ON THREE-DIMENSIONAL LORENTZIAN $\alpha$-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study three-dimensional Lorentzian $\alpha$-Sasakian manifolds which are Ricci-semisymmetric, locally $\phi$-symmetric and have $\eta$-parallel Ricci tensor. An example of a three-dimensional Lorentzian $\alpha$-Sasakian manifold is given which verifies all the Theorems.

1. Introduction

The product of an almost contact manifold $M$ and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes $M$ to be an almost contact metric manifold and supposes that the product metric $G$ on $M \times \mathbb{R}$ is Kaehlerian, then the structure on $M$ is cosymplectic [9] and not Sasakian. On the other hand Oubina [13] pointed out that if the conformally related metric $e^{2t}G$, $t$ being the coordinate on $\mathbb{R}$, is Kaehlerian, then $M$ is Sasakian and conversely.

In [15], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing $\xi$ is a constant, say $c$. He showed that they can be divided into three classes:

(i) homogeneous normal contact Riemannian manifolds with $c > 0$,
(ii) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$,
(iii) a warped product space if $c < 0$.

It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, $W_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [4]. An almost contact metric structure on a manifold $M$...
is called a trans-Sasakian structure [13, 1] if the product manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_6 \oplus C_5$ [12] coincides with the class of the trans-Sasakian structures of type $(\alpha, \beta)$. In fact, in [12], local nature of the two subclasses, namely, $C_5$ and $C_6$, structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [1], $\beta$-Kenmotsu [8] and $\alpha$-Sasakian [8], respectively. In [17] it is proved that trans-Sasakian structures are generalized quasi-Sasakian [8]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures. Then, in [13], Yıldız and Murathan introduced Lorentzian $\alpha$-Sasakian manifolds.

Also, three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [6], De and Sarkar [5] and many others. Also three-dimensional Lorentzian Para-Sasakian manifolds have been studied by Shaikh and De [14].

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ is called a trans-Sasakian structure [13] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [7], where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (X - f, \eta(X)\frac{d}{dt}),$$

for all vector fields $X$ on $M$ and smooth functions $f$ on $M \times \mathbb{R}$, and $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

(1.1)

for some smooth functions $\alpha$ and $\beta$ on $M$, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. From the formula (1.1) it follows that

$$\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X))\xi,$$

(1.2)

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

(1.3)

More generally one has the notion of an $\alpha$-Sasakian structure [8] which may be defined by

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X),$$

(1.4)

where $\alpha$ is a non-zero constant. From the condition one may readily deduce that

$$\nabla_X \xi = -\alpha \phi X,$$

(1.5)

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y).$$

(1.6)

Thus $\beta = 0$ and therefore a trans-Sasakian structure of type $(\alpha, \beta)$ with $\alpha$ a non-zero constant is always $\alpha$-Sasakian [8]. If $\alpha = 1$, then $\alpha$-Sasakian manifold is a Sasakian manifold.

Let $(x, y, z)$ be Cartesian coordinates in $\mathbb{R}^3$, then $(\phi, \xi, \eta, g)$ given by

$$\xi = \partial/\partial z, \quad \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \quad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix},$$

is a trans-Sasakian structure of type $(-1/(2e^z), 1/2)$ in $\mathbb{R}^3$ [2]. In general, in a three-dimensional $K$-contact manifold with structure tensors $(\phi, \xi, \eta, g)$ for a non-constant function $f$, if we define $g' = fg + (1 - f)\eta \otimes \eta$; then $(\phi, \xi, \eta, g')$ is a trans-Sasakian structure of type $(1/f, (1/2)\xi(ln f))$ [3, 8, 11].
Proposition 1.1. [11] A trans-Sasakian manifold of dimension $\geq 5$ is either $\alpha$-Sasakian, $\beta$-Kenmotsu or cosymplectic.

The relation between trans-Sasakian, $\alpha$-Sasakian and $\beta$-Kenmotsu structures was discussed by Marrero [11].

