SPACES OF $D_{L^p}$-TYPE AND A CONVOLUTION PRODUCT ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

C.BACCAR AND L.T.RACHDI

Abstract. We define and study the spaces $M_p(\mathbb{R}^2)$, $1 \leq p \leq +\infty$, that are of $D_{L^p}$-type. Using the harmonic analysis related to the Fourier transform connected with the Riemann-Liouville operator, we give a new characterization of the dual space $M_p'(\mathbb{R}^2)$ and we describe its bounded subsets. Next, we define a convolution product in $M_p'(\mathbb{R}^2) \times M_r(\mathbb{R}^2)$, $1 \leq r \leq p < +\infty$, where $M_r(\mathbb{R}^2)$ is the closure of the space $S^\ast(\mathbb{R}^2)$ in $M_r(\mathbb{R}^2)$ and we prove some new results.

1. Introduction

The space $D_{L^p}(\mathbb{R}^n)$; $1 \leq p \leq +\infty$; is formed by the measurable functions on $\mathbb{R}^n$ such that for all $\alpha \in \mathbb{N}^n$; the function $D^\alpha(f)$ belongs to $L^p(\mathbb{R}^n, dx)$ (the space of functions with $p^{th}$ power integrable on $\mathbb{R}^n$ with respect to the Lebesgue measure $dx$ on $\mathbb{R}^n$).

Many aspects of these spaces have been studied [1, 2, 6, 24]. In [8]; the authors have defined some spaces of functions that are of $D_{L^p}$-type but replacing the usual derivative by the Bessel operator $\frac{1}{r^{2a+1}} \frac{d}{dr}(r^{2a+1} \frac{d}{dr})$, and they have established many results for these spaces.

In [3]; we define the so-called Riemann-Liouville operator; defined on $C_0(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} f\left(\sqrt{1-t^2}, x + rt\right) \\
\times \left(1-t^2\right)^{\alpha - \frac{3}{2}} \left(1-s^2\right)^{\alpha - 1} dt ds; & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} f\left(\sqrt{1-t^2}, x + rt\right) \frac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha = 0. \end{cases}$$

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The dual operator $^t\mathcal{R}_0$ is defined by

$$^t\mathcal{R}_0(f)(r,x) = \begin{cases} \frac{2\alpha}{\pi} \int_r^{+\infty} \int_{\sqrt{u^2-r^2}}^{\infty} f(u,x+v) \\ \times (u^2 - u^2 - r^2)^{\alpha-1} \, u\, du\, dv; & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{\mathbb{R}} f \left( \sqrt{r^2 + (x-y)^2}, y \right) \, dy; & \text{if } \alpha = 0. \end{cases}$$

The operators $\mathcal{R}_0$ and $^t\mathcal{R}_0$ generalize the mean operator $\mathcal{R}_0$ and its dual $^t\mathcal{R}_0$ ([28]), defined respectively by

$$\mathcal{R}_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \sin \theta, x + r \cos \theta) \, d\theta,$$

and

$$^t\mathcal{R}_0(f)(r,x) = \frac{1}{\pi} \int_{\mathbb{R}} f \left( \sqrt{r^2 + (x-y)^2}, y \right) \, dy.$$
differential operator $\Delta_\alpha$ is continuous from $\mathcal{S}^\prime_c(\mathbb{R}^2)$ into itself. Moreover, for all $T \in \mathcal{S}^\prime(\mathbb{R}^2)$ we define $\Delta_\alpha(T)$ by
\[ \forall \varphi \in \mathcal{S}^\prime(\mathbb{R}^2); \quad \langle \Delta_\alpha(T), \varphi \rangle = \langle T, \Delta_\alpha(\varphi) \rangle, \]
then, $\Delta_\alpha$ is also continuous from $\mathcal{S}^\prime(\mathbb{R}^2)$ into itself.

The space $\mathcal{M}^p_\rho(\mathbb{R}^2)$ consists of all measurable functions on $\mathbb{R}^2$ even with respect to the first variable such that for all $k \in \mathbb{N}$, there exists a function denoted by $\Delta^k_\rho(f) \in L^p(\nu_\alpha)$ satisfying $\Delta^k_\rho(T_f) = T_{\Delta^k_\rho(f)}$; where
\[ T_f \text{ is the tempered distribution, even with respect to the first variable given by the function } f. \]

$L^p(\nu_\alpha)$, $1 \leq p < +\infty$, is the space of measurable functions $f$ on $[0, +\infty[ \times \mathbb{R}$, such that
\[ \|f\|_{p,\nu_\alpha} = \left( \int_0^{+\infty} \int_\mathbb{R} |f(r,x)|^p \, d\nu_\alpha(r,x) \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty; \]
\[ \|f\|_{\infty,\nu_\alpha} = \text{ess sup}_{(r,x) \in [0, +\infty[ \times \mathbb{R}} |f(r,x)| < \infty, \quad p = +\infty. \]

Using the convolution product and the Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann-Liouville operator, we establish firstly the following results which give a nice characterization of the elements of the dual space $\mathcal{M}^p_\rho(\mathbb{R}^2)$

- Let $T \in \mathcal{S}^\prime_c(\mathbb{R}^2)$. Then $T$ belongs to $\mathcal{M}^p_\rho(\mathbb{R}^2)$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0, \ldots, f_m\} \subset L^p(\nu_\alpha)$; $p^\prime = \frac{p}{p-1}$, such that $T = \sum_{k=0}^{m} \Delta^k_\rho(T_{f_k})$,

where
\[ \forall \varphi \in \mathcal{M}^p_\rho(\mathbb{R}^2); \quad \langle \Delta^k_\rho(T_f), \varphi \rangle = \int_0^{+\infty} \int_\mathbb{R} f(r,x) \Delta^k_\rho(\varphi)(r,x) \, d\nu_\alpha(r,x); \]

- Let $T \in \mathcal{S}^\prime(\mathbb{R}^2)$, $p \in [1, +\infty[\setminus\{1\}$, and $p^\prime = \frac{p}{p-1}$. Then $T$ belongs to $\mathcal{M}^p_\rho(\mathbb{R}^2)$ if, and only if for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$, the function $T \ast \varphi$ belongs to the space $L^{p^\prime}(\nu_\alpha)$,

where
\[ T \ast \varphi \text{ is the function defined by } T \ast \varphi(r,x) = \langle T, \tau_{(r,-x)}(\varphi) \rangle, \text{ with } \varphi(r,x) = \varphi(r,-x). \]
\[ \tau_{(r,x)} \text{ is the translation operator associated with the Riemann-Liouville transform, defined on } L^p(\nu_\alpha) \text{ by} \]
\[ \tau_{(r,x)} f(s,y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^{\pi} f \left( \sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y \right) \sin^{2\alpha}(\theta) \, d\theta. \]

$\mathcal{S}(\mathbb{R}^2)$ is the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.

Next, by the fact that a subset of $\mathcal{M}^p_\rho(\mathbb{R}^2)$ is bounded if, and only if it is equicontinuous, we show the coming result that is a good description of the bounded subsets of the dual space $\mathcal{M}^p_\rho(\mathbb{R}^2)$

- Let $p \in [1, +\infty[\setminus\{1\}$ and let $B'$ be a subset of $\mathcal{M}^p_\rho(\mathbb{R}^2)$. The following assertions are equivalent

  (i) $B'$ is weakly (equivalently strongly) bounded in $\mathcal{M}^p_\rho(\mathbb{R}^2)$,
(ii) there exist $C > 0$ and $m \in \mathbb{N}$, such that for every $T \in B'$, it is possible to find \( f_0, \ldots, f_m \in L^{p'}(d\nu_\alpha) \) satisfying
\[
T = \sum_{k=0}^{m} \Delta_k(T f_k) \text{ with } \max_{0 \leq k \leq m} \|f_k\|_{p', \nu_\alpha} \leq C;
\]

(iii) for every $\varphi \in \mathcal{D}_\ast(\mathbb{R}^2)$, the set \( \{T * \varphi, T \in B'\} \) is bounded in $L^{p'}(d\nu_\alpha)$.

Finally, we define and study a convolution product on the space $M'_p(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$, $1 \leq r \leq p < +\infty$, where $\mathcal{M}_r(\mathbb{R}^2)$ is the closure of the space $\mathcal{L}_r(\mathbb{R}^2)$ in $M'_p(\mathbb{R}^2)$. More precisely

- Let $p \in [1, +\infty]$. For every $(r,x) \in [0, +\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ is continuous from $M'_p(\mathbb{R}^2)$ into itself.
- Let $1 \leq r \leq p < +\infty$ and $q \in [1, +\infty]$, such that
\[
\frac{1}{q} = \frac{1}{r} - \frac{1}{p}.
\]

Then for every $T \in M'_p(\mathbb{R}^2)$, the mapping
\[
\phi \mapsto T * \phi
\]
is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into $M_q(\mathbb{R}^2)$.

2. The Fourier transform associated with the Riemann-Liouville operator

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Riemann-Liouville operator. For more details see [3, 5, 7, 16].

