A NEW SUBCLASS OF MEROMORPHIC FUNCTION WITH POSITIVE COEFFICIENTS

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Abstract. In the present investigation, the authors define a new class of meromorphic functions defined in the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also prove a Property using an integral operator and its inverse defined on the new class.

1. Introduction

Let $\Sigma$ denote the class of normalized meromorphic functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

(1.1)

defined on the punctured unit disk

$$\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

The Hadamard product or convolution of two functions $f(z)$ given by (1.1) and

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n$$

(1.2)

is defined by

$$(f \ast g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n g_n z^n.$$

A function $f \in \Sigma$ is meromorphic starlike of order $\alpha$ ($0 \leq \alpha < 1$) if

$$-\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \Delta := \Delta^* \cup \{0\}).$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. Similarly the class of convex functions of order $\alpha$ is defined. Let $\Sigma_P$ be the class of functions $f \in \Sigma$ with $a_n \geq 0$. The subclass of $\Sigma_P$ consisting of starlike functions of order $\alpha$ is denoted by $\Sigma_P^*(\alpha)$.

Now, we define a new class of functions in Definition 1.
Theorem 2.1. Let $M$ be in the class $\Sigma$. This shows that (2.1) holds.

Definition 1. Let $0 \leq \alpha < 1$. Further, let $f(z) \in \Sigma_{p}$ be given by (1.1). Then $f \in M_{p}(\alpha, \lambda)$ if and only if

$$M_{p}(\alpha, \lambda) = \left\{ f \in \Sigma_{p} : \Re \left( \frac{zf'(z)}{(\lambda - 1)f(z) + \lambda zf'(z)} \right) > \alpha \right\}. $$

Clearly, $M_{p}(\alpha, 0)$ reduces to the class $\Sigma_{p}(\alpha)$.

The class $\Sigma_{p}(\alpha)$ and various other subclasses of $\Sigma$ have been studied rather extensively by Clunie [4], Nehari and Netanyhau [5], Pommerenke ([9], [10]), Royset [11], and others (cf., e.g., Bajpai [2], Mogra et al. [7], Uralegaddi and Ganigi [16], Cho et al. [3], Aouf [3], and Uralegaddi and Somanatha [15]; see also Duren [5], pages 29 and 137, and Srivastava and Owa [13], pages 86 and 429) (see also [11]).

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $M_{p}(\alpha, \lambda)$. Properties of a certain integral operator and its inverse defined on the new class $M_{p}(\alpha, \lambda)$ are also discussed.

2. COEFFICIENTS INEQUALITIES

Our first theorem gives a necessary and sufficient condition for a function $f$ to be in the class $M_{p}(\alpha, \lambda)$.

Theorem 2.1. Let $f(z) \in \Sigma_{p}$ be given by (1.1). Then $f \in M_{p}(\alpha, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda (1 + n)\} a_{n} \leq 1 - \alpha. \quad (2.1)$$

Proof. If $f \in M_{p}(\alpha, \lambda)$, then

$$\Re \left( \frac{zf'(z)}{(\lambda - 1)f(z) + \lambda zf'(z)} \right) = \Re \left\{ \frac{-1 + \sum_{n=1}^{\infty} na_{n}z^{n+1}}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n)a_{n}z^{n+1}} \right\} \geq \alpha. $$

By letting $z \to 1^{-}$, we have

$$\left\{ \frac{-1 + \sum_{n=1}^{\infty} na_{n}}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n)a_{n}} \right\} \geq \alpha. $$

This shows that (2.1) holds.