The paper is organized as follows: After introduction in section 2, we introduce the notion of Lorentzian $\alpha$-Sasakian manifolds. In section 3, we study three-dimensional Lorentzian $\alpha$-Sasakian manifolds. In the next section we prove that a three-dimensional Ricci -semisymmetric Lorentzian $\alpha$-Sasakian manifold is a manifold of constant curvature and in section 5, it is shown that such a manifold is locally $\phi$-symmetric. In section 6, we prove that three-dimensional Lorentzian $\alpha$-Sasakian manifold with $\eta$-parallel Ricci tensor is also locally $\phi$-symmetric. In the last section, we give an example of a locally $\phi$-symmetric three-dimensional Lorentzian $\alpha$-Sasakian manifold.

2. Lorentzian $\alpha$-Sasakian manifolds

A differentiable manifold of dimension $(2n + 1)$ is called a Lorentzian $\alpha$-Sasakian manifold if it admits a $(1, 1)$-tensor field $\phi$, a contravariant vector field $\eta$, a covariant vector field $\xi$ and the Lorentzian metric $g$ which satisfy

\begin{align*}
\eta(\xi) &= -1, \quad (2.1) \\
\phi^2 &= I + \eta \otimes \xi, \quad (2.2) \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \quad (2.3) \\
g(X, \xi) &= \eta(X), \quad (2.4) \\
\phi \xi &= 0, \quad \eta(\phi X) = 0, \quad (2.5) \\
(\nabla_X \phi)Y &= \alpha(g(X, Y)\xi + \eta(Y)X), \quad (2.6)
\end{align*}

for all $X, Y \in TM$.

Also a Lorentzian $\alpha$-Sasakian manifold $M$ satisfies

\begin{align*}
\nabla_X \xi &= \alpha \phi X, \quad (2.7) \\
(\nabla_X \eta)Y &= \alpha g(X, \phi Y), \quad (2.8)
\end{align*}

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is constant.

On the other hand, on a Lorentzian $\alpha$-Sasakian manifold $M$ the following relations hold [13]:

\begin{align*}
R(\xi, X)Y &= \alpha^2(g(X, Y)\xi - \eta(Y)X), \quad (2.9) \\
R(X, Y)\xi &= \alpha^2(\eta(Y)X - \eta(X)Y), \quad (2.10) \\
R(\xi, X)\xi &= \alpha^2(\eta(X)\xi + X), \quad (2.11) \\
S(X, \xi) &= 2n\alpha^2\eta(X), \quad (2.12)
\end{align*}
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\[ Q\xi = 2\alpha^2\xi, \]  \hspace{1cm} (2.13)

\[ S(\xi, \xi) = -2\alpha^2, \]  \hspace{1cm} (2.14)

for any vector fields $X, Y, Z$, where $S$ is the Ricci curvature and $Q$ is the Ricci operator given by $S(X, Y) = g(QX, Y)$.

A Lorentzian $\alpha$-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

\[ S(X, Y) = a g(X, Y) + b\eta(X)\eta(Y), \]

for any vector fields $X, Y$, where $a, b$ are functions on $M^n$.

$\[ \text{3. Three-dimensional Lorentzian } \alpha\text{-Sasakian manifolds} \]

In a three-dimensional Lorentzian $\alpha$-Sasakian manifold the curvature tensor satisfies

\[ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \]

\[ -\frac{\tau}{2} [g(Y, Z)X - g(X, Z)Y], \]  \hspace{1cm} (3.1)

where $\tau$ is the scalar curvature.