Let $D$ and $\Xi$ be the singular partial differential operators defined by
\[
\begin{align*}
D &= \frac{\partial}{\partial x}; \\
\Xi &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}; \quad (r,x) \in [0, +\infty[\times\mathbb{R}, \alpha \geq 0.
\end{align*}
\]

For all $(\mu, \lambda) \in \mathbb{C}^2$, the system
\[
\begin{align*}
Du(r,x) &= -i\lambda u(r,x); \\
\Xi u(r,x) &= -\mu^2 u(r,x); \\
u(0,0) &= 1, \quad \frac{\partial u}{\partial r}(0,x) = 0; \quad \forall x \in \mathbb{R}.
\end{align*}
\]

admits a unique solution $\varphi_{\mu, \lambda}$, given by
\[
\forall (r,x) \in [0, +\infty[\times\mathbb{R}; \quad \varphi_{\mu, \lambda}(r,x) = j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) \exp(-i\lambda x),
\]

where $j_\alpha$ is the modified Bessel function defined by
\[
j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left( \frac{s}{2} \right)^{2k},
\]

\[
\Phi_{\alpha}(s) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left( \frac{s}{2} \right)^{2k}.
\]
The translation operator associated with Riemann-Liouville transform is defined
and $J$, Bessel function of first kind and index $\alpha$. The modified Bessel function $j_\alpha$ has the integral representation

$$j_\alpha(s) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{-\frac{1}{2}} \exp(-ist) \, dt.$$  \hspace{1cm} (2.2)

Consequently, for all $k \in \mathbb{N}$ and $s \in \mathbb{R}$; we have

$$|j_\alpha^{(k)}(s)| \leq 1.$$ \hspace{1cm} (2.3)

The eigenfunction function $\varphi_{\mu, \lambda}$ satisfies the following properties

- $\sup_{(r, x) \in \mathbb{R}^2} |\varphi_{\mu, \lambda}(r, x)| = 1$ if, and only if $(\mu, \lambda) \in \Gamma$, \hspace{1cm} (2.4)

where $\Gamma$ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \{ (i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda| \}.$$ \hspace{1cm} (2.5)

- The function $\varphi_{\mu, \lambda}$ has the following Mehler integral representation

$$\varphi_{\mu, \lambda}(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} \cos \left( \mu rs \sqrt{1 - t^2} \right) \exp \left(-i\lambda(x + rt) \right) \\ \times (1 - t^2)^{\alpha - \frac{1}{2}} (1 - s^2)^{\alpha - 1} \, dt \, ds; \quad \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} \cos \left( r \mu \sqrt{1 - t^2} \right) \exp(-i\lambda(x + rt)) \\ \times \frac{dt}{\sqrt{1 - t^2}}; \quad \text{if } \alpha = 0. \end{cases}$$

The precedent integral representation allows us to define the Riemann-Liouville transform $\mathcal{R}_\alpha$ associated with the operators $\Delta_1$ and $\Delta_2$ by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} \int_{-1}^{1} f \left( rs \sqrt{1 - t^2}, x + rt \right) \\ \times (1 - t^2)^{\alpha - \frac{1}{2}} (1 - s^2)^{\alpha - 1} \, dt \, ds; \quad \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} f \left( r \sqrt{1 - t^2}, x + rt \right) \frac{dt}{\sqrt{1 - t^2}}; \quad \text{if } \alpha = 0. \end{cases}$$

where $f$ is any continuous functions on $\mathbb{R}^2$, even with respect to the first variable.

- From the precedent integral representation of the eigenfunction $\varphi_{\mu, \lambda}$, we have

$$\forall (r, x) \in [0, +\infty[ \times \mathbb{R}, \quad \varphi_{\mu, \lambda}(r, x) = \mathcal{R}_\alpha(\cos(\mu) e^{-\lambda})(r, x).$$

In the following, we will define the convolution product and the Fourier transform associated with the Riemann-Liouville transform. For this, we use the product formula for the function $\varphi_{\mu, \lambda}$ given by

$$\varphi_{\mu, \lambda}(r, x) \varphi_{\mu, \lambda}(s, y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{0}^{\pi} \varphi_{\mu, \lambda}(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \times \sin^{2\alpha}(\theta) \, d\theta. \hspace{1cm} (2.6)$$

**Definition 2.1.**

(i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(\mu_\alpha)$, by for all $(r, x), (s, y) \in [0, +\infty[ \times \mathbb{R},$

$$\tau_{(r, x)} f(s, y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{0}^{\pi} f \left( \sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y \right) \sin^{2\alpha}(\theta) \, d\theta.$$
(ii) The convolution product of \( f, g \in L^1(d\nu_\alpha) \) is defined for all \((r, x) \in [0, +\infty[ \times \mathbb{R}, \) by
\[
 f \ast g(r, x) = \int_0^{+\infty} \int_\mathbb{R} \tau_{(r, -x)}(\tilde{f}(s, y))g(s, y)d\nu_\alpha(s, y),
\]
where \( \tilde{f}(s, y) = f(s, -y) \).

We have the following properties
- The product formula [2.6], can be written
  \[
  \tau_{(r, x)}(\varphi_{\mu, \lambda})(s, y) = \varphi_{\mu, \lambda}(r, x)\varphi_{\mu, \lambda}(s, y).
  \]
- For all \( f \in L^p(d\nu_\alpha), 1 \leq p \leq +\infty \), and for all \((r, x) \in [0, +\infty[ \times \mathbb{R}, \) the function \( \tau_{(r, x)}(f) \) belongs to \( L^p(d\nu_\alpha) \) and we have
  \[
  \|\tau_{(r, x)}(f)\|_{p, \nu_\alpha} \leq \|f\|_{p, \nu_\alpha} \tag{2.7}
  \]
- For \( f, g \in L^1(d\nu_\alpha) \), the function \( f \ast g \) belongs to \( L^1(d\nu_\alpha) \); the convolution product is commutative, associative and we have
  \[
  \|f \ast g\|_{1, \nu_\alpha} \leq \|f\|_{1, \nu_\alpha} \|g\|_{1, \nu_\alpha}. \tag{2.8}
  \]

Moreover, if \( 1 \leq p, q, r \leq +\infty \) are such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \) and if \( f \in L^p(d\nu_\alpha), g \in L^q(d\nu_\alpha) \), then the function \( f \ast g \) belongs to \( L^r(d\nu_\alpha) \), and we have
\[
\|f \ast g\|_{r, \nu_\alpha} \leq \|f\|_{p, \nu_\alpha} \|g\|_{q, \nu_\alpha}. \tag{2.9}
\]

In the sequel, we use the following notations
- \( \Gamma_+ \) is the subset of \( \Gamma \) given by
  \[
  \Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x); \ (t, x) \in \mathbb{R}^2; \ 0 \leq t \leq |x|\}.
  \]
- \( \mathcal{B}_{\Gamma_+} \) is the \( \sigma \)-algebra defined on \( \Gamma_+ \) by
  \[
  \mathcal{B}_{\Gamma_+} = \{\theta^{-1}(B); \ B \in \mathcal{B}_{\text{Bor}}([0, +\infty[ \times \mathbb{R})\},
  \]
  where \( \theta \) is the bijective function defined on the set \( \Gamma_+ \) by
  \[
  \theta(\mu, \lambda) = (\sqrt{\mu^2 + \lambda^2}, \lambda). \tag{2.10}
  \]
- \( d_{\gamma_\alpha} \) is the measure defined on \( \mathcal{B}_{\Gamma_+} \) by
  \[
  \forall A \in \mathcal{B}_{\Gamma_+}; \ \gamma_\alpha(A) = \nu_\alpha(\theta(A)).
  \]
- \( L^p(d_{\gamma_\alpha}), p \in [1, +\infty[ \) is the space of measurable functions \( f \) on \( \Gamma_+ \), such that
  \[
  \|f\|_{p, \gamma_\alpha} = \left( \int_{\Gamma_+} |f(\mu, \lambda)|^p d_{\gamma_\alpha}(\mu, \lambda) \right)^{\frac{1}{p}} < \infty, \ \text{if} \ p \in [1, +\infty[,
  \]
  \[
  \|f\|_{\infty, \gamma_\alpha} = \text{ess sup}_{(r, x) \in [0, +\infty[ \times \mathbb{R}} |f(r, x)| < +\infty, \ \text{if} \ p = +\infty.
  \]

**Proposition 2.2.**

i) For all non negative measurable function \( g \) on \( \Gamma_+ \), we have
\[
\int_{\Gamma_+} g(\mu, \lambda)d\gamma_\alpha(\mu, \lambda) = \frac{1}{2^\alpha \Gamma(\alpha + 1)\sqrt{2\pi}} \left( \int_0^{+\infty} \int_\mathbb{R} g(\mu, \lambda)(\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda \right.
  \]
\[
+ \left. \int_\mathbb{R} \int_0^{\left|\lambda\right|} g(i\mu, \lambda)(\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda \right).
\]
ii) For all nonnegative measurable function \( f \) on \([0, +\infty] \times \mathbb{R}\) (respectively integrable on \([0, +\infty] \times \mathbb{R}\) with respect to the measure \(d\nu_\alpha\)) \( f \circ \theta \) is a nonnegative measurable function on \( \Gamma_+ \) (respectively integrable on \( \Gamma_+ \) with respect to the measure \(d\gamma_\alpha\)) and we have
\[
\int_{\Gamma_+} \left( f \circ \theta \right)(\mu, \lambda)d\gamma_\alpha(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x)d\nu_\alpha(r, x).
\]

**Definition 2.3.** The Fourier transform associated with the Riemann-Liouville operator is defined on \( L^1(d\nu_\alpha) \), by
\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x)\varphi_{\mu, \lambda}(r, x)d\nu_\alpha(r, x),
\]
where \( \varphi_{\mu, \lambda} \) is the eigenfunction given by the relation (2.1) and \( \Gamma \) is the set defined by the relation (2.5).

