CERTAIN CLASSES OF $k$-UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS AND OTHER RELATED FUNCTIONS DEFINED BY USING THE DZIOK-SRIVASTAVA OPERATOR

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Abstract. Several interesting classes of $k$-uniformly close-to-convex functions and $k$-uniformly quasi-convex functions are defined here by using the Dziok-Srivastava operator. We provide necessary and sufficient coefficient conditions, extreme points, integral representations, and distortion bounds for functions belonging to each of these classes of $k$-uniformly close-to-convex functions and $k$-uniformly quasi-convex functions.

1. Introduction, Definitions and Preliminaries

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$ 

Also let $A^-$ denote a subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

which are analytic in $U$.

A function $f(z) \in A$ is said to be in the class of $k$-uniformly convex functions of order $\beta \ (0 \leq \beta < 1)$, denoted by $UK(k, \beta)$ (cf. [10]; see also [9] and [3]) if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|^\beta \quad (k \geq 0; \ 0 \leq \beta < 1; \ z \in U).$$  (1.3)
A corresponding class of $k$-uniformly starlike functions, denoted by $\mathcal{US}(k, \beta)$, consists of functions $f(z) \in \mathcal{A}$ such that
\[ \Re \left( \frac{zf'(z)}{g(z)} \right) > k \left| \frac{zf'(z)}{g(z)} - 1 \right| + \beta \quad (k \geq 0; \ 0 \leq \beta < 1; \ z \in \mathbb{U}). \]  
(1.4)

It is obvious from the inequalities in (1.3) and (1.4) that (see [10])
\[ f(z) \in \mathcal{UK}(k, \beta) \iff zf'(z) \in \mathcal{US}(k, \beta). \]  
(1.5)

Each of the function classes $\mathcal{UK}(k, \beta)$ and $\mathcal{US}(k, \beta)$ provides unifications and generalizations various other (known or new) subclasses of $\mathcal{A}$. Several properties of some of the subclasses of the function classes $\mathcal{UK}(k, \beta)$ and $\mathcal{US}(k, \beta)$ were studied recently in [9] (see also [6] and [8]).

**Definition 1** (see [1]). Define $\mathcal{UC}(k, \gamma, \beta)$ to be the family of functions $f(z) \in \mathcal{A}$ such that
\[ \Re \left( \frac{zf'(z)}{g(z)} \right) > k \left| \frac{zf'(z)}{g(z)} - 1 \right| + \gamma \quad (k \geq 0; \ \gamma \in [0, 1); \ z \in \mathbb{U}) \]  
(1.6)

for some function $g(z) \in \mathcal{US}(k, \beta)$.

**Definition 2** (see [1]). Define $\mathcal{UQ}(k, \gamma, \beta)$ to be the family of functions $f(z) \in \mathcal{A}$ such that
\[ \Re \left( \frac{(zf'(z))'}{g'(z)} \right) > k \left| \frac{(zf'(z))'}{g'(z)} - 1 \right| + \gamma \quad (k \geq 0; \ \gamma \in [0, 1); \ z \in \mathbb{U}) \]  
(1.7)

for some function $g(z) \in \mathcal{UK}(k, \beta)$.

It readily follows from Definitions 1 and 2 that
\[ f(z) \in \mathcal{UQ}(k, \gamma, \beta) \iff zf'(z) \in \mathcal{UC}(k, \gamma, \beta). \]  
(1.8)

We say that $\mathcal{UC}(0, \gamma, \beta)$ is the class of close-to-convex functions of order $\gamma$ and type $\beta$ in $\mathbb{U}$ and that $\mathcal{UQ}(0, \gamma, \beta)$ is the class of quasi-convex functions of order $\gamma$ and type $\beta$ in $\mathbb{U}$.

**Definition 3.** For functions $f(z) \in \mathcal{A}$ given by (1.1), and $g(z) \in \mathcal{A}$ given by
\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \]  
(1.9)

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by
\[ (f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \quad (z \in \mathbb{U}). \]  
(1.10)

For complex parameters $\alpha_j \in \mathbb{C} \ (j = 1, \cdots, l)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ (j = 1, \cdots, m; \mathbb{Z}_0^- := \{0, -1, -2, \cdots\})$, the generalized hypergeometric function $_{l}F_m$ (with $l$ numerator and $m$ denominator parameters) is defined by
\[ _{l}F_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \cdot \frac{z^n}{n!} \]  
(1.11)
(l \leq m + l; \ l, m \in \mathbb{N}_0 := \{0, 1, 2, \cdots \} = \mathbb{N} \cup \{0\}),

where \((\lambda)\nu\) denotes the Pochhammer symbol (or the shifted factorial, since \((1)_n = n!\) for \(n \in \mathbb{N}\)) defined, in terms of the familiar Gamma functions, by

\[
(\lambda)\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 
1 & (\nu = 0; \ \lambda \in \mathbb{C} \setminus \{0\}) \\
(\lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \ \lambda \in \mathbb{C}).
\end{cases}
\]