Conversely assume that (2.1) holds. It is sufficient to show that

$$\left| \frac{zf'(z) - \{(\lambda - 1)f(z) + \lambda zf'(z)\}}{zf'(z) + (1 - 2\alpha) \{(\lambda - 1)f(z) + \lambda zf'(z)\}} \right| < 1 \quad (z \in \Delta).$$

Using (2.1), we see that

$$\left| \frac{zf'(z) - \{(\lambda - 1)f(z) + \lambda zf'(z)\}}{zf'(z) + (1 - 2\alpha) \{(\lambda - 1)f(z) + \lambda zf'(z)\}} \right| = \left| \frac{\sum_{n=1}^{\infty} (1 - \lambda) (n + 1) a_{n} z^{n+1}}{-2(1 - \alpha) + \sum_{n=1}^{\infty} [(1 + (1 - 2\alpha) \lambda) n + (1 - 2\alpha)(\lambda - 1)] a_{n} z^{n+1}} \right| \leq \frac{2(1 - \alpha) - \sum_{n=1}^{\infty} [(1 + (1 - 2\alpha) \lambda) n + (1 - 2\alpha)(\lambda - 1)] a_{n}}{2(1 - \alpha)} \leq 1. $$

Thus we have $f \in M_{p}(\alpha, \lambda)$.

For the choice of $\lambda = 0$, we get the following.
Remark 2.2. Let \( f(z) \in \Sigma_P \) be given by (1.1). Then \( f \in \Sigma^*_P(\alpha) \) if and only if
\[
\sum_{n=1}^{\infty} (n + \alpha) a_n \leq 1 - \alpha.
\]

Our next result gives the coefficient estimates for functions in \( M_P(\alpha, \lambda) \).

Theorem 2.3. If \( f \in M_P(\alpha, \lambda) \), then
\[
a_n \leq \frac{1 - \alpha}{\{n + \alpha - \alpha\lambda(1+n)\}}, \quad n = 1, 2, 3, \ldots.
\]
The result is sharp for the functions \( F_n(z) \) given by
\[
F_n(z) = \frac{1}{z} + \frac{1 - \alpha}{\{n + \alpha - \alpha\lambda(1+n)\}} z^n, \quad n = 1, 2, 3, \ldots.
\]

Proof. If \( f \in M_P(\alpha, \lambda) \), then we have, for each \( n \),
\[
\{n + \alpha - \alpha\lambda(1+n)\} a_n \leq \sum_{n=1}^{\infty} \{n + \alpha - \alpha\lambda(1+n)\} a_n \leq 1 - \alpha.
\]
Therefore we have
\[
a_n \leq \frac{1 - \alpha}{\{n + \alpha - \alpha\lambda(1+n)\}}.
\]
Since
\[
F_n(z) = \frac{1}{z} + \frac{1 - \alpha}{\{n + \alpha - \alpha\lambda(1+n)\}} z^n
\]
satisfies the conditions of Theorem 2.1, \( F_n(z) \in M_P(\alpha, \lambda) \) and the equality is attained for this function. \( \square \)

For \( \lambda = 0 \), we have the following corollary.

Remark 2.4. If \( f \in \Sigma^*_P(\alpha) \), then
\[
a_n \leq \frac{1 - \alpha}{n + \alpha}, \quad n = 1, 2, 3, \ldots.
\]

Theorem 2.5. If \( f \in M_P(\alpha, \lambda) \), then
\[
\frac{1}{r} - \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} r \leq |f(z)| \leq \frac{1}{r} + \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} r \quad (|z| = r).
\]
The result is sharp for
\[
f(z) = \frac{1}{z} + \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} z.
\]

Proof. Since \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \), we have
\[
|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n.
\]
Since,
\[
\sum_{n=1}^{\infty} a_n \leq \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda}.
\]
Using this, we have
\[
|f(z)| \leq \frac{1}{r} + \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} r.
\]
Similarly
\[ |f(z)| \geq \frac{1}{r} - \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} r. \]

The result is sharp for \( f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} z \). \( \square \)

Similarly we have the following:

**Theorem 2.6.** If \( f \in M_P(\alpha, \lambda) \), then
\[ \frac{1}{r^2} - \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda} \quad (|z| = r). \]

The result is sharp for the function given by (2.2).