We have the following properties:

- From the relation (2.4), we deduce that for \( f \in L^1(d\nu_\alpha) \) the function \( \mathcal{F}_\alpha(f) \) belongs to the space \( L^\infty(d\nu_\alpha) \) and we have
\[
\|\mathcal{F}_\alpha(f)\|_{\infty, \nu_\alpha} \leq \|f\|_{1, \nu_\alpha}.
\]

- For \( f \in L^1(d\nu_\alpha) \), we have
\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \mathcal{F}_\alpha(f) \circ \theta(\mu, \lambda), \tag{2.11}
\]
where
\[
\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x)j_\alpha(\mu r)\exp(-i\lambda x)d\nu_\alpha(r, x), \tag{2.12}
\]
and \( \theta \) is the function defined by (2.9).

- Let \( f \in L^1(d\nu_\alpha) \) such that the function \( \mathcal{F}_\alpha(f) \) belongs to the space \( L^1(d\gamma_\alpha) \), then we have the following inversion formula for \( \mathcal{F}_\alpha \), for almost every \((r, x) \in [0, +\infty] \times \mathbb{R},
\[
\mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu, \lambda)\varphi_{\mu, \lambda}(r, x)d\gamma_\alpha(\mu, \lambda).
\]

- Let \( f \in L^1(d\nu_\alpha) \). For all \((s, y) \in [0, +\infty] \times \mathbb{R}, \) we have
\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(\tau_{s, y}(f)) (\mu, \lambda) = \overline{\varphi_{\mu, \lambda}(s, y)}\mathcal{F}_\alpha(f)(\mu, \lambda).
\]

- For \( f, g \in L^1(d\nu_\alpha) \), we have
\[
\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f * g)(\mu, \lambda) = \mathcal{F}_\alpha(f)(\mu, \lambda)\mathcal{F}_\alpha(g)(\mu, \lambda).
\]

- Let \( p \in [1, +\infty], \) the function \( f \) belongs to \( L^p(d\nu_\alpha) \) if, and only if the function \( f \circ \theta \) belongs to the space \( L^p(d\gamma_\alpha) \) and we have
\[
\|f \circ \theta\|_{p, \gamma_\alpha} = \|f\|_{p, \nu_\alpha}. \tag{2.13}
\]

Since the mapping \( \mathcal{F}_\alpha \) is an isometric isomorphism from \( L^2(d\nu_\alpha) \) onto itself, then the relations (2.11) and (2.13) show that the Fourier transform \( \mathcal{F}_\alpha \) is an isometric isomorphism from \( L^2(d\nu_\alpha) \) into \( L^2(d\gamma_\alpha) \), namely, for every \( f \in L^2(d\nu_\alpha) \), the function \( \mathcal{F}_\alpha(f) \) belongs to the space \( L^2(d\gamma_\alpha) \) and we have
\[
\|\mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha} = \|f\|_{2, \nu_\alpha}. \tag{2.14}
\]
Proposition 2.4. For all \( f \) in \( L^p(d\nu_\alpha) \), \( p \in [1,2] \); the function \( \mathcal{F}_\alpha(f) \) lies in \( L^{p'}(d\gamma_\alpha) \), \( p' = \frac{p}{p-1} \), and we have

\[
\|\mathcal{F}_\alpha(f)\|_{p',\gamma_\alpha} \leq \|f\|_{p,\nu_\alpha}.
\]

Proof. The result follows from the relations (2.10), (2.14) and the Riesz-Thorin theorem’s [25, 26]. □

We denote by

- \( \mathcal{S}_*(\Gamma) \) (see [3, 28]) the space of functions \( f : \Gamma \rightarrow \mathbb{C} \) infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that means for all \( k_1, k_2, k_3 \in \mathbb{N} \),

\[
\sup_{(\mu, \lambda) \in \Gamma} (1 + \mu^2 + 2\lambda^2)^{k_1} \left| \left( \frac{\partial}{\partial \mu} \right)^{k_2} \left( \frac{\partial}{\partial \lambda} \right)^{k_3} f(\mu, \lambda) \right| < +\infty,
\]

where

\[
\frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} 
\frac{\partial}{\partial r}(f(r, \lambda)), & \text{if } \mu = r \in \mathbb{R}; \\
1 \frac{\partial}{\partial t}(f(it, \lambda)), & \text{if } \mu = it, |t| \leq |\lambda|. 
\end{cases}
\]

- \( \mathcal{S}'(\mathbb{R}^2) \) and \( \mathcal{S}'(\Gamma) \) are respectively the dual spaces of \( \mathcal{S}_*(\mathbb{R}^2) \) and \( \mathcal{S}_*(\Gamma) \). Each of these spaces is equipped with its usual topology.

Remark 2.5. (See [3]) The Fourier transform \( \mathcal{F}_\alpha \) is a topological isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) onto \( \mathcal{S}_*(\Gamma) \). The inverse mapping is given by for all \( (r, x) \in \mathbb{R}^2 \),

\[
\mathcal{F}_\alpha^{-1}(f)(r, x) = \int \int_{\Gamma_+} f(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda).
\]

Definition 2.6. The Fourier transform \( \mathcal{F}_\alpha \) is defined for all \( T \in \mathcal{S}_*(\mathbb{R}^2) \) by

\[
\langle \mathcal{F}_\alpha(T), \varphi \rangle = \langle T, \mathcal{F}_\alpha^{-1}(\varphi) \rangle, \quad \varphi \in \mathcal{S}_*(\Gamma).
\]

Since the Fourier transform \( \mathcal{F}_\alpha \) is an isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) into \( \mathcal{S}_*(\Gamma) \), we deduce that \( \mathcal{F}_\alpha \) is also an isomorphism from \( \mathcal{S}'(\mathbb{R}^2) \) into \( \mathcal{S}'(\Gamma) \).

3. The space \( \mathcal{M}_p(\mathbb{R}^2) \)

We denote by

- \( \Delta_\alpha \) the partial differential operator defined by

\[
\Delta_\alpha = - \left( \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} \right).
\]

- For all \( f \in L^p(d\nu_\alpha) \), \( p \in [1, +\infty] \), \( T_f \) is the element of \( \mathcal{S}_*(\mathbb{R}^2) \) defined by

\[
\forall \varphi \in \mathcal{S}_*(\mathbb{R}^2), \quad \langle T_f, \varphi \rangle = \int_0^{+\infty} \int_\mathbb{R} f(r, x) \varphi(r, x) d\nu_\alpha(r, x).
\]

- For all \( g \in L^p(d\gamma_\alpha) \), \( p \in [1, +\infty] \), \( T_g \) is the element of \( \mathcal{S}_*(\Gamma) \) defined by

\[
\forall \psi \in \mathcal{S}_*(\Gamma), \quad \langle T_g, \psi \rangle = \int \int_{\Gamma_+} g(\mu, \lambda) \psi(\mu, \lambda) d\gamma_\alpha(\mu, \lambda),
\]

where \( \psi, \varphi \in \mathcal{S}_*(\mathbb{R}^2) \) and \( g, f \in \mathcal{S}_*(\mathbb{R}^2) \).
From proposition 2.4 and remark 2.5 we deduce that for all \( f \in L^p(d\nu), \ 1 \leq p \leq 2, \) the function \( \mathcal{F}_\alpha(f) \) belongs to the space \( L^p(d\gamma) \) and we have
\[
\mathcal{F}_\alpha(T_f) = T_{\mathcal{F}_\alpha(f)},
\] (3.1) with \( p' = \frac{p}{p-1} \).

On the other hand, since the operator \( \Delta_\alpha \) is continuous from \( \mathcal{S}_*(\mathbb{R}^2) \) into itself; we define \( \Delta_\alpha \) on \( S'_*(\mathbb{R}^2) \) by setting
\[
\forall \varphi \in \mathcal{S}_*(\mathbb{R}^2), \ \forall T \in S'_*(\mathbb{R}^2); \ \langle \Delta_\alpha(T), \varphi \rangle = \langle T, \Delta_\alpha(\varphi) \rangle.
\]

Then \( \Delta_\alpha \) becomes a continuous operator from \( \mathcal{S}_*(\mathbb{R}^2) \) into itself; moreover for all \( f \in \mathcal{S}_*(\mathbb{R}^2) \) and for all integer \( k \) we have
\[
\Delta_\alpha^k(T_f) = T\Delta_\alpha^k(f).
\]

**Definition 3.1.** Let \( p \in [1, +\infty] \). We define \( \mathcal{M}_p(\mathbb{R}^2) \) to be the space of measurable functions \( f \) on \( \mathbb{R}^2 \), even with respect to the first variable, and such that for all \( k \in \mathbb{N} \) there exists a function \( \Delta_\alpha^k(f) \in L^p(d\nu) \), satisfying
\[
\Delta_\alpha^k(T_f) = T\Delta_\alpha^k(f), \ \text{in} \ \mathcal{S}_*(\mathbb{R}^2).
\]

The space \( \mathcal{M}_p(\mathbb{R}^2) \) is equipped with the topology generated by the family of norms
\[
\gamma_{m,p}(f) = \max_{0 \leq k \leq m} \|\Delta_\alpha^k(f)\|_{p,\nu}, \quad m \in \mathbb{N},
\]
Also, we define a distance \( d_p \), on \( \mathcal{M}_p(\mathbb{R}^2) \) by
\[
\forall (f, g) \in \mathcal{M}_p(\mathbb{R}^2), \quad d_p(f, g) = \sum_{m=0}^{\infty} 2^m \frac{\gamma_{m,p}(f-g)}{1 + \gamma_{m,p}(f-g)}.
\]

Then, a sequence \( (f_k)_{k \in \mathbb{N}} \) converges to 0 in \( \mathcal{M}_p(\mathbb{R}^2), d_p \) if, and only if
\[
\forall m \in \mathbb{N}; \quad \gamma_{m,p}(f_k) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]

In the following, we will give some properties of the space \( \mathcal{M}_p(\mathbb{R}^2) \).