Now, corresponding to the function

\[
h(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) = z \cdot F_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m),
\]

the Dziok-Srivastava linear operator (see [3, 4, 5] and [11]; see also [7, 14] and [15])

\[
H^l_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)
\]

is defined as follows by using the Hadamard product (or convolution):

\[
H^l_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z) = h(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) \ast f(z)
\]

\[
= z + \sum_{n=2}^{\infty} \varphi_n(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) a_n z^n,
\]

(1.12)

where, for convenience,

\[
\varphi_n(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)
\]

is given by

\[
\varphi_n(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) := \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \cdot \frac{1}{(n-1)!}.
\]

(1.13)

It is well known (see, for example, [5]) that

\[
\alpha_1 H^l_m(\alpha_1 + 1, \alpha_2, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z) = z \cdot (H^l_m(\alpha_1 + 1, \alpha_2, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z))' + (\alpha_1 - 1) H^l_m(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z).
\]

(1.14)

For notational simplification in our investigation, we write

\[
H^l_m[\alpha_1] f(z) = H^l_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z).
\]

(1.15)

We now define the linear operator \(L^{\tau, \alpha_l}_{\lambda, \beta_m}\) as follows:

\[
L^0_{\lambda, \alpha_l} f(z) = f(z),
\]

(1.16)

\[
L^1_{\lambda, \beta_m} f(z) = (1 - \lambda) H^l_m[\alpha_1] f(z) + \lambda z (H^l_m[\alpha_1] f(z))' = L^1_{\lambda, \beta_m} f(z) \quad (\lambda \geq 0),
\]

(1.17)

\[
L^2_{\lambda, \beta_m} f(z) = L^1_{\lambda, \beta_m} f(z),
\]

(1.18)

and, in general,

\[
L^{\tau, \alpha_l}_{\lambda, \beta_m} f(z) = L^{\alpha_l}_{\lambda, \beta_m} (L^{-1, \alpha_l}_{\lambda, \beta_m} f(z)) \quad (l \leq m + l; \ l, m \in \mathbb{N}_0; \ \tau \in \mathbb{N}).
\]

(1.19)

If the function \(f(z)\) is given by (1.1), then we see from (1.12), (1.13), (1.17) and (1.19) that

\[
L^{\tau, \alpha_l}_{\lambda, \beta_m} f(z) = z + \sum_{n=2}^{\infty} \phi_n^\tau(\alpha_1, \lambda, l, m)a_n z^n \quad (\tau \in \mathbb{N}_0),
\]

(1.20)
where
\[
\phi_n^\tau(\alpha_1, \lambda, l, m) = \left( \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \cdot \frac{[1 + \lambda(n - 1)]}{(n - 1)!} \right)^\tau \quad (1.21)
\]

When
\[\tau = 1 \quad \text{and} \quad \lambda = 0,\]
the linear operator \( L^{\tau, \alpha_1}_{\lambda, j, m} \) would reduce to the familiar Dziok-Srivastava linear operator given by (1.12) above (see, for example, [3]). For a linear operator which is essentially analogous to the Dziok-Srivastava operator in (1.12), but uses instead the Fox-Wright generalization of the hypergeometric function \( {}_pF_q \) defined here by (1.11), the interested reader may be referred to the recent works [2] and [12] as well as to the closely-related works cited in each of these recent works.

By applying the general operator \( L^{\tau, \alpha_1}_{\lambda, j, m} \), we define the following subclasses of the function class \( A \).

I. Let \( US_m^l(\tau, \lambda, k, \beta) \) be the class of functions \( f(z) \in A \) satisfying the following inequality:
\[
\Re \left( \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))'}{L^{\tau, \alpha_1}_{\lambda, j, m} f(z)} \right) > k \left| \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))'}{L^{\tau, \alpha_1}_{\lambda, j, m} f(z)} - 1 \right| + \beta \quad (k \geq 0; \ \beta \in [0, 1)). \quad (1.22)
\]
Observe that
\[L^{\tau, \alpha_1}_{\lambda, j, m} f(z) \in US(k, \beta).\]

II. Let \( UK_m^l(\tau, \lambda, k, \beta) \) be the class of functions \( f(z) \in A \) satisfying the following inequality:
\[
\Re \left( 1 + \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))''}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} \right) > k \left| \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))''}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} - 1 \right| + \beta \quad (k \geq 0; \ \beta \in [0, 1)). \quad (1.23)
\]
Observe that
\[L^{\tau, \alpha_1}_{\lambda, j, m} f(z) \in UK(k, \beta).\]