3. Neighborhoods for the class \( M_p^{(\gamma)}(\alpha, \lambda) \)

In this section, we determine the neighborhood for the class \( M_p^{(\gamma)}(\alpha, \lambda) \), which we define as follows:

**Definition 2.** A function \( f \in \Sigma_p \) is said to be in the class \( M_p^{(\gamma)}(\alpha, \lambda) \) if there exists a function \( g \in M_p(\alpha, \lambda) \) such that
\[ \frac{|f(z)|}{|g(z)|} - 1 < 1 - \gamma, \quad (z \in \Delta, 0 \leq \gamma < 1). \] (3.1)

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [14], we define the \( \delta \)-neighborhood of a function \( f \in \Sigma_p \) by
\[ N_\delta(f) := \left\{ g \in \Sigma_p : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \] (3.2)

**Theorem 3.1.** If \( g \in M_p(\alpha, \lambda) \) and
\[ \gamma = 1 - \frac{\delta(1 + \alpha - 2\alpha\lambda)}{2\alpha - 2\alpha\lambda}, \] (3.3)
then
\[ N_\delta(g) \subset M_p^{(\gamma)}(\alpha, \lambda). \]

**Proof.** Let \( f \in N_\delta(g) \). Then we find from (3.2) that
\[ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \] (3.4)
which implies the coefficient inequality
\[ \sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}). \] (3.5)

Since \( g \in M_p(\alpha, \lambda) \), we have [cf. equation (2.1)]
\[ \sum_{n=1}^{\infty} b_n \leq \frac{1 - \alpha}{1 + \alpha - 2\alpha\lambda}, \] (3.6)
so that
\[
\frac{|f(z)|}{|g(z)|} - 1 < \frac{\sum_{n=1}^\infty |a_n - b_n|}{1 - \sum_{n=1}^\infty b_n}
\]
\[
= \frac{\delta(1 + \alpha - 2\alpha\lambda)}{2\alpha - 2\alpha\lambda}
\]
\[
= 1 - \gamma,
\]
provided \(\gamma\) is given by (3.3). Hence, by definition, \(f \in M_p^{(\gamma)}(\alpha, \lambda)\) for \(\gamma\) given by (3.3), which completes the proof. \(\square\)

4. Closure Theorems

Let the functions \(F_k(z)\) be given by
\[
F_k(z) = \frac{1}{z} + \sum_{n=1}^\infty f_{n,k}z^n, \quad k = 1, 2, ..., m. \tag{4.1}
\]
We shall prove the following closure theorems for the class \(M_P(\alpha, \lambda)\).

**Theorem 4.1.** Let the function \(F_k(z)\) defined by (4.1) be in the class \(M_P(\alpha, \lambda)\) for every \(k = 1, 2, ..., m\). Then the function \(f(z)\) defined by
\[
f(z) = \frac{1}{z} + \sum_{n=1}^\infty a_n z^n \quad (a_n \geq 0)
\]
belongs to the class \(M_P(\alpha, \lambda)\), where \(a_n = \frac{1}{m} \sum_{k=1}^m f_{n,k} \quad (n = 1, 2, ..)
\]
Proof. Since \(F_n(z) \in M_P(\alpha, \lambda)\), it follows from Theorem 2.1 that
\[
\sum_{n=1}^\infty \{n + \alpha - \alpha \lambda(1 + n)\} f_{n,k} \leq 1 - \alpha \tag{4.2}
\]
for every \(k = 1, 2, ..., m\). Hence
\[
\sum_{n=1}^\infty \{n + \alpha - \alpha \lambda(1 + n)\} a_n = \sum_{n=1}^\infty \{n + \alpha - \alpha \lambda(1 + n)\} \left(\frac{1}{m} \sum_{k=1}^m f_{n,k}\right)
\]
\[
= \frac{1}{m} \sum_{k=1}^m \left(\sum_{n=1}^\infty \{n + \alpha - \alpha \lambda(1 + n)\} f_{n,k}\right)
\]
\[
\leq 1 - \alpha.
\]
By Theorem 2.1 it follows that \(f(z) \in M_P(\alpha, \lambda)\). \(\square\)