**Proposition 3.2.** \( \mathcal{M}_p(\mathbb{R}^2), d_p \) is a Fréchet space.

**Proof.** Let \( (f_m)_{m \in \mathbb{N}} \) be a Cauchy sequence in \( \mathcal{M}_p(\mathbb{R}^2), d_p \) and \( (\Delta_\alpha^k(f_m))_{m \in \mathbb{N}} \subset L^p(d\nu) \), such that
\[
\Delta_\alpha^k(T_{f_m}) = T\Delta_\alpha^k(f_m), \quad k \in \mathbb{N}.
\]

Then, for all \( k \in \mathbb{N} \); \( (\Delta_\alpha^k(f_m))_{m \in \mathbb{N}} \) is a Cauchy sequence in \( L^p(d\nu) \). We put
\[
g_k = \lim_{m \rightarrow +\infty} \Delta_\alpha^k(f_m), \quad k \in \mathbb{N}, \quad (3.2)
\]
in \( L^p(d\nu) \). Thus,
\[
\forall k \in \mathbb{N}; \quad \lim_{m \rightarrow +\infty} \Delta_\alpha^k(T_{f_m}) = \lim_{m \rightarrow +\infty} T\Delta_\alpha^k(f_m) = T_{g_k}, \ \text{in} \ \mathcal{S}_*(\mathbb{R}^2).
\]

Since the operator \( \Delta_\alpha \) is continuous from \( \mathcal{S}_*(\mathbb{R}^2) \) into itself and using the relation (3.2) we deduce that for all \( k \in \mathbb{N} \)
\[
\Delta_\alpha^k(T_{g_k}) = T_{g_k}
\]
This equality shows that the function \( g_k \) belongs to the space \( \mathcal{M}_p(\mathbb{R}^2) \) and that for all \( k \in \mathbb{N}, \Delta_\alpha^k(g_k) = g_k \).
Now, the relation (3.2) implies that the sequence \((f_m)_m\) converges to \(g_0\) in \((\mathcal{M}_p(\mathbb{R}^2), d_p)\).

**Remark 3.3.** The operator \(\Delta_\alpha\) is continuous from \(\mathcal{M}_p(\mathbb{R}^2)\) into itself. Moreover, for all \(m \in \mathbb{N}\), we have
\[
\forall f \in \mathcal{M}_p(\mathbb{R}^2), \quad \gamma_{m,p}(\Delta_\alpha(f)) \leq \gamma_{m+1,p}(f).
\]

We denote by
- \(\mathcal{C}_*(\mathbb{R}^2)\) the space of continuous functions on \(\mathbb{R}^2\), even with respect to the first variable.
- \(\mathcal{E}_*(\mathbb{R}^2)\) the subspace of \(\mathcal{C}_*(\mathbb{R}^2)\) consisting of infinitely differentiable functions on \(\mathbb{R}^2\).

**Proposition 3.4.** Let \(p \in [1, 2]\) and \(f \in \mathcal{M}_p(\mathbb{R}^2)\) then
\[
\begin{align*}
\text{(i)} & \quad \text{For all } k \in \mathbb{N}, \text{ the function } \varphi \rightarrow (1 + \mu^2 + 2\lambda^2)^k \mathcal{F}_\alpha(f)(\mu, \lambda) \\
& \quad \text{belongs to the space } L^{p'}(d\gamma_\alpha), \text{ with } p' = \frac{p}{p-1}.
\end{align*}
\]
\[
\begin{align*}
\text{(ii)} & \quad \mathcal{M}_p(\mathbb{R}^2) \cap \mathcal{C}_*(\mathbb{R}^2) \subset \mathcal{E}_*(\mathbb{R}^2).
\end{align*}
\]

**Proof.** (i) Let \(f \in \mathcal{M}_p(\mathbb{R}^2)\), \(1 \leq p \leq 2\). From the relation (3.1), we have
\[
\mathcal{F}_\alpha(\Delta_\alpha^k(T_f)) = \mathcal{F}_\alpha(T_{\Delta_\alpha^k(f)}) = T_{\mathcal{F}_\alpha(\Delta_\alpha^k(f))}.
\]
On the other hand,
\[
\mathcal{F}_\alpha(\Delta_\alpha^k(T_f)) = (\mu^2 + 2\lambda^2)^k \mathcal{F}_\alpha(T_f),
\]
\[
= T_{(\mu^2 + 2\lambda^2)^k} \mathcal{F}_\alpha(f),
\]
hence,
\[
(\mu^2 + 2\lambda^2)^k \mathcal{F}_\alpha(f) = \mathcal{F}_\alpha(\Delta_\alpha^k(f)),
\]
this equality, together with the fact that the function \(\mathcal{F}_\alpha(\Delta_\alpha^k(f))\) belongs to the space \(L^{p'}(d\gamma_\alpha)\) implies (i).

(ii) Let \(f \in \mathcal{M}_p(\mathbb{R}^2) \cap \mathcal{C}_*(\mathbb{R}^2)\). From the assertion (i) and the relations (2.11) and (2.13), we deduce that for all \(k \in \mathbb{N}\), the function
\[
(\mu, \lambda) \rightarrow (\mu^2 + \lambda^2)^k \tilde{\mathcal{F}}_\alpha(f)(\mu, \lambda),
\]
belongs to the space \(L^{p'}(dv_\alpha)\). Hence, by using H"{o}lder’s inequality, we deduce that the function \(\tilde{\mathcal{F}}_\alpha(f)\) belongs to the space \(L^1(dv_\alpha) \cap L^2(dv_\alpha)\).

On the other hand, the transform \(\mathcal{F}_\alpha\) is an isometric isomorphism from \(L^2(dv_\alpha)\) onto itself, then from the inversion formula for \(\mathcal{F}_\alpha\), and using the continuity of the function \(f\), we have for all \((r, x) \in \mathbb{R}^2\),
\[
f(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} \tilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) j_\alpha(rp) \exp(i\lambda x) dv_\alpha(\mu, \lambda).
\]
(3.3)
Then, the result follows from the derivative theorem, the relations (2.3) and (3.3).

**Proposition 3.5.** Let \(p \in [1, 2]\). Then, for all \(\eta \in [2, +\infty]\),
\[
\mathcal{M}_p(\mathbb{R}^2) \cap \mathcal{C}_*(\mathbb{R}^2) \subset \mathcal{M}_\eta(\mathbb{R}^2).
\]
Proof. Let \( f \in \mathcal{M}_p(\mathbb{R}^2) \cap \mathcal{C}_\infty(\mathbb{R}^2) \), \( p \in [1, 2] \), \( \eta \geq 2 \) and \( \eta' = \eta/(\eta - 1) \). From proposition 3.4, we deduce that \( f \in \mathcal{E}_\eta(\mathbb{R}^2) \) and that for all \( k \in \mathbb{N} \), the function 
\[
(\mu, \lambda) \mapsto (\mu^2 + \lambda^2)^k \tilde{F}_\alpha(f)(\mu, \lambda),
\]
belongs to the space \( L^{\eta'}(dv_\alpha) \). By applying Hölder’s inequality it follows that this last function belongs to the space \( L^{\eta'}(dv_\alpha) \).

On the other hand, for all \( (r, x) \in \mathbb{R}^2 \),
\[
\Delta^k_\alpha(f)(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} (\mu^2 + \lambda^2)^k \tilde{F}_\alpha(f)(\mu, \lambda) j_\alpha(r\mu) \exp(i\lambda x)dv_\alpha(\mu, \lambda),
\]
\[
= \tilde{F}_\alpha ( (\mu^2 + \lambda^2)^k \tilde{F}_\alpha(f) ) (r, x).
\]

From proposition 2.4 and the fact that for all \( g \in L^n(\mathbb{R}^2) \),
\[
\| F_\alpha(g) \|_{n, \gamma} = \| F_\alpha(g) \|_{n, \nu_\alpha},
\]
we deduce that for all \( k \in \mathbb{N} \), the function \( \Delta^k_\alpha(f) \) belongs to the space \( L^n(dv_\alpha) \). \( \square \)

4. The dual space \( \mathcal{M}_p^*(\mathbb{R}^2) \)

In this section, we will give a new characterization of the dual space \( \mathcal{M}_p^*(\mathbb{R}^2) \) of \( \mathcal{M}_p(\mathbb{R}^2) \).

It is well known that for every \( f \in \mathcal{M}_p(\mathbb{R}^2) \), the family \( \{ \mathfrak{N}_{m, p, \varepsilon}(f), m \in \mathbb{N}, \varepsilon > 0 \} \), defined by
\[
\mathfrak{N}_{m, p, \varepsilon}(f) = \{ g \in \mathcal{M}_p(\mathbb{R}^2), \gamma_{m, p}(f - g) < \varepsilon \}
\]
is a basis of neighborhoods of \( f \) in \( (\mathcal{M}_p(\mathbb{R}^2), d_p) \).