Then putting $Z = \xi$ in (3.1) and using (2.4) and (2.12), we have

\[ R(X, Y)\xi = \eta(Y)QX - \eta(X)QY - \left[ \frac{\tau}{2} - 2\alpha^2 \right] [\eta(Y)X - \eta(X)Y]. \]  \hspace{1cm} (3.2)

Using (2.10) in (3.2), we get

\[ \eta(Y)QX - \eta(X)QY = \left[ \frac{\tau}{2} - \alpha^2 \right] [\eta(Y)X - \eta(X)Y]. \]  \hspace{1cm} (3.3)

Putting $Y = \xi$ in (3.3), we obtain

\[ QX = \left[ \frac{\tau}{2} - \alpha^2 \right] X + \left[ \frac{\tau}{2} - 3\alpha^2 \right] \eta(X)\xi. \]  \hspace{1cm} (3.4)

Then from (3.4), we get

\[ S(X, Y) = \left[ \frac{\tau}{2} - \alpha^2 \right] g(X, Y) + \left[ \frac{\tau}{2} - 3\alpha^2 \right] \eta(X)\eta(Y). \]  \hspace{1cm} (3.5)

From (3.5), it follows that a Lorentzian $\alpha$-Sasakian manifold is an $\eta$-Einstein manifold.

Lemma 3.1. A three-dimensional Lorentzian $\alpha$-Sasakian manifold is a manifold of constant curvature if and only if the scalar curvature is $6\alpha^2$.

Proof. Using (3.4) and (3.5) in (3.1), we get

\[ R(X, Y)Z = \left[ \frac{\tau}{2} - 2\alpha^2 \right] [g(Y, Z)X - g(X, Z)Y] \]

\[ + \left[ \frac{\tau}{2} - 3\alpha^2 \right] [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \]

\[ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \]  \hspace{1cm} (3.6)
From (3.6) the Lemma 3.1 is obvious.

4. THREE-DIMENSIONAL RICCI-SEMSYMMETRIC LORENTZIAN $\alpha$-SASAKIAN MANIFOLDS

**Definition 4.1.** A Lorentzian $\alpha$-Sasakian manifold is said to be Ricci-semisymmetric if the Ricci tensor $S$ satisfies

$$R(X, Y) \cdot S = 0,$$

(4.1)

where $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the manifold.

Let us consider a three-dimensional Lorentzian $\alpha$-Sasakian manifold which satisfies the condition (4.1). Hence, we can write

$$(R(X, Y) \cdot S)(U, V) = R(X, Y)S(U, V)$$

(4.2)

$$-S(R(X, Y)U, V) - S(U, R(X, Y)V) = 0.$$

Then from (4.2), we have

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.$$  

(4.3)

Putting $X = \xi$ in (4.3) and using (2.9) and (2.12), we get

$$S(R(\xi, Y)U, V) = 2\alpha^2 g(Y, U)\eta(V) - \alpha^2 S(Y, V)\eta(U),$$

(4.4)

and

$$S(U, R(\xi, Y)V) = 2\alpha^2 g(Y, V)\eta(U) - \alpha^2 S(Y, U)\eta(V).$$

(4.5)

Using (4.4) and (4.5) in (4.3), we get

$$2\alpha^2 g(Y, U)\eta(V) - S(Y, V)\eta(U)$$

(4.6)

$$+2\alpha^2 g(Y, V)\eta(U) - S(Y, U)\eta(V) = 0, \quad \alpha \neq 0.$$

Let $\{e_1, e_2, \xi\}$ be an orthonormal basis of the tangent space at each point of the three-dimensional Lorentzian $\alpha$-Sasakian manifold. Then we can write

$$\begin{cases}
g(e_i, e_j) = \delta_{ij}, & i, j = 1, 2 \\
g(\xi, \xi) = \eta(\xi) = -1, \\
\eta(e_i) = 0, & i = 1, 2
\end{cases}.$$  

(4.7)

Putting $Y = U = e_i$ in (4.6) and using (4.7), we obtain

$$\eta(V)[2\alpha^2 g(e_i, e_i) - S(e_i, e_i)] = 0,$$

where since $S(e_i, e_i) = [\frac{\tau}{2} - \alpha^2]g(e_i, e_i)$, we get

$$[3\alpha^2 - \frac{\tau}{2}]g(e_i, e_i) = 0.$$

This gives

$$\tau = 6\alpha^2,$$

since $g(e_i, e_i) \neq 0$

which implies by Lemma 3.1 that the manifold is of constant curvature.