**Theorem 4.2.** The class \(M_P(\alpha, \lambda)\) is closed under convex linear combination.

Proof. Let the function \(F_k(z)\) given by (4.1) be in the class \(M_P(\alpha, \lambda)\). Then it is enough to show that the function
\[
H(z) = \lambda F_1(z) + (1 - \lambda) F_2(z) \quad (0 \leq \lambda \leq 1)
\]
is also in the class \(M_P(\alpha, \lambda)\). Since for \(0 \leq \lambda \leq 1\),
\[
H(z) = \frac{1}{z} + \sum_{n=1}^\infty [\lambda f_{n,1} + (1 - \lambda) f_{n,2}] z^n,
\]
we observe that
\[
\sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda(1 + n)\} [\lambda f_{n,1} + (1 - \lambda) f_{n,2}]
\]
\[
= \lambda \sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda(1 + n)\} f_{n,1} + (1 - \lambda) \sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda(1 + n)\} f_{n,2}
\]
\[
\leq 1 - \alpha.
\]
By Theorem 2.1, we have \(H(z) \in M_P(\alpha, \lambda)\). \(\square\)

**Theorem 4.3.** Let \(F_0(z) = \frac{1}{z}\) and \(F_n(z) = \frac{1}{z} + \frac{1}{n+\alpha - \alpha \lambda(1+n)} z^n\) for \(n = 1, 2, \ldots\). Then \(f(z) \in M_P(\alpha, \lambda)\) if and only if \(f(z)\) can be expressed in the form \(f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)\) where \(\lambda_n \geq 0\) and \(\sum_{n=0}^{\infty} \lambda_n = 1\).

**Proof.** Let
\[
f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)
\]
\[
= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda_n(1 - \alpha)}{\{n + \alpha - \alpha \lambda(1 + n)\}} z^n.
\]
Then
\[
\sum_{n=1}^{\infty} \lambda_n \frac{1 - \alpha}{\{n + \alpha - \alpha \lambda(1 + n)\}} \frac{n + \alpha - \alpha \lambda(1 + n)}{(1 - \alpha)} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.
\]
By Theorem 2.1, we have \(f(z) \in M_P(\alpha, \lambda)\).

Conversely, let \(f(z) \in M_P(\alpha, \lambda)\). From Theorem 2.3, we have
\[
a_n \leq \frac{1 - \alpha}{\{n + \alpha - \alpha \lambda(1 + n)\}} \quad \text{for} \quad n = 1, 2, \ldots
\]
we may take
\[
\lambda_n = \frac{n + \alpha - \alpha \lambda(1 + n)}{1 - \alpha} a_n \quad \text{for} \quad n = 1, 2, \ldots
\]
and
\[
\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.
\]
Then
\[
f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z).
\]
5. Partial Sums

Silverman \cite{12} determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman \cite{12} and Cho and Owa \cite{3} we will investigate the ratio of a function of the form

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (5.1) \]

to its sequence of partial sums

\[ f_1(z) = \frac{1}{z} \quad \text{and} \quad f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \quad (5.2) \]

when the coefficients are sufficiently small to satisfy the condition analogous to

\[ \sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda (1 + n)\} \ a_n \leq 1 - \alpha. \]

For the sake of brevity we rewrite it as

\[ \sum_{n=1}^{\infty} d_n |a_n| \leq 1 - \alpha, \quad (5.3) \]

where

\[ d_n := n + \alpha - \alpha \lambda (1 + n) \quad (5.4) \]

More precisely we will determine sharp lower bounds for \( \Re \{ f(z) / f_k(z) \} \) and \( \Re \{ f_k(z) / f(z) \} \).

