma \frac{z\rho'(z)}{\rho(z)} - z\rho'(z)) \} \times \left( q(z) + \gamma zq'(z) \right) \]

holds. Then 

\[ \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu \prec q(z) \]

and \( q \) is the best dominant.

**Proof.** Setting 

\[ p(z) := \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu . \]

Our aim is to applied Lemma 1.2. Let \( \psi := 1 \), since 

\[ p(z) + \gamma z\rho'(z) = \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu + \gamma z(\left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu)' \]

\[ = \left[ \frac{I^\alpha H^\alpha_m[\alpha_1]f(z)}{\rho_a(z)} \right]^\mu \{ 1 + \mu z(\gamma \left[ I^\alpha H^\alpha_m[\alpha_1]f(z) \right]^\gamma \frac{z\rho'(z)}{\rho(z)} - z\rho'(z)) \} \]

\[ < q(z) + \gamma zq'(z) \]
then, in view of Lemma 1.2, \( p(z) \prec q(z) \) and \( q \) is the best dominant.

**Corollary 2.3.** Let \( f, g \) be analytic in \( U, \ -1 \leq B \leq A \leq 1, \ q(z) := \left[ \frac{1 + A z}{1 + B z} \right]^\mu \) with \( \Re \{ 1 + \frac{z g'(z)}{q(z)} + \frac{1}{z} \} \), \( \gamma \in \mathbb{C} \) and \( \left[ I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu} \) be analytic in \( U \). If the subordination

\[
\left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu} \left\{ 1 + \mu \left( \frac{z I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z)'}{I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z)} - \frac{z p'(z)}{\rho(z)} \right) \right\} \prec \left[ 1 + A z \right]^{\mu} \frac{1 + \frac{z \gamma z (A - B)}{(1 + A z)(1 + B z)}}{1 + B z}^{\mu}, -1 \leq B < A \leq 1
\]

holds. Then

\[
\left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu} \prec \left[ \frac{1 + A z}{1 + B z} \right]^\mu, -1 \leq B < A \leq 1
\]

and \( \left[ \frac{1 + A z}{1 + B z} \right]^\mu \) is the best dominant.

Next, applying Lemma 1.3 and Lemma 1.4 respectively, to obtain the following theorems.

**Theorem 2.4.** Let \( f, g \) be analytic in \( U, \left[ I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu} \) be convex univalent in \( U \) such that \( I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \neq 0, \ z \left[ I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu} \) be starlike univalent in \( U \) and \( \left( z I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right)^{\mu} \) be univalent in \( U \). If the subordination

\[
\left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu} \left\{ 1 + \mu \left( \frac{z I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z)'}{I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z)} - \frac{z p'(z)}{\rho(z)} \right) \right\} \prec \left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu} \left\{ 1 + \mu \left( \frac{z I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z)'}{I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z)} - \frac{z p'(z)}{\rho(z)} \right) \right\}
\]

holds and \( \left( z^{-1} I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right)^{\mu} \in \mathcal{H}[0, 1] \cap Q \). Then

\[
\left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu} \prec \left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu}
\]

and \( \left[ I_{m}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu} \) is the best subordinate.

**Proof.** Setting

\[
p(z) := \left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]f(z) \right]^{\mu}, \ q(z) := \left[ I_{z}^{\alpha} H_{m}^{\alpha}[\alpha_{1}]g(z) \right]^{\mu}.
\]

Our aim is to apply Lemma 1.3. By taking

\[
\theta(\omega) := \omega \text{ and } \varphi(\omega) := 1,
\]

it can easily observed that \( \theta, \varphi \) are analytic in \( \mathbb{C} \). Thus

\[
\Re \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = 1 > 0.
\]

Now we must show that

\[
q(z) + z q'(z) \prec p(z) + z p'(z).
\]
a computation shows that
\[ q(z) + zq'(z) = \left[ \frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
\[ \prec \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
\[ = p(z) + zp'(z). \]

Thus in view of Lemma 1.3, \( q(z) \prec p(z) \) and \( p \) is the best subordinant.

**Theorem 2.5.** Let \( f, g \) be analytic in \( U \), \( q \) be convex univalent in \( U \), \( |I_z^\alpha H_m^l[\alpha_1]f(z)|^\mu \in \mathcal{H}[0,1] \cap Q \) and
\[ \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}) \}, \ \Re\{\tau\} > 0, \]
be univalent in \( U \). If the subordination
\[ q(z) + \gamma zq'(z) \prec \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
holds. Then
\[ q(z) \prec \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu \]
and \( q \) is the best subordinant.