III. Let \( UC_m^l(\tau, \lambda, k, \gamma, \beta) \) be the class of functions \( f \in A \) such that
\[
\Re \left( \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))'}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} \right) > k \left| \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))'}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} - 1 \right| + \gamma \quad (k \geq 0; \ \gamma \in [0, 1)) \quad (1.24)
\]
for some function \( g(z) \in US_m^l(\tau, k, \beta) \). Observe that
\[L^{\tau, \alpha_1}_{\lambda, j, m} f(z) \in UC(k, \gamma, \beta).\]

IV. Let \( UQ_m^l(\tau, \lambda, k, \gamma, \beta) \) be the class of functions \( f \in A \) such that
\[
\Re \left( 1 + \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))''}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} \right) > k \left| \frac{z(L^{\tau, \alpha_1}_{\lambda, j, m} f(z))''}{L^{\tau, \alpha_1}_{\lambda, j, m} g(z)} - 1 \right| + \gamma \quad (k \geq 0; \ \gamma \in [0, 1)) \quad (1.25)
\]
for some function \( g(z) \in UK_m^l(\tau, \lambda, k, \beta) \). Observe that
\[L^{\tau, \alpha_1}_{\lambda, j, m} f(z) \in UQ(k, \gamma, \beta).\]
It is clear from two of the above definitions that
\[
  f(z) \in \mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta) \iff z f'(z) \in \mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta).
\]

(1.26)

Finally, in terms of the above-defined function classes, we write
\[
  \mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta) = \mathcal{A}^- \cap \mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta),
\]
\[
  \mathcal{U}^l_{l,m}(\tau, \lambda, \kappa, \beta) = \mathcal{A}^- \cap \mathcal{U}^l_{l,m}(\tau, \lambda, \kappa, \beta),
\]
\[
  \mathcal{U}^r_{l,m}(\tau, \lambda, \kappa, \beta, \gamma) = \mathcal{A}^- \cap \mathcal{U}^r_{l,m}(\tau, \lambda, \kappa, \beta, \gamma),
\]
and
\[
  \mathcal{U}^q_{l,m}(\tau, \lambda, \kappa, \beta, \gamma) = \mathcal{A}^- \cap \mathcal{U}^q_{l,m}(\tau, \lambda, \kappa, \beta, \gamma).
\]

The various properties and characteristics of functions in the class \(\mathcal{U}^k_{l,m}(1, 0, k, \beta)\) were investigated by Dziok and Srivastava [3]. In this paper, we obtain several relationships and properties of the convolution operators considered here. Our paper mainly studies the functions in the class \(\mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta)\). We first prove a sufficient condition for a function \(f \in \mathcal{A}\) to be in the class \(\mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta)\). We then provide necessary and sufficient coefficient conditions, extreme points, integral representations, distortion bounds, radii of starlikeness and convexity for functions in the class \(\mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta)\).

2. FIRST SET OF MAIN RESULTS

First of all, we obtain a sufficient condition for a function \(f \in \mathcal{A}\) to be in the class \(\mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \beta)\).

**Theorem 1.** Let \(f(z) \in \mathcal{A}\) be given by (1.1). Suppose also that \(\phi_n^*(\alpha_1, \lambda, l, m)\) is given by (2.1). If
\[
  k \geq 0, \quad \beta \in [0, 1), \quad \gamma \in [0, 1), \quad \lambda \geq 0, \quad \tau \in \mathbb{N}_0
\]
and
\[
  \sum_{n=2}^{\infty} \left[ 2k|na_n - b_n| + (1 - \gamma)|b_n| \right] \phi_n^*(\alpha_1, \lambda, l, m) < 1 - \gamma,
\]
then \(f(z) \in \mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \gamma, \beta)\).

**Proof.** By the definition of the function class \(\mathcal{U}^k_{l,m}(\tau, \lambda, \kappa, \gamma, \beta)\), it suffices to show for a function \(f(z) \in \mathcal{A}\) given by (1.1) that
\[
  k \left| \frac{z(L_{\lambda,j,m}^* f(z))'}{L_{\lambda,j,m}^* g(z)} - 1 \right| - \Re \left( \frac{z(L_{\lambda,j,m}^* f(z))'}{L_{\lambda,j,m}^* g(z)} - \gamma \right) \leq 2k \left| \frac{z(L_{\lambda,j,m}^* f(z))'}{L_{\lambda,j,m}^* g(z)} - 1 \right|
\]
\[
  \leq 2k \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m)|na_n - b_n| \cdot |z|^{n-1}
\]
\[
  1 - \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m)|b_n| \cdot |z|^{n-1}.
\]