Hence, \( T \in \mathcal{M}_p^*(\mathbb{R}^2) \) if, and only if there exist \( m \in \mathbb{N} \) and \( C > 0 \), such that
\[
\forall f \in \mathcal{M}_p(\mathbb{R}^2); \quad |\langle T, f \rangle| \leq C \gamma_{m, p}(f). \tag{4.1}
\]

For \( f \in L^{\eta'}(dv_\alpha) \) and \( \varphi \in \mathcal{M}_p(\mathbb{R}^2) \), we put
\[
\langle \Delta^k_\alpha(T_f), \varphi \rangle = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \Delta^k_\alpha(\varphi)(r, x)dv_\alpha(r, x), \tag{4.2}
\]
with \( \Delta^k_\alpha(T_\varphi) = T_{\Delta^k_\alpha(\varphi)} \). Then
\[
|\langle \Delta^k_\alpha(T_f), \varphi \rangle| \leq \| f \|_{p', \nu_\alpha} \| \Delta^k_\alpha(\varphi) \|_{p, \nu_\alpha} \leq \| f \|_{p', \nu_\alpha} \gamma_{k, p}(\varphi),
\]
this proves that for all \( f \in L^{\eta'}(dv_\alpha) \) and \( k \in \mathbb{N} \), the functional \( \Delta^k_\alpha(T_f) \) defined by the relation (4.2), belongs to the space \( \mathcal{M}_p^*(\mathbb{R}^2) \).

In the following, we will prove that every element of \( \mathcal{M}_p^*(\mathbb{R}^2) \) is also of this type.

Theorem 4.1. Let \( T \in \mathcal{M}_p^*(\mathbb{R}^2) \). Then \( T \) belongs to \( \mathcal{M}_p^*(\mathbb{R}^2) \), \( 1 \leq p < +\infty \), if and only if there exist \( m \in \mathbb{N} \) and \( \{ f_0, \ldots, f_m \} \subset L^{\eta'}(dv_\alpha) \), such that
\[
T = \sum_{k=0}^{m} \Delta^k_\alpha(T_{f_k}), \tag{4.3}
\]
where \( \Delta^k_\alpha(T_{f_k}) \) is given by the relation (4.2).
Proof. It is clear that if 
\[ T = \sum_{k=0}^{m} \Delta_k^k(T_{f_k}), \quad \{f_0, \ldots, f_m\} \subset L^p'(dw), \]
then \( T \) belongs to the space \( \mathcal{M}'(\mathbb{R}^2) \). Conversely, suppose that \( T \in \mathcal{M}'(\mathbb{R}^2) \). From the relation (4.1), there exist \( m \in \mathbb{N} \) and \( C > 0 \), such that 
\[ \forall \varphi \in \mathcal{M}(\mathbb{R}^2), \quad |\langle T, \varphi \rangle| \leq C \gamma_m, \varphi. \]
Let \( \left(L^p(dw)\right)^{m+1} = \{(f_0, \ldots, f_m), f_k \in L^p(dw), 0 \leq k \leq m\} \), equipped with the norm 
\[ \left\| (f_0, \ldots, f_m) \right\|_{\left(L^p(dw)\right)^{m+1}} = \max_{0 \leq k \leq m} \|f_k\|_{p, w}. \]
We consider the mappings 
\[ \mathcal{A} : \mathcal{M}(\mathbb{R}^2) \rightarrow \left(L^p(dw)\right)^{m+1} \]
\[ \varphi \mapsto (\varphi, \Delta_\alpha^k(\varphi), \ldots, \Delta_m^\alpha(\varphi)), \]
and 
\[ \mathcal{B} : \mathcal{A}(\mathcal{M}(\mathbb{R}^2)) \rightarrow \mathbb{C}; \quad \mathcal{B}(\mathcal{A}) = \langle T, \varphi \rangle. \]
From the relation (4.1), we deduce that 
\[ \left| \mathcal{B}(\mathcal{A}) \right| = \left| \langle T, \varphi \rangle \right| \leq C \left\| \mathcal{A}(\varphi) \right\|_{\left(L^p(dw)\right)^{m+1}}, \]
this means that \( \mathcal{B} \) is a continuous functional on the subspace \( \mathcal{A}(\mathcal{M}(\mathbb{R}^2)) \) of the space \( \left(L^p(dw)\right)^{m+1} \). From Hahn Banach theorem, there exists a continuous extension of \( \mathcal{B} \) to \( \left(L^p(dw)\right)^{m+1} \), denoted again by \( \mathcal{B} \).

By Riesz theorem, there exist \( (f_0, \ldots, f_m) \in \left(L^p'(dw)\right)^{m+1} \), such that for all \( (\varphi_0, \ldots, \varphi_m) \in \left(L^p(dw)\right)^{m+1} \),
\[ \mathcal{B}(\varphi_0, \ldots, \varphi_m) = \sum_{k=0}^{m} \int_0^{+\infty} \int_\mathbb{R} f_k(r, x) \varphi_k(r, x) dw(r, x). \]
By means of the relation (4.2), we deduce that for \( \varphi \in \mathcal{M}(\mathbb{R}^2) \), we have
\[ \langle T, \varphi \rangle = \sum_{k=0}^{m} \int_0^{+\infty} \int_\mathbb{R} f_k(r, x) \Delta_k^\alpha(\varphi)(r, x) dw(r, x) = \sum_{k=0}^{m} \left(\Delta_k^\alpha(T_{f_k}), \varphi \right). \]

□

**Proposition 4.2.** Let \( p \geq 2 \). Then for all \( T \in \mathcal{M}'(\mathbb{R}^2) \), there exist \( m \in \mathbb{N} \) and \( F \in L^p(d\gamma) \); such that 
\[ \mathcal{F}_m(T) = T_{1+\mu^2+2\lambda^2} = F. \]
Proof. Let $T \in \mathcal{M}'(\mathbb{R}^2)$. From Theorem 4.1 there exist $m \in \mathbb{N}$ and $(f_0, \ldots, f_m) \subset (L^p(d\nu_\alpha))^{m+1}$, $p' = \frac{p}{p-1}$, such that

$$T = \sum_{k=0}^{m} \Delta^k_\alpha(T_{f_k}).$$

Consequently,

$$\mathcal{F}_\alpha(T) = \sum_{k=0}^{m} \mathcal{F}_\alpha(\Delta^k_\alpha(T_{f_k})) = \sum_{k=0}^{m} (\mu^2 + 2\lambda^2)^k \mathcal{F}_\alpha(T_{f_k}).$$

From the relation (3.1), we get

$$\mathcal{F}_\alpha(T) = T_{(1 + \mu^2 + 2\lambda^2)^m F},$$

where

$$F = \sum_{k=0}^{m} \frac{(\mu^2 + 2\lambda^2)^k}{(1 + \mu^2 + 2\lambda^2)^m} \mathcal{F}_\alpha(\hat{f}_k).$$

□

Proposition 4.3. Let $T \in \mathcal{H}'(\mathbb{R}^2)$, then $T \in \mathcal{M}'_2(\mathbb{R}^2)$ if, and only if there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma_\alpha)$, such that

$$\mathcal{F}_\alpha(T) = T_{(1 + \mu^2 + 2\lambda^2)^m F}. \quad (4.4)$$

Proof. From Proposition 4.2, we deduce that if $T \in \mathcal{M}'_2(\mathbb{R}^2)$, then there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma_\alpha)$ verifying (4.4).

Conversely, suppose that (4.4) holds with $F \in L^2(d\gamma_\alpha)$. Since $\mathcal{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$, then there exists $G \in L^2(d\nu_\alpha)$, such that $\mathcal{F}_\alpha(G) = F$ and from the relation (3.1), we have

$$\mathcal{F}_\alpha(T_G) = T_F.$$

Consequently,

$$\mathcal{F}_\alpha(T) = \mathcal{F}_\alpha((I + \Delta_\alpha)^m(T_G)),$$

thus,

$$T = \sum_{k=0}^{m} C^k_m \Delta^k_\alpha(T_G),$$

and Theorem 4.1 implies that $T$ belongs to $\mathcal{M}'_2(\mathbb{R}^2)$. □

We denote by

- $\mathcal{D}_a(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support, equipped with its usual topology.
- For $a > 0$, $\mathcal{D}_{*,a}(\mathbb{R}^2)$ the subspace of $\mathcal{D}_a(\mathbb{R}^2)$, consisting of function $f$, such that $\text{supp} f \subset B(0, a) = \{(r, x) \in \mathbb{R}^2, r^2 + x^2 \leq a^2\}$.
- For $a > 0$, $\mathcal{D}'_{*,a}(\mathbb{R}^2)$ the dual space of $\mathcal{D}_{*,a}(\mathbb{R}^2)$.
- For $a > 0$ and $m \in \mathbb{N}$, $\mathcal{W}_a^m(\mathbb{R}^2)$ the space of function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ of class $C^{2m}$ on $\mathbb{R}^2$, even with respect to the first variable and with support in $B(0, a)$, normed by

$$\mathcal{N}_{\infty, m}(f) = \max_{0 \leq k \leq m} \|\Delta^k_\alpha(f)\|_{\infty, \nu_\alpha}.$$
Lemma 4.4. For all \( m \in \mathbb{N} \), there exists \( \beta \in \mathbb{N} \) sufficiently large, such that the function
\[
(\mu, \lambda) \mapsto g_\beta(\mu, \lambda) = \tilde{\mathcal{F}}_\alpha \left( \frac{1}{(1 + r^2 + x^2)^{\frac{3}{2}}} \right)(\mu, \lambda),
\] (4.5)
is of class \( C^{2m} \) on \( \mathbb{R}^2 \), even with respect to the first variable, and infinitely differentiable on \( \mathbb{R}^2 \setminus \{(0,0)\} \).