**Proof.** Setting
\[ p(z) := \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu. \]

Our aim is to apply Lemma 1.4. Since
\[ q(z) + \gamma zq'(z) = \left[ \frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
\[ \prec \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
\[ = p(z) + \gamma zp'(z) \]
then, in view of Lemma 1.4, \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

Combining the results of differential subordination and superordination, we state the following sandwich theorems.

**Theorem 2.6.** Let \( f, g_1, g_2 \) be analytic in \( U \), \( |I_z^\alpha H_m^l[\alpha_1]g(z)|^\mu \) be convex univalent in \( U \) such that \( \frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \neq 0, z(\frac{I_z^\alpha H_m^l[\alpha_1]g_1(z)}{\rho_\alpha(z)})', z(\frac{I_z^\alpha H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)})' \) be starlike univalent in \( U \) and let \( (z\frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)})' \), \( |I_z^\alpha H_m^l[\alpha_1]g(z)|^\mu \) be univalent in \( U \). If the subordination
\[ \left[ \frac{I_z^\alpha H_m^l[\alpha_1]g(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]
\[ \prec \left[ \frac{I_z^\alpha H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \{1 + \mu(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\rho'(z)}{\rho(z)}) \} \]

Theorem 3.1. In other words, every solution of the equation (3.1) is also a solution of the initial value problem (1.1). Let
\[ u(z) = \frac{1}{\rho_\alpha(z)} \int_0^z (z - \zeta)^{\alpha-1} \frac{H_m^l[\alpha_1]f(\zeta)}{\Gamma(\alpha)} d\zeta, \] holds, \[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right] = 1 + \mu \left( \frac{z I^{\alpha} H_m^l[\alpha_1]g_2(z)}{I^{\alpha} H_m^l[\alpha_1]g_2(z)} - \frac{z \rho'(z)}{\rho(z)} \right) \]
holds, \[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right] \in \mathcal{H}[0,1] \cap Q \] and
\[ \Re \left\{ \frac{z G_2(z)}{G_2(z)} + (\mu - 1) \frac{z G_2(z) \rho_\alpha(z)}{I^{\alpha} H_m^l[\alpha_1]g_2(z)} \right\} > 0, \ z \in U, \]
where
\[ G_2(z) := \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right]. \]
Then
\[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right] \leq \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] < \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right] \]
and \[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]g_2(z)}{\rho_\alpha(z)} \right]^\mu \] and \[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu \] are respectively the best subordinant and dominant.

Theorem 2.7. Let \( f, q_1, q_2 \in A \), \( q_1, q_2 \) be convex univalent in \( U \), with \( \Re \{ 1 + \frac{z q_2'(z)}{q_2(z)} + \gamma, \gamma \in \mathbb{C}, \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right] \in \mathcal{H}[0,1] \cap Q \) and analytic in \( U \) and
\[ \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu \{ 1 + \mu \left( \frac{z I^{\alpha} H_m^l[\alpha_1]f'}{I^{\alpha} H_m^l[\alpha_1]f(z)} - \frac{z \rho'(z)}{\rho(z)} \right) \}, \ \Re \{ \gamma \} > 0, \]
be univalent in \( U \). If the subordination
\[ q_1(z) + \gamma z q_1'(z) < \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu \{ 1 + \mu \left( \frac{z I^{\alpha} H_m^l[\alpha_1]f'}{I^{\alpha} H_m^l[\alpha_1]f(z)} - \frac{z \rho'(z)}{\rho(z)} \right) \} \]
holds. Then
\[ q_1(z) < \left[ \frac{I^{\alpha} H_m^l[\alpha_1]f(z)}{\rho_\alpha(z)} \right]^\mu \]
and \( q_1, q_2 \) are respectively the best subordinant and the best dominant.


Let \( B := C[U, \mathbb{C}] \) be a Banach space of all continuous functions on \( U \) endowed with the sup. norm
\[ \| u \| := \sup_{z \in U} |u(z)|. \]
By using the properties in Theorem 2.1, we can easily obtain the following result:

Lemma 3.1. If the function \( f \in A \), then the initial value problem (1.1) is equivalent to the nonlinear integral equation
\[ u(z) = \frac{1}{\rho_\alpha(z)} \int_0^z (z - \zeta)^{\alpha-1} \frac{H_m^l[\alpha_1]f(\zeta)}{\Gamma(\alpha)} d\zeta. \] \hfill (3.1)
In other words, every solution of the equation (3.1) is also a solution of the initial value problem (1.1) and vice versa.