Hence we can state the following:
Theorem 4.2. A three-dimensional Ricci-semisymmetric Lorentzian α-Sasakian manifold is a manifold of constant curvature.

5. Locally φ-symmetric three-dimensional Lorentzian α-Sasakian manifolds

Definition 5.1. A Lorentzian α-Sasakian manifold is said to be locally φ-symmetric if
\[ \phi^2(\nabla_W R)(X, Y)Z = 0, \] (5.1)
for all vector fields \( W, X, Y, Z \) orthogonal to \( \xi \).

This notion was introduced by Takahashi for Sasakian manifolds [16].

Let us consider a three-dimensional Lorentzian α-Sasakian manifold. Firstly, differentiating (3.6) covariantly with respect to \( W \), we get
\[
(\nabla_W R)(X, Y)Z = \frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y]
+ \frac{d\tau(W)}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]
+ \left[\frac{\tau}{2} - 3\alpha^2\right][g(Y, Z)(\nabla_W \eta)(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi
+ g(Y, Z)\eta(X)\nabla_W \xi - g(X, Z)\eta(Y)\nabla_W \xi + (\nabla_W \eta)(Y)\eta(Z)X
+ \eta(Y)(\nabla_W \eta)(Z)X - (\nabla_W \eta)X\eta(Y)Z - \eta(X)(\nabla_W \eta)(Z)Y].
\]

Taking \( X, Y, Z, W \) orthogonal to \( \xi \) in the above equation, we have
\[
(\nabla_W R)(X, Y)Z = \frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y] \quad \text{(5.2)}
+ \left[\frac{\tau}{2} - 3\alpha^2\right][g(Y, Z)(\nabla_W \eta)(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi].
\]

Using the equation (2.8) in (5.2), we obtain
\[
(\nabla_W R)(X, Y)Z = \frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y] \quad \text{(5.3)}
+ \left[\frac{\tau}{2} - 3\alpha^2\right][g(Y, Z)g(W, X)\xi + g(X, Z)g(W, Y)\xi].
\]

From (5.3), it follows that
\[
\phi^2(\nabla_W R)(X, Y)Z = \frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y]. \quad \text{(5.4)}
\]

Thus, we obtain the following:

Theorem 5.2. A three-dimensional Lorentzian α-Sasakian manifold is locally φ-symmetric if and only if the scalar curvature \( \tau \) is constant.

Again if the manifold is Ricci-semisymmetric, then we have seen that \( \tau = 6\alpha^2 \), i.e., \( \tau = \text{constant} \) and hence from Theorem 5.2, we can state the following:

Theorem 5.3. A three-dimensional Ricci-semisymmetric Lorentzian α-Sasakian manifold is locally φ-symmetric.
6. Three-dimensional Lorentzian $\alpha$-Sasakian manifolds with $\eta$-parallel Ricci Tensor

**Definition 6.1.** The Ricci tensor $S$ of a Lorentzian $\alpha$-Sasakian manifold $M$ is called $\eta$-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0,$$  

(6.1)

for all vector fields $X, Y$ and $Z$.

The notion of Ricci $\eta$-parallelity for Sasakian manifolds was introduced by Kon [10].

Now let us consider three-dimensional Lorentzian $\alpha$-Sasakian manifold with $\eta$-parallel Ricci tensor. Then from (3.5), we have

$$S(\phi X, \phi Y) = \left[\frac{\tau}{2} - \alpha^2\right]g(\phi X, \phi Y),$$  

(6.2)

where, using (2.3), we get

$$S(\phi X, \phi Y) = \left[\frac{\tau}{2} - \alpha^2\right](g(X, Y) + \eta(X)\eta(Y)).$$  

(6.3)

Differentiating (6.3) covariantly along $Z$, we obtain

$$(\nabla_Z S)(\phi X, \phi Y) = \frac{d\tau(Z)}{2}(g(X, Y) + \eta(X)\eta(Y)) + (\frac{\tau}{2} - \alpha^2)(\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y)).$$  

(6.4)

Using (6.1) in (6.4), yields

$$\frac{1}{2}[d\tau(Z)(g(X, Y) + \eta(X)\eta(Y)) + (\tau - 2\alpha^2)(\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y))] = 0.$$  

(6.5)

Taking a frame field, we get from (6.5), $d\tau(Z) = 0$, for all $Z$.