(2.1)

Now the last expression in (2.1) is bounded above by \(1 - \gamma\) if and only if
\[
  \sum_{n=2}^{\infty} \left[ 2k|na_n - b_n| + (1 - \gamma)|b_n| \right] \phi_n^*(\alpha_1, \lambda, l, m) < 1 - \gamma,
\]
which evidently completes the proof of Theorem 1. □

We next provide a necessary and sufficient coefficient bound for a given function \( f(z) \in \mathcal{A}^- \) to belong to the class \( \mathcal{UC}_{l,m}^- (\tau, \lambda, k, \gamma, \beta) \).

**Theorem 2.** Let \( f(z) \in \mathcal{A}^- \) be given by (1.2). Also let \( \phi_n^*(\alpha_1, \lambda, l, m) \) be given by (1.21). Then \( f \in \mathcal{UC}_{l,m}^- (\tau, \lambda, k, \gamma, \beta) \) if and only if

\[
\sum_{n=2}^{\infty} \left[ n(1+k)a_n - (k+\gamma)b_n \right] \phi_n^*(\alpha_1, \lambda, l, m) < 1 - \gamma. \tag{2.2}
\]

**Proof.** Suppose that \( f(z) \in \mathcal{UC}_{l,m}^- (\tau, \lambda, k, \gamma, \beta) \). Then, making use of the fact that \( \Re(\omega) > k |\omega - 1| + \gamma \iff \Re(e^{i\phi}) > \gamma \quad (\gamma \in \mathbb{R}) \) and letting

\[
\omega = \frac{z \left( L^*_{\lambda,j,m} f(z) \right)'}{L^*_{\lambda,j,m} g(z)}
\]

in (1.3), we obtain

\[
\Re \left( \frac{z \left( L^*_{\lambda,j,m} f(z) \right)'}{L^*_{\lambda,j,m} g(z)} (1 + e^{i\phi}) - e^{i\phi} \right) > \gamma
\]

or, equivalently,

\[
\Re \left( \frac{(1 + e^{i\phi}) z \left( z - \sum_{n=2}^{\infty} n \phi_n^*(\alpha_1, \lambda, l, m) a_n z^n \right)' - (e^{i\phi} + \gamma) \left( z - \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m) b_n z^n \right)}{1 - \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m) b_n z^n} \right) > 0,
\]

which holds true for all \( z \in \mathbb{U} \). By letting \( z \to 1^- \) through real values, we thus find that

\[
\Re \left( (1 - \gamma) - (1 + e^{i\phi}) \sum_{n=2}^{\infty} n \phi_n^*(\alpha_1, \lambda, l, m) a_n + (\gamma + e^{i\phi}) \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m) b_n \right) > 0,
\]

and so (by the mean value theorem) we have

\[
\Re \left( (1 - \beta) - (1 + e^{i\gamma}) \sum_{n=2}^{\infty} n \phi_n^*(\alpha_1, \lambda, l, m) a_n + (\beta + e^{i\phi}) \sum_{n=2}^{\infty} \phi_n^*(\alpha_1, \lambda, l, m) b_n \right) > 0.
\]

Therefore, we get

\[
\sum_{n=2}^{\infty} \left[ n(1+k)a_n - (k+\gamma)b_n \right] \phi_n^*(\alpha_1, \lambda, l, m) < 1 - \gamma,
\]

which proves the first part of Theorem 2.

Conversely, we let the inequality (2.2) hold true.

Then, in light of the fact that

\[
\Re(\omega) > \gamma \iff |\omega - (1+\gamma)| < |\omega + (1-\gamma)| \quad (\gamma \in \mathbb{R}),
\]

which completes the proof.
we need only to show that
\[
\left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} - \left(1 + k \left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} - 1 \right| + \gamma \right) \right| < \left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} + \left(1 - k \left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} - 1 \right| - \gamma \right) \right|
\]