In this connection we make use of the well known results that

\[ \Re \left\{ \frac{1 + w(z)}{1 - w(z)} \right\} > 0 \quad (z \in \Delta) \]

if and only if \( \omega(z) = \sum_{n=1}^{\infty} c_n z^n \) satisfies the inequality \( |\omega(z)| \leq |z| \). Unless otherwise stated, we will assume that \( f \) is of the form (1.1) and its sequence of partial sums is denoted by \( f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \).

**Theorem 5.1.** Let \( f(z) \in MP(\alpha, \lambda) \) be given by (5.1) satisfies condition, \( \square \) \( \square \)

\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{d_{k+1}(\lambda, \alpha) - 1 + \alpha}{d_{k+1}(\lambda, \alpha)} \quad (z \in U) \quad (5.5) \]

where

\[ d_n(\lambda, \alpha) \geq \begin{cases} 1 - \alpha, & \text{if} \quad n = 1, 2, 3, \ldots, k \\ d_{k+1}(\lambda, \alpha), & \text{if} \quad n = k + 1, k + 2, \ldots. \end{cases} \quad (5.6) \]

The result (5.5) is sharp with the function given by

\[ f(z) = \frac{1}{z} + \frac{1 - \alpha}{d_{k+1}(\lambda, \alpha)} z^{k+1}. \quad (5.7) \]

**Proof.** Define the function \( w(z) \) by

\[ \frac{1 + w(z)}{1 - w(z)} = \frac{d_{k+1}(\lambda, \alpha)}{1 - \alpha} \left[ \frac{f(z)}{f_k(z)} - \frac{d_{k+1}(\lambda, \alpha) - 1 + \alpha}{d_{k+1}(\lambda, \alpha)} \right] \]
\[ w(z) = \frac{1 + \sum_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \sum_{n=k+1}^\infty a_n z^{n+1}}{1 + \sum_{n=1}^{k} a_n z^{n+1}} \] (5.8)

It suffices to show that \(|w(z)| \leq 1\). Now, from (5.8) we can write

\[ w(z) = \frac{1}{2} \left( \frac{d_{k+1}(\lambda, \alpha)}{1-\alpha} \right) \sum_{n=k+1}^\infty a_n z^{n+1} - \left(\frac{d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \sum_{n=k+1}^\infty \left| a_n \right| \]

Hence we obtain

\[ |w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \sum_{n=k+1}^\infty \left| a_n \right|}{2 - 2 \sum_{n=1}^{k} \left| a_n \right| - \left(\frac{d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \sum_{n=k+1}^\infty \left| a_n \right|} \]

Now \(|w(z)| \leq 1\) if

\[ 2 \left(\frac{d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \sum_{n=k+1}^\infty \left| a_n \right| \leq 2 - 2 \sum_{n=1}^{k} \left| a_n \right| \]

or, equivalently,

\[ \sum_{n=1}^{k} \left| a_n \right| + \frac{d_{k+1}(\lambda, \alpha)}{1-\alpha} \sum_{n=k+1}^\infty \left| a_n \right| \leq 1. \]

From the condition (2.1), it is sufficient to show that

\[ \sum_{n=1}^{k} \left| a_n \right| + \frac{d_{k+1}(\lambda, \alpha)}{1-\alpha} \sum_{n=k+1}^\infty \left| a_n \right| \leq \sum_{n=1}^\infty \frac{d_n(\lambda, \alpha)}{1-\alpha} \left| a_n \right| \]

which is equivalent to

\[ \sum_{n=1}^{k} \left(\frac{d_n(\lambda, \alpha) - 1 + \alpha}{1-\alpha}\right) \left| a_n \right| + \sum_{n=k+1}^\infty \left(\frac{d_n(\lambda, \alpha) - d_{k+1}(\lambda, \alpha)}{1-\alpha}\right) \left| a_n \right| \geq 0 \] (5.9)