Proof. From the relation (2.12), we have for all \((\mu, \lambda) \in \mathbb{R}^2\),
\[
g_\beta(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} j_\alpha(r \mu) \exp(-i \lambda x) \frac{r}{(1 + r^2 + x^2)^{\beta}} \, dr \, dx.
\]
Using the relation (2.2) and the derivative theorem, we can choose \( \beta \) sufficiently large, such that the function \( g_\beta \) is of class \( C^{2m} \) on \( \mathbb{R}^2 \).

On the other hand, from the integral representation (2.2) of the function \( j_\alpha \) and applying the Fubini’s theorem, we get
\[
g_\beta(\mu, \lambda) = \frac{1}{\pi^{2\alpha+\frac{1}{2}}} \frac{1}{\Gamma(\alpha + \frac{1}{2})} \int_{\mathbb{R}} \int_0^{+\infty} \frac{r}{(1 + r^2 + x^2)^{\beta}} \left( \int_0^r (r^2 - t^2)^{\alpha - \frac{1}{2}} \cos(\mu t) \, dt \right) \exp(-i \lambda x) \, dr \, dx.
\]
Using the change of variables \( s = \frac{r^2 - x^2}{1 + r^2 + x^2} \), we obtain
\[
g_\beta(\mu, \lambda) = \frac{\Gamma(\beta - \alpha - \frac{1}{2})}{\pi^{2\alpha+1/2} \Gamma(\beta)} \int_0^{+\infty} \frac{\cos(\mu t) \exp(-i \lambda x)}{(1 + t^2 + x^2)^{\beta - \alpha - 1}} \frac{tdt}{t^{\alpha + \frac{1}{2}}},
\]
again, from the relation (2.2), we deduce that for all \( x \in \mathbb{R} \),
\[
\int_0^{+\infty} j_\alpha(tx) \frac{tx}{(1 + t^2)^{\beta - \alpha - \frac{1}{2}}} \, dt = \int_0^{+\infty} \frac{\cos(sx)}{(1 + s^2)^{\beta - \alpha - 1}} \left( \int_s^{+\infty} \frac{(t^2 - s^2)^{-1/2}}{(1 + t^2)^{\beta - \alpha - 1}} \, dt \right) ds,
\]
\[
= \frac{\sqrt{\pi} \Gamma(\beta - \alpha - 1)}{2} \frac{\Gamma(\beta - \alpha - 1/2)}{\Gamma(\beta - \alpha - 1/2)} \int_0^{+\infty} \frac{\cos(sx)}{(1 + s^2)^{\beta - \alpha - 1}} \, ds,
\]
and therefore for all \((\mu, \lambda) \in \mathbb{R}^2\),
\[
g_\beta(\mu, \lambda) = \frac{\sqrt{\pi} \Gamma(\beta - \alpha - 1)}{2^{\alpha+1/2} \Gamma(\beta)} \int_0^{+\infty} \frac{\cos(s\sqrt{\mu^2 + \lambda^2})}{(1 + s^2)^{\beta - \alpha - 1}} \, ds.
\]
Now, from [10] [29] it follows that for all \((\mu, \lambda) \in \mathbb{R}^2 \setminus \{(0,0)\}\),
\[
g_\beta(\mu, \lambda) = \frac{1}{2^{\beta+1} \Gamma(\beta)} \left( \sqrt{\mu^2 + \lambda^2} \right)^{\beta - \alpha - \frac{1}{2}} K_{\beta - \alpha - \frac{1}{2}} \left( \sqrt{\mu^2 + \lambda^2} \right)
\]
where \( K_{\beta - \alpha - \frac{1}{2}} \) is the Bessel function of second kind and index \( \beta - \alpha - \frac{1}{2} \), called also the Mac-Donald function.

This shows that the function \( g_\beta \) is infinitely differentiable on \( \mathbb{R}^2 \setminus \{(0,0)\} \), even with respect to the first variable.

Proposition 4.5. Let $a > 0$ and $m \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, $n \geq n_0$, it is possible to find $\varphi_n \in \mathcal{D}_{*,a}(\mathbb{R}^2)$ and $\psi_n \in \mathcal{W}_a^m(\mathbb{R}^2)$ satisfying

$$\delta = (I + \Delta_a)^n T_{\varphi_n} + T_{\psi_n}$$

in $\mathcal{S}'(\mathbb{R}^2)$. Where $\delta$ is the Dirac distribution.

Proof. Let $\kappa \in \mathcal{D}_{*,a}(\mathbb{R}^2)$, such that

$$\forall (r, x) \in \mathbb{R}^2, \quad r^2 + x^2 \leq \frac{a^2}{4}, \quad \kappa(r, x) = 1.$$  

From Lemma 4.4, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, the function $g_n$ is of class $C^{2m} \in \mathbb{R}^2$, even with respect to the first variable and infinitely differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Since the transform $\mathcal{F}_\alpha$ defined by relation (2.12), is an isomorphism from $\mathcal{S}'(\mathbb{R}^2)$ onto itself, and for all $\varphi \in \mathcal{S}_*(\mathbb{R}^2)$, $(r, x) \in \mathbb{R}^2$, we have

$$\varphi(r, x) = \int_0^{+\infty} \int_\mathbb{R} \mathcal{F}_\alpha(\varphi)(s, y) j_\alpha(rs) \exp(ixy) d\nu_\alpha(s, y). \quad (4.6)$$

Then, from the relations (4.5) and (4.6), we deduce that for all $\varphi \in \mathcal{S}_*(\mathbb{R}^2)$, we have

$$\left<(I + \Delta_a)^n T_{\varphi_n}, \varphi\right> = \left<T_{\varphi_n}, (I + \Delta_a)^n \varphi\right> = \int_0^{+\infty} \int_\mathbb{R} g_n(r, x)(I + \Delta_a)^n \varphi(r, x) d\nu_\alpha(r, x)$$

$$= \int_0^{+\infty} \int_\mathbb{R} \frac{1}{(1 + r^2 + x^2)^n} \mathcal{F}_\alpha((I + \Delta_a)^n)(\varphi)(r, x) d\nu_\alpha(r, x),$$

$$= \int_0^{+\infty} \int_\mathbb{R} \mathcal{F}_\alpha(\varphi)(r, x) d\nu_\alpha(r, x),$$

$$= \varphi(0, 0).$$

This means that for all $n \geq n_0$: $(I + \Delta_a)^n T_{\varphi_n} = \delta$. Then

$$\kappa(I + \Delta_a)^n T_{\varphi_n} = (I + \Delta_a)^n T_{\varphi_n} = \delta. \quad (4.7)$$

Using the fact that the function $g_n$ is infinitely differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$, even with respect to the first variable, we deduce that the function

$$\varphi_n(r, x) = (\kappa - 1)(I + \Delta_a)^n g_n + (I + \Delta_a)^n (1 - \kappa) g_n, \quad (4.8)$$

belongs to the space $\mathcal{D}_{*,a}(\mathbb{R}^2)$. From the relation (4.7), we have

$$T_{(\kappa - 1)(I + \Delta_a)^n g_n} = (\kappa - 1)(I + \Delta_a)^n T_{g_n} = 0,$$

and this implies, by using the relation (4.8) that

$$T_{\varphi_n} = T_{(I + \Delta_a)^n (I - \kappa) g_n} = (I + \Delta_a)^n (T_{(1 - \kappa) g_n}).$$

Hence,

$$T_{\varphi_n} + (I + \Delta_a)^n T_{\kappa g_n} = (I + \Delta_a)^n T_{g_n} = \delta,$$

and this completes the proof of the proposition if we pick $\psi_n = \kappa g_n$. 

In the following, we will prove that the elements of all bounded subset $\mathcal{B}' \subset \mathcal{D}'_{*,a}(\mathbb{R}^2)$, can be continuously extended to the space $\mathcal{W}_a^m(\mathbb{R}^2)$. For this we define some new families of norms on the space $\mathcal{D}_{*,a}(\mathbb{R}^2)$.