By setting
\[
\frac{L_{\lambda,j,m}^{\alpha_1} g(z)}{L_{\lambda,j,m}^{\alpha_1}} = e^{i\phi},
\]
we may write
\[
\mathcal{E} = \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} + \left(1 - k \left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} - 1 \right| - \gamma \right)
\]
\[
= \frac{|z|}{L_{\lambda,j,m}^{\alpha_1} g(z)} \left( (2 - \gamma) - \sum_{n=2}^{\infty} [na_n + (1 - \gamma)b_n] \phi_n^\alpha (\alpha_1, \lambda, l, m) z^{n-1} - e^{i\phi} \sum_{n=2}^{\infty} (kna_n - kb_n) \phi_n^\alpha (\alpha_1, \lambda, l, m) z^{n-1} \right)
\]
\[
> \frac{|z|}{L_{\lambda,j,m}^{\alpha_1} g(z)} \left( (2 - \gamma) - \sum_{n=2}^{\infty} (n(1 + k)a_n + (1 - k - \gamma)b_n) \phi_n^\alpha (\alpha_1, \lambda, l, m) \right)
\]
and
\[
\mathcal{E} = \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} - \left(1 - k \left| \frac{z (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime}{L_{\lambda,j,m}^{\alpha_1} g(z)} + 1 \right| + \gamma \right)
\]
\[
= \frac{|z|}{L_{\lambda,j,m}^{\alpha_1} g(z)} \left( (L_{\lambda,j,m}^{\alpha_1} f(z))^\prime - (1 + \gamma) \frac{L_{\lambda,j,m}^{\alpha_1} g(z)}{z} - k \left| (H_{m,\alpha_1} f(z))^\prime - \frac{L_{\lambda,j,m}^{\alpha_1} g(z)}{z} \right| \right)
\]
\[
= \frac{|z|}{L_{\lambda,j,m}^{\alpha_1} g(z)} \left( (1 - \gamma) - \sum_{n=2}^{\infty} [na_n - (1 + \gamma)b_n] \phi_n^\alpha (\alpha_1, \lambda, l, m) z^{n-1} - e^{i\phi} \sum_{n=2}^{\infty} (kna_n - kb_n) \phi_n^\alpha (\alpha_1, \lambda, l, m) z^{n-1} \right)
\]
\[
< \frac{|z|}{L_{\lambda,j,m}^{\alpha_1} g(z)} \left( (1 - \gamma) + \sum_{n=2}^{\infty} [n(1 + k)a_n - (1 + k + \gamma)b_n] \phi_n^\alpha (\alpha_1, \lambda, l, m) \right).
\]
It is easy to verify that
\[
\mathcal{E} - \bar{\mathcal{E}} > 0
\]
in case the inequality (2.2) holds true. The proof of Theorem 2 is thus completed.
When
\[ f(z) = g(z) \quad (z \in \mathbb{U}), \]
Theorem 2 would yield the following corollary.

**Corollary 1.** Let \( g(z) \in A^- \) be given by
\[
g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0), \tag{2.3}
\]
Then \( g(z) \in US_{l,m}^- (\tau, \lambda, k, \beta) \) if and only if
\[
\sum_{n=2}^{\infty} \frac{[(n-1)k+n-\beta]b_n \phi_n^+(\alpha_1, \lambda, l, m)}{1-\beta} < 1.
\]

**Corollary 2.** If \( g(z) \in US_{l,m}^- (\tau, \lambda, k, \beta) \) is given by (2.3), then
\[
\sum_{n=2}^{\infty} b_n < \frac{1-\beta}{(2+k-\beta)\phi_2^+(\alpha_1, \lambda, l, m)}.
\]

**Proof.** Since \( g(z) \in US_{l,m}^- (\tau, \lambda, k, \beta) \) is given by (2.3), we can apply Corollary 1 to obtain
\[
(k + 2 - \beta)\phi_2^+(\alpha_1, \lambda, l, m) \sum_{n=2}^{\infty} b_n
\]
\[
\leq \sum_{n=2}^{\infty} b_n [(n-1)k+n-\beta] \phi_n^+(\alpha_1, \lambda, l, m)
\]
\[
< 1 - \beta.
\]
We thus find that
\[
\sum_{n=2}^{\infty} b_n < \frac{1-\beta}{(2+k-\beta)\phi_2^+(\alpha_1, \lambda, l, m)},
\]
which proves Corollary 2. \( \square \)

**Corollary 3.** If \( g(z) \in US_{l,m}^- (\tau, \lambda, k, \beta) \) is given by (2.3), then
\[
b_n < \frac{1-\beta}{(2+k-\beta)\phi_2^+(\alpha_1, \lambda, l, m)}.
\]

3. **FURTHER RESULTS AND CONSEQUENCES**

In this section, several further results involving the various function classes which were introduced in Section 1.