To see that the function given by (5.7) gives the sharp result, we observe that for \(z = re^{i\pi/k}\)

\[ \frac{f(z)}{f_k(z)} = 1 + \frac{1 - \alpha}{d_{k+1}(\lambda, \alpha)} z^n \to 1 - \frac{1 - \alpha}{d_{k+1}(\lambda, \alpha)} \]

\[ = \frac{d_{k+1}(\lambda, \alpha) - 1 + \alpha}{d_{k+1}(\lambda, \alpha)} \text{ when } r \to 1^{-}. \]
which shows the bound \( (5.5) \) is the best possible for each \( k \in \mathbb{N} \). □

The proof of the next theorem is much akin to that of the earlier theorem and hence we state the theorem without proof.

**Theorem 5.2.** Let \( f(z) \in M_P(\alpha, \lambda) \) be given by \( (5.1) \) satisfies condition, \( (2.1) \)

\[
\text{Re} \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{d_{k+1}(\lambda, \alpha)}{d_{k+1}(\lambda, \alpha) + 1 - \alpha} \quad (z \in U) \tag{5.10}
\]

where

\[
d_{k+1}(\lambda, \alpha) \geq 1 - \alpha \\
d_n(\lambda, \alpha) \geq \begin{cases} 
1 - \alpha, & \text{if } n = 1, 2, 3, \ldots, k \\
d_{k+1}(\lambda, \alpha), & \text{if } n = k + 1, k + 2, \ldots 
\end{cases} \tag{5.11}
\]

The result \( (5.10) \) is sharp with the function given by

\[
f(z) = \frac{1}{z} + \frac{1 - \alpha}{d_{k+1}(\lambda, \alpha)} z^{k+1}. \tag{5.12}
\]

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6. **Radius of meromorphic starlikeness and meromorphic convexity**

The radii of starlikeness and convexity for the class are given by the following theorems for the class \( M_P(\alpha, \lambda) \).

**Theorem 6.1.** Let the function \( f \) be in the class \( M_P(\alpha, \lambda) \). Then \( f \) is meromorphically starlike of order \( \rho (0 \leq \rho < 1) \), in \( |z| < r_1(\alpha, \lambda, \rho) \), where

\[
r_1(\alpha, \lambda, \rho) = \inf_{n \geq 1} \left[ \frac{(1 - \rho)(1 - \alpha)}{(n + 2 - \rho)(n + \alpha - \alpha \lambda(1 + n))} \right]^{\frac{1}{\pi^+}}, \tag{6.1}
\]

**Proof.** Since,

\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,
\]

we get

\[
f'(z) = -\frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}.
\]

It is sufficient to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \tag{6.2}
\]

or equivalently

\[
\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \sum_{n=1}^{\infty} \frac{(n+1)a_n z^n}{1 - \sum_{n=1}^{\infty} a_n z^n} \right| \leq 1 - \rho
\]

or

\[
\sum_{n=1}^{\infty} \left( \frac{n+2-\rho}{1-\rho} \right) a_n |z|^{n+1} \leq 1,
\]

for \( 0 \leq \rho < 1 \), and \( |z| < r_1(\alpha, \lambda, \rho) \). By Theorem \( 2.1 \) \( (6.2) \) will be true if

\[
\left( \frac{n+2-\rho}{1-\rho} \right) \frac{1 - \alpha}{n + \alpha - \alpha \lambda(1 + n)} \leq \frac{1 - \alpha}{n + \alpha - \alpha \lambda(1 + n)}.
\]
or, if
\[|z| \leq \left[ \frac{(1 - \rho)(1 - \alpha)}{(n + 2 - \rho) (n + \alpha - \alpha\lambda(1 + n))} \right]^{\frac{1}{n+1}}, \quad n \geq 1. \quad (6.3)\]
This completes the proof of Theorem 6.1.