Theorem 3.1. (Existence) Assume that \( \frac{1}{|\rho_\alpha(z)|} \leq M; M > 0 \). Then there exists a univalent function \( u : U \to \mathbb{C} \) solving the problem (1.1).

Proof. Define an operator \( P : \mathbb{C} \to \mathbb{C} \)
\[ (Pu)(z) := \frac{1}{\rho_\alpha(z)} \int_0^z (z - \zeta)^{\alpha-1} \frac{H_m^l[\alpha_1]f(\zeta)}{\Gamma(\alpha)} d\zeta. \] \hfill (3.2)
Denotes $B_n := \frac{(\alpha_i)_{n-1}}{(n-1)!}$. Our aim is to apply Theorem 2.1. First we show that $P$ is bounded operator:

$$|(Pu)(z)| = \left| \frac{1}{\rho_n(z)} \int_0^z \frac{(z - \zeta)^{\alpha - 1}}{\Gamma(\alpha)} H_n^1[\alpha_1]f(\zeta)d\zeta \right|$$

$$\leq \frac{1}{\rho_n(z)} \int_0^z \frac{(z - \zeta)^{\alpha - 1}}{\Gamma(\alpha)} H_n^1[\alpha_1]f(\zeta)d\zeta$$

$$< M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n]) \int_0^z (z - \zeta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} d\zeta$$

$$= M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n]) \frac{|z^\alpha|}{\Gamma(\alpha + 1)}$$

Thus we obtain that

$$\|P\| < \frac{M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n])}{\Gamma(\alpha + 1)} := r$$

that is $P : B_r \to B_r$. Then $P$ maps $B_r$ into itself. Now we proceed to prove that $P$ is equicontinuous. For $z_1, z_2 \in U$ such that $z_1 \neq z_2$, $|z_2 - z_1| < \delta$, $\delta > 0$ Then for all $u \in S$, where

$$S := \{u \in \mathbb{C} : |u| \leq \frac{M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n])}{\Gamma(\alpha + 1)} := r, r > 0\},$$

we obtain

$$|(Pu)(z_1) - (Pu)(z_2)|$$

$$\leq M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n]) \int_0^{z_1} (z_1 - \zeta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} d\zeta - \int_0^{z_2} (z_2 - \zeta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} d\zeta$$

$$\leq M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n]) \int_0^{z_1} ((z_1 - \zeta)^{\alpha - 1} - (z_2 - \zeta)^{\alpha - 1}) \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} d\zeta + \int_{z_1}^{z_2} (z_2 - \zeta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} d\zeta$$

$$= \frac{M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n])}{\Gamma(\alpha + 1)} \left[|2(z_2 - z_1)^\alpha + z_2^\alpha - z_1^\alpha|\right]$$

$$< \frac{2M(1 + \sum_{n=2}^{\infty} B_n[\alpha_n])}{\Gamma(\alpha + 1)} \left|z_2 - z_1\right|^\alpha$$

which is independent on $u$. Hence $P$ is an equicontinuous mapping on $S$. By the assumption of the theorem we can show that $P$ is a univalent function (see [17]).

The Arzela-Ascoli theorem yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that $P$ has a fixed point. By construction, a fixed point of $P$ is a univalent solution of the initial value problem (1.1).

The next theorems show the relation between univalent solutions and the subordination for a class of fractional differential problem.

**Theorem 3.2.** Let the assumptions of Theorem 2.6 be satisfied. Then univalent solutions $u_1, u, u_2$, of the problem

$$D_2^\alpha u(z) = F(z, u(z)),$$

(3.3)
subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$ and $F : U \times \mathbb{C} \to \mathbb{C}$, is an analytic functions in $z \in U$, are satisfying the subordination $u_1 \prec u \prec u_2$.

**Proof.** Setting $\mu = 1$ and let $F(z, u_1(z)) := \frac{H_1^\mu[a_1]g_1(z)}{\rho_a(z)}$, $F(z, u(z)) := \frac{H_1^\mu[a_1]f(z)}{\rho_a(z)}$, and $F(z, u_2(z)) := \frac{H_1^\mu[a_1]g_2(z)}{\rho_a(z)}$ where $\rho_a(z) \neq 0, \forall z \in U$.

**Theorem 3.3.** Let the assumptions of Theorem 2.7 be satisfied. Then every univalent solution $u(z)$ of the problem (3.3) satisfies the subordination $q_1(z) \prec u(z) \prec q_2(z)$, where $q_1(z)$ and $q_2(z)$ are univalent function in $U$.

**Proof.** Setting $\mu = 1$, $F(z, u(z)) := \frac{H_1^1[a_1]f(z)}{\rho_a(z)}$.

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