**Theorem 3.** If \( g(z) \in US_{l,m}^- (\tau, \lambda, k, \beta) \), then
\[
L_{\lambda, j, m}^{r, \alpha_1} g(z) = \exp \left( \int_0^z \frac{k - \beta Q(t)}{t[k - Q(t)]} dt \right) \quad (|Q(z)| < 1; \ z \in \mathbb{U}) \tag{3.1}
\]
and
\[
L_{\lambda, j, m}^{r, \alpha_1} g(z) = \exp \left( \int_{|x|=1} \log \left[ (k - xz)^{-1-\beta} \right] d\mu(x) \right), \tag{3.2}
\]
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where $\mu(x)$ is a probability measure on the set:

$$X = \{ x : |x| = 1 \}.$$  

**Proof.** The case $k = 0$ of the assertion (3.1) if Theorem 3 is obvious. Let $k \neq 0$. Then, for

$$g(z) \in US_{l,m}^-(k, \beta) \quad \text{and} \quad \omega = \frac{z(L_{\lambda,j,m}^\alpha g(z))'}{L_{\lambda,j,m}^\alpha g(z)},$$

we have

$$\Re(\omega) > k|\omega - 1| + \beta.$$  

We thus find that

$$\left| \frac{\omega - 1}{\omega - \beta} \right| < \frac{1}{k} \quad \text{and} \quad \frac{\omega - 1}{\omega - \beta} = \frac{Q(z)}{k} \quad (|Q(z)| < 1; \ z \in U),$$

which readily yields

$$\frac{z(L_{\lambda,j,m}^\alpha g(z))'}{L_{\lambda,j,m}^\alpha g(z)} = \frac{k - \beta Q(z)}{z[k - Q(z)]}$$

and, therefore,

$$L_{\lambda,j,m}^\alpha g(z) = \exp \left( \int_0^z \frac{k - \beta Q(t)}{t[k - Q(t)]} dt \right).$$

In order to derive the second representation (3.2), corresponding to the set:

$$X = \{ x : |x| = 1 \},$$

we observe that

$$\frac{\omega - 1}{\omega - \beta} < \frac{1}{k} z$$

or, equivalently, that

$$\frac{z(L_{\lambda,j,m}^\alpha g(z))'}{L_{\lambda,j,m}^\alpha g(z)} = \frac{k - \beta Q(z)}{z[k - Q(z)]}$$

$$\implies \log \left( \frac{H_m^l \alpha_1 g(z)}{z} \right) = -(1 + \beta) \log(k - xz).$$

Thus, if $\mu(x)$ is the probability measure on $X$, then

$$L_{\lambda,j,m}^\alpha g(z) = \exp \left( \int_{|x|=1} \log \left( (k - xz)^{-1-\beta} \right) d\mu(x) \right).$$

\[ \square \]

**Theorem 4.** If $f(z) \in UC_{l,m}^-(\tau, \lambda, k, \gamma, \beta)$, then

$$L_{\lambda,j,m}^\alpha f(z) = \int_0^z \left[ \frac{k - \gamma Q(t)}{k - Q(t)} \exp \left( \int_{|x|=1} \log \left( (k - xz)^{-1-\beta} \right) d\mu(x) \right) \right] dt,$$

where $\mu(x)$ is a probability measure on the following set:

$$X = \{ x : |x| = 1 \}.$$
Proof. The case $k = 0$ of the assertion (3.3) of Theorem 4 is obvious. Let $k \neq 0$. Then, for

$$f \in \mathcal{UC}_{l,m}^{-}(\tau, \lambda, k, \beta) \quad \text{and} \quad \omega = \frac{z(L_{\lambda,j,m}^{\tau,\alpha_1} f(z))'}{L_{\lambda,j,m}^{\tau,\alpha_1} g(z)},$$

we have

$$\Re(\omega) > k|\omega - 1| + \gamma.$$

We thus find that

$$\left| \frac{\omega - 1}{\omega - \gamma} \right| < \frac{1}{k} \quad \text{and} \quad \frac{\omega - 1}{\omega - \gamma} = \frac{Q(z)}{k} \quad (|Q(z)| < 1; \, z \in \mathbb{U}),$$

which easily yields

$$z\left(L_{\lambda,j,m}^{\tau,\alpha_1} f(z)\right)' = \frac{k - \gamma Q(z)}{z[k - Q(z)]}. \quad (3.4)$$

Moreover, from Theorem 3, we have

$$L_{\lambda,j,m}^{\tau,\alpha_1} g(z) = \exp \left( \int_{|x|=1} \log \left( (k - xz)^{-1-\beta} \right) \, d\mu(x) \right), \quad (3.5)$$

where $\mu(x)$ is a probability measure on the set:

$$X = \{ x : |x| = 1 \}.$$

The assertion (3.3) of Theorem 4 would now follow from (3.4) and (3.5). \qed

Next we obtain a distortion bounds for the functions $f(z)$ and $g(z)$.