**Theorem 6.2.** Let the function \(f\) in the class \(M_p(\alpha, \lambda)\). Then \(f\) is meromorphically convex of order \(\rho\), \((0 \leq \rho < 1)\), in \(|z| < r_2(\alpha, \lambda, \rho)\), where
\[r_2(\alpha, \lambda, \rho) = \inf_{n \geq 1} \left[ \frac{(1 - \rho)(1 - \alpha)}{n(n + 2 - \rho) (n + \alpha - \alpha\lambda(1 + n))} \right]^{\frac{1}{n+1}}, \quad n \geq 1, \quad (6.4)\]

**Proof.** Since,
\[f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n,\]
we get
\[f'(z) = -\frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}.\]
It is sufficient to show that
\[-1 - \frac{zf''(z)}{f'(z)} - 1 \leq 1 - \rho \quad \text{or equivalently} \quad (6.5)\]
\[\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \sum_{n=1}^{\infty} \frac{n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}} \right| \leq 1 - \rho \quad \text{or}\]
\[\sum_{n=1}^{\infty} \left( \frac{n(n+2-\rho)}{1-\rho} \right) a_n |z|^{n+1} \leq 1,\]
for \(0 \leq \rho < 1\), and \(|z| < r_2(\alpha, \lambda, \rho)\). By Theorem \ref{thm:rho_convex}, \ref{thm:convexity} will be true if
\[\left( \frac{n(n+2-\rho)}{1-\rho} \right) |z|^{n+1} \leq \frac{(1-\alpha)}{(n + \alpha - \alpha\lambda(1 + n))}\]
or, if
\[|z| \leq \left[ \frac{(1 - \rho)(1 - \alpha)}{(n + 2 - \rho) (n + \alpha - \alpha\lambda(1 + n))} \right]^{\frac{1}{n+1}}, \quad n \geq 1. \quad (6.6)\]
This completes the proof of Theorem 6.2.

7. **Integral Operators**

In this section, we consider integral transforms of functions in the class \(M_p(\alpha, \lambda)\).

**Theorem 7.1.** Let the function \(f(z)\) given by \ref{eqn:fn} be in \(M_p(\alpha, \lambda)\). Then the integral operator
\[F(z) = c \int_0^1 u^{-\gamma} f(uz) du \quad (0 < u \leq 1, 0 < c < \infty)\]
is in \(M_p(\delta, \lambda)\), where
\[\delta = \frac{(c + 2) \{1 + \alpha - 2\alpha\lambda\} - c(1-\alpha)}{c(1-\alpha) \{1 - 2\lambda\} + (1 + \alpha) \{1 - 2\lambda\} (c + 2)}.\]
The result is sharp for the function \( f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha-2\alpha z} z \).

Proof. Let \( f(z) \in M_p(\alpha, \lambda) \). Then

\[
F(z) = c \int_0^1 u^\epsilon f(uz) du = c \int_0^1 \left( \frac{u^{\epsilon-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+\epsilon} z^n \right) du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c + n + 1} f_n z^n.
\]

It is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{c \{ n + \delta - \delta \lambda (1 + n) \}}{(c + n + 1)(1 - \delta)} a_n \leq 1. \tag{7.1}
\]

Since \( f \in M_p(\alpha, \lambda) \), we have

\[
\sum_{n=1}^{\infty} \frac{\{ n + \alpha - \alpha \lambda (1 + n) \}}{(1 - \alpha)} a_n \leq 1.
\]

Note that (7.1) is satisfied if

\[
\frac{c \{ n + \delta - \delta \lambda (1 + n) \}}{(c + n + 1)(1 - \delta)} \leq \frac{n + \alpha - \alpha \lambda (1 + n)}{(1 - \alpha)}.
\]

Rewriting the inequality, we have

\[
c \{ n + \delta - \delta \lambda (1 + n) \} (1 - \alpha) \leq (c + n + 1)(1 - \delta) \{ n + \alpha - \alpha \lambda (1 + n) \}.
\]

Solving for \( \delta \), we have

\[
\delta \leq \frac{(c + n + 1) \{ n + \alpha - \alpha \lambda (1 + n) \} - cn(1 - \alpha)}{c(1 - \alpha) \{ 1 - \lambda (1 + n) \} + \{(n + \alpha - \alpha \lambda (1 + n)) (c + n + 1) = F(n).}
\]

A simple computation will show that \( F(n) \) is increasing and \( F(n) \geq F(1) \). Using this, the results follows. \( \square \)

For the choice of \( \lambda = 0 \), we have the following result of Uralegaddi and Ganigi \[15\].