For $f \in \mathcal{D}_{*,a}(\mathbb{R}^2)$, $a > 0$, we denote by
• $P_m(f) = \max_{k_1+k_2 \leq m} \left\| \left( \frac{\partial}{\partial r} \right)^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} f \right\|_{\infty,v_\alpha}$,

• $\bar{P}_m(f) = \max_{k_1+k_2 \leq m} \left\| \ell_{\alpha}^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} f \right\|_{\infty,v_\alpha}$,

• $N_{p,m}(f) = \max_{0 \leq k \leq m} \| \nabla_k^m(f) \|_{p,v_\alpha}$, $p \in [1, +\infty]$.

where $\ell_{\alpha}$ is the Bessel operator defined by

$$\ell_{\alpha} = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r}.$$ 

**Lemma 4.6.** (i) For all $m \in \mathbb{N}$, there exists $C_1 > 0$, such that

$$\forall \varphi \in D_{*,a}(\mathbb{R}^2), \quad P_m(\varphi) \leq C_1 \bar{P}_m(\varphi).$$

(ii) For all $m \in \mathbb{N}$, there exist $C_2 > 0$ and $m' \in \mathbb{N}$, such that

$$\forall \varphi \in D_{*,a}(\mathbb{R}^2), \quad \bar{P}_m(\varphi) \leq C_2 N_{p,m'}(\varphi).$$

**Proof.** (i) Let $\varphi \in D_{*,a}(\mathbb{R}^2)$. By induction on $k_1$, we have

$$\left( \frac{\partial}{\partial r} \right)^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi(r, x) = \sum_{n=0}^{k_1} P_n(r) \left( \frac{\partial}{\partial r^2} \right)^n \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi(r, x), \quad (4.9)$$

where $\frac{\partial}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r}$ and $P_n$ is a real polynomial. Also, by induction, for all $n \geq 1$, we get

$$\left( \frac{\partial}{\partial r^2} \right)^n \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi(r, x) = \int_0^1 \cdots \int_0^t \ell_{\alpha}^{k_2} \varphi(rt_1 \ldots t_n, x) t_1^{2\alpha+1+2(n-1)} \ldots t_n^{2\alpha+1} dt_1 \ldots dt_n, \quad (4.10)$$

from the relations $(4.9)$ and $(4.10)$, it follows that for all $m \in \mathbb{N}$,

$$P_m(\varphi) \leq C_1 \bar{P}_m(\varphi).$$

(ii) Let $p \in [1, +\infty]$, $m \in \mathbb{N}$ and $m_1 \in \mathbb{N}$, such that

$$\left\| \frac{1}{(1 + r^2 + x^2)^{m_1}} \right\|_{1,v_\alpha} < +\infty,$$
then, for all $\varphi \in \mathcal{D}_{*,a}$ and $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 \leq m$, we have
\[
\left\| f_{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi \right\|_{\infty, \nu_a} = \left\| \mathcal{F}_a^{-1} \left( \mathcal{F}_a \left( f_{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi \right) \right) \right\|_{\infty, \nu_a}
\leq \left\| \mathcal{F}_a \left( f_{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi \right) \right\|_{1, \nu_a}
= \mu^{2k_1} \lambda^{k_2} \mathcal{F}_a(\varphi)_{1, \nu_a}
\leq \left\| (1 + \mu^2 + \lambda^2)^m \mathcal{F}_a(\varphi) \right\|_{1, \nu_a}
= \left\| \frac{1}{(1 + \mu^2 + \lambda^2)^m} \mathcal{F}_a \left( (I + \Delta_a)^{m + m_1} \varphi \right) \right\|_{1, \nu_a}
\leq \left\| \frac{1}{(1 + \mu^2 + \lambda^2)^m} \mathcal{F}_a \left( (I + \Delta_a)^{m + m_1} \varphi \right) \right\|_{\infty, \nu_a},
\]
and by Hölder’s inequality, we get
\[
\left\| f_{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} \varphi \right\|_{\infty, \nu_a} \leq \left\| \frac{1}{(1 + \mu^2 + \lambda^2)^m} \left( \nu(B(0, a)) \right)^{\frac{1}{p}} \left( (I + \Delta_a)^{m + m_1} \varphi \right) \right\|_{p, \nu_a},
\leq \left\| \frac{1}{(1 + \mu^2 + \lambda^2)^m} \left( \nu(B(0, a)) \right)^{\frac{1}{p}} 2^{m + m_1} N_{p, m + m_1}(\varphi).\right\|
\]
which implies that
\[
\mathcal{F}_m(\varphi) \leq 2^{m + m_1} \left( \nu(B(0, a)) \right)^{\frac{1}{p}} \left\| \frac{1}{(1 + \mu^2 + \lambda^2)^m} \right\|_{1, \nu_a} N_{p, m + m_1}(\varphi).
\]

\[\square\]

**Theorem 4.7.** Let $a > 0$ and let $B'$ be a weakly bounded set of $\mathcal{D}'_{*,a}(\mathbb{R}^2)$. Then, there exists $m \in \mathbb{N}$, such that the elements of $B'$ can be continuously extended to $\mathcal{W}^m_a(\mathbb{R}^2)$. Moreover, the family of these extensions is equicontinuous.

**Proof.** Let $p \in [1, +\infty[$. Since $B'$ is weakly bounded in $\mathcal{D}'_{*,a}(\mathbb{R}^2)$, then from [27] and Lemma 4.6, there exist a positive constant $C$ and $m \in \mathbb{N}$, such that
\[
\forall T \in B', \forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R}^2), \quad |\langle T, \varphi \rangle| \leq C N_{p, m}(\varphi). \tag{4.11}
\]
We consider the mappings
\[
A : \mathcal{W}^m_a(\mathbb{R}^2) \longrightarrow \left( L^p(d\nu_a) \right)^{m + 1},
\varphi \longmapsto (\Delta^k_a(\varphi))_{0 \leq k \leq m},
\]
and for all $T \in B'$,
\[
\mathcal{E}_T : A(\mathcal{D}_{*,a}(\mathbb{R}^2)) \longrightarrow \mathbb{C}; \quad \langle \mathcal{E}_T, A\varphi \rangle = \langle T, \varphi \rangle.
\]
From the relation (4.11), we deduce that
\[
\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R}^2); \quad |\langle \mathcal{E}_T, A\varphi \rangle| \leq C \| A\varphi \|_{L^p(d\nu_a)^{m + 1}},
\]

this means that $\mathcal{L}_T$ is a continuous functional on the subspace $A(\mathcal{D}_{+,a}(\mathbb{R}^2))$ of the space $\left(\mathcal{L}^p(d\nu_a)\right)^{m+1}$, and that for all $T \in B'$,

$$\|\mathcal{L}_T\|_{A(\mathcal{D}_{+,a}(\mathbb{R}^2))} = \sup_{\|A\| \leq m+1} |\langle \mathcal{L}_T, A\phi \rangle| \leq C$$

From the Hahn Banach theorem, $\mathcal{L}_T$ can be continuously extended to the space $\left(\mathcal{L}^p(d\nu_a)\right)^{m+1}$, denoted again by $\mathcal{L}_T$. Furthermore, for all $T \in B'$

$$\|\mathcal{L}_T\|_{(\mathcal{L}^p(d\nu_a))^{m+1}} = \sup_{\|\psi\| \leq m+1} |\langle \mathcal{L}_T, \psi \rangle| = \|\mathcal{L}_T\|_{A(\mathcal{D}_{+,a}(\mathbb{R}^2))} \leq C. \quad (4.12)$$

Now, from the Riesz theorem, for all $T \in B'$, there exists $(f_{k,T})_{0 \leq k \leq m} \in \mathcal{L}^p(d\nu_a)$, such that for all $\psi = (\psi_0, ..., \psi_m) \in \left(\mathcal{L}^p(d\nu_a)\right)^{m+1}$,

$$\langle \mathcal{L}_T, \psi \rangle = \sum_{k=0}^m \int_0^{+\infty} \int_\mathbb{R} f_{k,T}(r,x)\psi_k(r,x)d\nu_a(r,x);$$

with

$$\|\mathcal{L}_T\|_{(\mathcal{L}^p(d\nu_a))^{m+1}} = \max_{0 \leq k \leq m} \|f_{k,T}\|_{p',\nu_a}.$$ 

Thus, from the relation $(4.12)$ it follows that

$$\forall T \in B', \forall k \in \mathbb{N}, 0 \leq k \leq m; \quad \|f_{k,T}\|_{p',\nu_a} \leq C. \quad (4.13)$$

In particular, for $\phi \in \mathcal{W}^m_{a}(\mathbb{R}^2)$, we have

$$\langle \mathcal{L}_T, A\phi \rangle = \sum_{k=0}^m \int_0^{+\infty} \int_\mathbb{R} f_{k,T}(r,x)\Delta_a^k(\phi)(r,x)d\nu_a(r,x).$$

Using Hölder’s inequality and the relation $(4.13)$, we get for all $T \in B'$ and $\phi \in \mathcal{W}^m_{a}(\mathbb{R}^2)$,

$$|\langle \mathcal{L}_T, A\phi \rangle| \leq C(m+1) \left(\nu_a(B(0,a))\right)^{1/p} \mathcal{N}_{\infty,m}(\phi),$$

this shows that the mapping $\mathcal{L}_T o A$ is a continuous extension of $T$ on $\mathcal{W}^m_{a}(\mathbb{R}^2)$, and that the family $\{\mathcal{L}_T o A\}_{T \in B'}$ is equicontinuous, when applied to $\mathcal{W}^m_{a}(\mathbb{R}^2)$.

In the following, we will give a new characterization of the space $\mathcal{M}^p_{\mathcal{L}}(\mathbb{R}^2)$.

**Theorem 4.8.** Let $T \in \mathcal{S}'(\mathbb{R}^2)$, $p \in [1, +\infty]$, and $p' = \frac{p}{p-1}$. Then $T$ belongs to the space $\mathcal{M}^p_{\mathcal{L}}(\mathbb{R}^2)$ if, and only if for every $\phi \in \mathcal{D}_{+}(\mathbb{R}^2)$, the function $T \ast \phi$ belongs to the space $\mathcal{L}^p(d\nu_a)$, where

$$T \ast \phi(r,x) = \langle T, \tau_{(r,-x)}(\hat{\phi}) \rangle.$$ 

**Proof.** Let $T \in \mathcal{M}^p_{\mathcal{L}}(\mathbb{R}^2)$. From Theorem 1.1 there exist $m \in \mathbb{N}$ and $f_0, ..., f_m \in \mathcal{L}^p(d\nu_a)$, such that

$$T = \sum_{k=0}^m \Delta_a^k(T_{f_k}),$$
in \( \mathcal{M}_p'(\mathbb{R}^2) \). Thus, for every \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \):

\[
T \ast \varphi = \sum_{k=0}^{m} T f_k \ast \Delta^k_n (\varphi) = \sum_{k=0}^{m} f_k \ast \Delta^k_n (\varphi).
\]

Since, for all \( k \in \mathbb{N}, \ 0 \leq k \leq m \), \( f_k \in L^p'(dv_\alpha) \) and \( \Delta^k_n (\varphi) \in L^1(dv_\alpha) \); then from the inequality (2.8), we deduce that \( f_k \ast \Delta^k_n (\varphi) \in L^p'(dv_\alpha) \). This implies that the function \( T \ast \varphi \) belongs to the space \( L^p'(dv_\alpha) \).