**Theorem 5.** If $g(z) \in \mathcal{US}_{l,m}^{-}(\tau, \lambda, k, \beta)$, then

$$|z| - \frac{1 - \beta}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z|^2 < |g(z)| < |z| + \frac{1 - \beta}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z|^2 \quad (z \in \mathbb{U}) \quad (3.6)$$

and

$$1 - \frac{2(1 - \beta)}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z| < |g'(z)| < 1 + \frac{2(1 - \beta)}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z| \quad (z \in \mathbb{U}). \quad (3.7)$$

**Proof.** For $g(z) \in \mathcal{US}_{l,m}^{-}(\tau, \lambda, k, \beta)$ given by (2.3), we find from Corollary 2 that

$$\sum_{n=2}^{\infty} b_n < \frac{1 - \beta}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)}, \quad (3.8)$$

which implies that

$$|g(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} b_n < |z| + \frac{1 - \beta}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z|^2 \quad (z \in \mathbb{U})$$

and

$$|g(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} b_n > |z| - \frac{1 - \beta}{(2 + k - \beta)\phi_2^\tau(\alpha_1, \lambda, l, m)} |z|^2 \quad (z \in \mathbb{U}).$$
Thus the assertion (3.6) of Theorem 5 follows at once.

In a similar manner, for the derivative \(g'(z)\), the following inequalities:

\[
|g'(z)| \leq 1 + \sum_{n=2}^{\infty} nb_n|z|^{n-1} < 1 + |z| \sum_{n=2}^{\infty} nb_n \quad (z \in \mathbb{U})
\]

and

\[
\sum_{n=2}^{\infty} nb_n < \frac{2(1 - \beta)}{2 + k - \beta) \phi_2^\alpha(\alpha_1, \lambda, l, m)}
\]

lead us immediately to the assertion (3.7) of Theorem 5. This completes the proof of Theorem 5. \(\square\)

**Theorem 6.** If \(f \in UC_{l,m}(\tau, \lambda, k, \gamma, \beta)\), then

\[
|z| - \frac{1 - \gamma}{2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right) |z|^2 < |f(z)|
\]

\[
< |z| + \frac{1 - \gamma}{2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right) |z|^2 \quad (z \in \mathbb{U}) \quad (3.9)
\]

and

\[
1 - \frac{1 - \gamma}{(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right) < |f'(z)|
\]

\[
< 1 + \frac{1 - \gamma}{(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \left[1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right] |z| \quad (z \in \mathbb{U}). \quad (3.10)
\]

**Proof.** For \(f \in UC_{l,m}(\tau, \lambda, k, \gamma, \beta)\) given by (1.2), by using Theorem 1, we obtain

\[
2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m) \sum_{n=2}^{\infty} a_n < \sum_{n=2}^{\infty} n(1 + k) a_n \phi_n^\alpha(\alpha_1, \lambda, l, m)
\]

\[
< 1 - \gamma + \sum_{n=2}^{\infty} (k + \gamma)b_n \phi_n^\alpha(\alpha_1, \lambda, l, m), \quad (3.11)
\]

which immediately yields

\[
\sum_{n=2}^{\infty} a_n < \frac{1 - \gamma}{2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)}
\]

\[
+ \frac{k + \gamma}{2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \sum_{n=2}^{\infty} b_n \phi_n^\alpha(\alpha_1, \lambda, l, m). \quad (3.12)
\]

Also, by applying Corollary 1, we have

\[
\sum_{n=2}^{\infty} b_n \phi_n^\alpha(\alpha_1, \lambda, l, m) < \frac{1 - \beta}{2 + k - \beta},
\]

so that

\[
|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n
\]

\[
< |z| + \frac{1 - \gamma}{2(1 + k) \phi_2^\alpha(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right) |z|^2 \quad (z \in \mathbb{U}).
\]
Similarly, we can show that

\[
|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n
\]

\[
> |z| - \frac{1 - \gamma}{2(1 + k)\phi_2^*(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right)|z|^2 \quad (z \in \mathbb{U}).
\]

We thus have proved the assertion (3.9) of Theorem 6.

In a similar manner, for the derivative \(f'(z)\), the following inequalities:

\[
|f'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1} < 1 + |z| \sum_{n=2}^{\infty} na_n \quad (z \in \mathbb{U})
\]

and

\[
\sum_{n=2}^{\infty} na_n < \frac{1 - \gamma}{(1 + k)\phi_2^*(\alpha_1, \lambda, l, m)} \left[1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right]
\]

lead us to the assertion (3.12) of Theorem 6. This evidently completes the proof of Theorem 6. □

It is not difficult to deduce Corollary 4 below.