**Proposition 6.2.** If a three-dimensional Lorentzian $\alpha$-Sasakian manifold has $\eta$-parallel Ricci tensor, then the scalar curvature $\tau$ is constant.

From Theorem 5.2 and Proposition 6.2 we have the following:

**Theorem 6.3.** A three-dimensional Lorentzian $\alpha$-Sasakian manifold with $\eta$-parallel Ricci tensor is locally $\phi$-symmetric.

7. Example

We consider the three-dimensional manifold $M = \{(x_1, x_2, x_3): x_i \in \mathbb{R}^3\}$, where $(x_1, x_2, x_3)$ are the standard coordinates of $\mathbb{R}^3$. The vector fields

$$e_1 = e^{x_3} \frac{\partial}{\partial x_2}, \quad e_2 = e^{x_3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \quad e_3 = \alpha \frac{\partial}{\partial x_3},$$

are linearly independent at each point of $M$, where $\alpha$ is constant. Let $g$ be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1,$$
that is, the form of the metric becomes
\[ g = \frac{1}{(e^{x_3})^2} (dx^2)^2 - \frac{1}{\alpha^2} (dx^3)^2, \]
which is a Lorentzian metric.

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1,1)\)-tensor field defined by
\[ \phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0. \]
Then using the linearity of \( \phi \) and \( g \), we have
\[ \eta(e_3) = -1, \quad \phi^2 Z = Z + \eta(Z)e_3 \quad \text{and} \quad g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W), \]
for any \( Z, W \in \chi(M) \). Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \( M \). Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have
\[ [e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2. \]
Koszul’s formula is defined by
\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]). \] (7.1)
Using (7.1) for the Lorentzian metric \( g \), we can easily calculate that
\[
\begin{align*}
\nabla_{e_1} e_1 &= -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1, \\
\nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2, \\
\nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\end{align*}
\]
Hence the structure \((\phi, \xi, \eta, g)\) is a Lorentzian \( \alpha \)-Sasakian manifold. Now using the above results, we obtain
\[
\begin{align*}
R(e_1, e_2)e_3 &= 0, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2, \\
R(e_1, e_3)e_3 &= -\alpha^2 e_1, \quad R(e_1, e_2)e_2 = \alpha^2 e_1, \\
R(e_2, e_3)e_2 &= -\alpha^2 e_3, \quad R(e_1, e_2)e_1 = -\alpha^2 e_2, \\
R(e_3, e_1)e_1 &= \alpha^2 e_3, \quad R(e_2, e_1)e_1 = \alpha^2 e_2, \\
R(e_3, e_2)e_2 &= \alpha^2 e_3.
\end{align*}
\] (7.2)
From which it follows that
\[ \phi^2(\nabla_W R)(X, Y)Z = 0. \]
Hence, the three-dimensional Lorentzian \( \alpha \)-Sasakian manifold is locally \( \phi \)-symmetric.

Also from the above expressions of the curvature tensor we obtain
\[ S(e_1, e_1) = S(e_2, e_2) = 0 \quad \text{and} \quad S(e_3, e_3) = -2\alpha^2. \] (7.3)
Hence
\[ \tau = -2\alpha^2, \]
which is a constant. Thus Theorem 5.3 is verified.

Next from the expressions of the Ricci tensor we find that the manifold is Ricci-semisymmetric. Also from (7.2) we see that the manifold is a manifold of constant curvature \( \alpha^2 \). Hence Theorem 4.2 is verified.
Finally from (7.3) it follows that
\[(\nabla_Z S)(\phi X, \phi Y) = 0,\]
for all \(X, Y, Z\). Therefore Theorem 6.3 is also verified.

**References**


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