Remark 7.2. Let the function \( f(z) \) defined by (1) be in \( \Sigma_p^*(\alpha) \). Then the integral operator

\[
F(z) = c \int_0^1 u^\epsilon f(uz) du \quad (0 < u \leq 1, 0 < c < \infty)
\]

is in \( \Sigma_p(\delta) \), where \( \delta = \frac{1+n+\alpha}{1+\alpha+c} \). The result is sharp for the function

\[
f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha z}.
\]

Also we have the following:
Theorem 7.3. Let \( f(z) \), given by (1), be in \( M_p(\alpha, \lambda) \),

\[
F(z) = \frac{1}{c}[(c + 1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c + n + 1}{c} f_n z^n, \quad c > 0. \tag{7.2}
\]

Then \( F(z) \) is in \( M_p(\alpha, \lambda) \) for \( |z| \leq r(\alpha, \lambda, \beta) \) where

\[
r(\alpha, \lambda, \beta) = \inf_n \left( \frac{c(1 - \beta)\{n + \alpha - \alpha \lambda(1 + n)\}}{(1 - \alpha)(c + n + 1)\{n + \beta - \beta \lambda(1 + n)\}} \right)^{1/(n+1)}, \quad n = 1, 2, 3, \ldots.
\]

The result is sharp for the function \( f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n + \alpha - \alpha \lambda(1 + n)} z^n, \quad n = 1, 2, 3, \ldots \)

Proof. Let \( w = \frac{zf'(z)}{(\lambda-1)f(z) + zf'(z)} \). Then it is sufficient to show that

\[
\left| \frac{w - 1}{w + 1 - 2\beta} \right| < 1.
\]

A computation shows that this is satisfied if

\[
\sum_{n=1}^{\infty} \frac{\{n + \beta - \beta \lambda(1 + n)\} (c + n + 1)}{(1 - \beta)c} a_n |z|^{n+1} \leq 1. \tag{7.3}
\]

Since \( f \in M_p(\alpha, \lambda) \), by Theorem 2.1 we have

\[
\sum_{n=1}^{\infty} \{n + \alpha - \alpha \lambda(1 + n)\} a_n \leq 1 - \alpha.
\]

The equation (7.3) is satisfied if

\[
\frac{\{n + \beta - \beta \lambda(1 + n)\} (c + n + 1)}{(1 - \beta)c} a_n |z|^{n+1} \leq \frac{n + \alpha - \alpha \lambda(1 + n)}{1 - \alpha} a_n.
\]

Solving for \( |z| \), we get the result. \( \square \)

For the choice of \( \lambda = 0 \), we have the following result of Uralegaddi and Ganigi [15].

Remark 7.4. Let the function \( f(z) \) defined by (1) be in \( \Sigma_p^*(\alpha) \) and \( F(z) \) given by (7.2). Then \( F(z) \) is in \( \Sigma_p^*(\alpha) \) for \( |z| \leq r(\alpha, \beta) \) where

\[
r(\alpha, \beta) = \inf_n \left( \frac{c(1 - \beta)(n + \alpha)}{(1 - \alpha)(c + n + 1)(n + \beta)} \right)^{1/(n+1)}, \quad n = 1, 2, 3, \ldots.
\]

The result is sharp for the function \( f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n + \alpha} z^n, \quad n = 1, 2, 3, \ldots \)

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