**Corollary 4.** Let \(f \in \mathcal{UC}_{1,m}^{-}(\tau, \lambda, k, \gamma, \beta)\). Then

\[
\left\{ \omega : |\omega| < 1 - \frac{1 - \gamma}{(1 + k)\phi_2^*(\alpha_1, \lambda, l, m)} \left(1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right) \right\} \subset f(\mathbb{U})
\]

\[
\subset \left\{ \omega : |\omega| < 1 + \frac{1 - \gamma}{(1 + k)\phi_2^*(\alpha_1, \lambda, l, m)} \left[1 + \frac{(k + \gamma)(1 - \beta)}{(1 - \gamma)(2 + k - \beta)}\right] \right\}. \quad (3.13)
\]

Theorem 7 below follows easily from Corollary 1. In fact, the proof of Theorem 7 is essentially analogous to that of Theorem 8, which we have chosen to present here in detail.

**Theorem 7.** Let

\[
g_m(z) = z - \sum_{n=2}^{\infty} b_{j,m}z^j \in \mathcal{US}_{1,m}^{-}(\tau, \lambda, k, \gamma, \beta) \quad (m = 1, 2).
\]

Then

\[
g(z) = (1 - \xi)g_1(z) + \xi g_2(z) = z - \sum_{j=2}^{\infty} b_j z^j
\]

\[
\in \mathcal{US}_{1,m}^{-}(\tau, \lambda, k, \gamma, \beta) \quad (0 \leq \xi \leq 1). \quad (3.14)
\]

**Theorem 8.** Let

\[
f_m(z) = z - \sum_{n=2}^{\infty} a_{j,m}z^j \in \mathcal{UC}_{1,m}^{-}(\tau, \lambda, k, \gamma, \beta) \quad (m = 1, 2).
\]
Then
\[ f(z) = (1 - \xi)f_1(z) + \xi f_2(z) = z - \sum_{j=2}^{\infty} a_j z^j \]
\[ \in \mathcal{UC}_{l,m}^{-}(\tau, \lambda, k, \gamma, \beta) \quad (0 \leq \xi \leq 1). \]
(3.15)

Proof. Since
\[ f_m(z) \in \mathcal{UC}_{l,m}^{-}(\tau, \lambda, k, \gamma, \beta) \quad (m = 1, 2), \]
by using Theorem 2, we get the following coefficient inequalities:
\[
\sum_{j=2}^{\infty} [(1 + k)ja_{j,1}\phi_j^*(\alpha_1, \lambda, l, m) - (k + \gamma)b_{j,1}\phi_j^*(\alpha_1, \lambda, l, m)] < 1 - \gamma
\]
and
\[
\sum_{j=2}^{\infty} [(1 + k)ja_{j,2}\phi_j^*(\alpha_1, \lambda, l, m) - (k + \gamma)b_{j,2}\phi_j^*(\alpha_1, \lambda, l, m)] < 1 - \gamma.
\]
Furthermore, in view of the following obvious relationships:
\[ a_j = (1 - \xi)a_{j,1} + \xi a_{j,2} \quad \text{and} \quad b_j = (1 - \xi)b_{j,1} + \xi b_{j,2} \]
\[ (j \in \mathbb{N} \setminus \{1\}; \ 0 \leq xi \leq 1), \]
we thus find that
\[
\sum_{j=2}^{\infty} [(1 + k)ja_{j,1}\phi_j^*(\alpha_1, \lambda, l, m) - (k + \gamma)b_{j,1}\phi_j^*(\alpha_1, \lambda, l, m)]
\]
\[ = \sum_{j=2}^{\infty} (1 + k)j\phi_j^*(\alpha_1, \lambda, l, m) [(1 - \xi)a_{j,1}(z) + \xi a_{j,2}(z)] \\
- \sum_{j=2}^{\infty} (k + \gamma)b_{j,1}\phi_j^*(\alpha_1, \lambda, l, m) [(1 - \xi)b_{j,1}(z) + \xi b_{j,2}(z)] \\
= \sum_{j=2}^{\infty} (1 - \xi) [(1 + k)ja_{j,1}\phi_j^*(\alpha_1, \lambda, l, m) - (k + \beta)b_{j,1}\phi_j^*(\alpha_1, \lambda, l, m)] \\
+ \sum_{j=2}^{\infty} \xi[(1 + k)ja_{j,2}\phi_j^*(\alpha_1, \lambda, l, m) - (k + \gamma)b_{j,2}\phi_j^*(\alpha_1, \lambda, l, m)] \\
\leq (1 - \xi)(1 - \gamma) + \xi(1 - \gamma) = 1 - \gamma.
\]
Thus, by using Theorem 2 again, we finally obtain
\[ f(z) \in \mathcal{UC}_{l,m}^{-}(\tau, \lambda, k, \gamma, \beta), \]
which completes the proof of Theorem 8. \( \square \)

We remark in conclusion that, by suitably specializing the parameters involved in the results presented in this paper, we can deduce numerous further corollaries and consequences of each of these results.
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CERTAIN CLASSES OF $k$-UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS

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