- Conversely, let \( T \in \mathcal{D}'(\mathbb{R}^2) \) such that for every \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \) the function \( T \ast \varphi \) belongs to the space \( L^p'(dv_\alpha) \). For \( \varphi, \psi \in \mathcal{D}_* (\mathbb{R}^2) \), we have

\[
\langle T_{T \ast \varphi}, \psi \rangle = \langle T, \varphi \ast \tilde{\psi} \rangle = \langle T, \psi \ast \tilde{\varphi} \rangle = \langle T_{T \ast \psi}, \varphi \rangle.
\]

Thus, from Hölder’s inequality and using the hypothesis, we obtain

\[
|\langle T_{T \ast \varphi}, \psi \rangle| \leq \| T \ast \psi \|_{p',v_\alpha} \| \varphi \|_{p,v_\alpha},
\]

from which, we deduce that the set

\[
B' = \{ T_{T \ast \varphi} : \varphi \in \mathcal{D}_* (\mathbb{R}^2); \| \varphi \|_{p,v_\alpha} \leq 1 \},
\]

is bounded in \( \mathcal{D}'(\mathbb{R}^2) \). Now, using Theorem 4.7, it follows that for all \( a > 0 \) there exists \( m \in \mathbb{N} \), such that for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \); \( \| \varphi \|_{p,v_\alpha} \leq 1 \), the mapping \( T_{T \ast \varphi} \) can be continuously extended to the space \( \mathcal{W}_a^m (\mathbb{R}^2) \) and the family of these extensions is equicontinuous, which means that there exists \( C > 0 \), such that for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \), \( \| \varphi \|_{p,v_\alpha} \leq 1 \), and \( \psi \in \mathcal{W}_a^m (\mathbb{R}^2) \),

\[
|\langle T_{T \ast \varphi}, \psi \rangle| \leq C \mathcal{N}_{\infty,m} (\psi).
\]

This involves that for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \), for all \( \psi \in \mathcal{W}_a^m (\mathbb{R}^2) \),

\[
|\langle T_{T \ast \varphi}, \psi \rangle| \leq C \mathcal{N}_{\infty,m} (\psi) \| \varphi \|_{p,v_\alpha}.
\]

(4.14)

On the other hand, we have for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \), and \( \psi \in \mathcal{W}_a^m (\mathbb{R}^2) \),

\[
\langle T_{T \ast \varphi}, \psi \rangle = \langle T \ast T_{\psi}, \varphi \rangle,
\]

(4.15)

where, for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \),

\[
\langle T \ast T_{\psi}, \varphi \rangle = \langle T, T_{\psi} \ast \varphi \rangle = \langle T, \psi \ast \varphi \rangle.
\]

The relations (4.14) and (4.15) lead to, for all \( \varphi \in \mathcal{D}_* (\mathbb{R}^2) \),

\[
|\langle T \ast T_{\psi}, \varphi \rangle| \leq C \mathcal{N}_{\infty,m} (\psi) \| \varphi \|_{p,v_\alpha}.
\]

This last inequality shows that the functional \( T \ast T_{\psi} \) can be continuously extended to the space \( L^p(dv_\alpha) \) and from Riesz theorem there exists \( g \in L^p'(dv_\alpha) \), such that

\[
T \ast T_{\psi} = T_g
\]

(4.16)

Furthermore, from proposition 4.5 there exist \( n \in \mathbb{N}, \psi_n \in \mathcal{W}_a^m (\mathbb{R}^2) \), and \( \varphi_n \in \mathcal{D}_* (\mathbb{R}^2) \) satisfying

\[
\delta = (I + L)^n T_{\psi_n} + T_{\varphi_n},
\]

then,

\[
T = (I + L)^n (T \ast T_{\psi_n}) + T \ast T_{\varphi_n} = (I + L)^n (T \ast T_{\psi_n}) + T_{T \ast \varphi_n}.
\]

We complete the proof by using the hypothesis, the relation (4.16) and Theorem 4.1.
Theorem 4.9. Let \( p \in [1, \infty) \) and let \( B' \) be a subset of \( M_p'(\mathbb{R}^2) \). The following assertions are equivalent

(i) the subset \( B' \) is weakly bounded in \( M_p'(\mathbb{R}^2) \),
(ii) there exist \( C > 0 \) and \( m \in \mathbb{N} \), such that for every \( T \in B' \), it is possible to find \( f_0, \ldots, f_m \in L^{p'}(dv_{\alpha}) \) satisfying

\[
T = \sum_{k=0}^{m} \Delta^k_\alpha(T f_k) \text{ with } \max_{0 \leq k \leq m} \| f_k \|_{p', \nu_\alpha} \leq C,
\]

(iii) for every \( \varphi \in \mathcal{P}(\mathbb{R}^2) \), the set \( \{ T \ast \varphi, T \in B' \} \) is bounded in \( L^{p'}(dv_{\alpha}) \).

Proof. (1) Suppose that the subset \( B' \) is weakly bounded in \( M_p'(\mathbb{R}^2) \), then from \( \quad B' \) is equicontinuous. There exist \( C > 0 \) and \( m \in \mathbb{N} \), such that

\[
\forall T \in B', \forall \ f \in M_p(\mathbb{R}^2), \quad |\langle T, f \rangle| \leq C \gamma_{m,p}(f). \tag{4.17}
\]

As in the proof of theorem 4.7, we consider the mappings

\[
A : M_p(\mathbb{R}^2) \rightarrow \left( L^p(dv_{\alpha}) \right)^{m+1},
\]

\[
f \mapsto (f, \Delta_\alpha(f), \ldots, \Delta^m_\alpha(f)),
\]

and for all \( T \in B' \),

\[
\mathcal{S}_T : A(M_p(\mathbb{R}^2)) \rightarrow \mathbb{C}; \quad \langle \mathcal{S}_T, A(f) \rangle = \langle T, f \rangle.
\]

Then, the relation (4.17) implies that for all \( \varphi \in M_p(\mathbb{R}^2) \),

\[
|\mathcal{S}_T(A \varphi)| \leq C \| A \varphi \|_{(L^p(dv_{\alpha}))^{m+1}}.
\]

Using Hahn Banach theorem and Riesz theorem, we deduce that \( \mathcal{S}_T \) can be continuously extended to \( \left( L^p(dv_{\alpha}) \right)^{m+1} \), denoted again by \( \mathcal{S}_T \), and that there exists \( (f_k)_{0 \leq k \leq m} \subset L^{p'}(dv_{\alpha}), \quad p' = \frac{p}{p-1} \), verifying for all \( \psi = (\psi_0, \ldots, \psi_m) \in \left( L^p(dv_{\alpha}) \right)^{m+1} \),

\[
\langle \mathcal{S}_T, \psi \rangle = \sum_{k=0}^{m} \int_{0}^{+\infty} \int_{\mathbb{R}} f_k(r, x) \psi_k(r, x) dv_{\alpha}(r, x),
\]

with \( \| \mathcal{S}_T \|_{(L^p(dv_{\alpha}))^{m+1}} = \max_{0 \leq k \leq m} \| f_k \|_{p', \nu_\alpha} \leq C. \)

In particular, if \( \psi = A(f), \quad f \in M_p(\mathbb{R}^2) \),

\[
\langle \mathcal{S}_T, A(f) \rangle = \langle T, f \rangle = \sum_{k=0}^{m} \langle \Delta^k_\alpha(T f_k), f \rangle,
\]

this proves that (i) implies (ii).

(2) Suppose that there exist \( C > 0 \) and \( m \in \mathbb{N} \), such that for every \( T \in B' \) one can find \( f_0, \ldots, f_m \in L^{p'}(dv_{\alpha}) \), satisfying

\[
T = \sum_{k=0}^{m} \Delta^k_\alpha(T f_k), \quad \max_{0 \leq k \leq m} \| f_k \|_{p', \nu_\alpha} \leq C,
\]
then, for all \( f \in \mathcal{M}_p(\mathbb{R}^2) \) and \( T \in B' \),

\[
\langle T, f \rangle = \sum_{k=0}^{m} \int_{0}^{+\infty} \int_{\mathbb{R}} f_k(r, x) g_k(r, x) dv_\alpha(r, x);
\]

consequently,

\[
|\langle T, f \rangle| \leq C(m + 1)\gamma_{m, p}(f),
\]

which means that the set \( B' \) is weakly bounded in \( \mathcal{M}'_p(\mathbb{R}^2) \) and proves that (ii) implies (i).

(3) Suppose that (ii) holds. Let \( \varphi \in \mathcal{D}_*(\mathbb{R}^2) \), then from Theorem 4.8 we know that for all \( T \in B' \) the function \( T \ast \varphi \) belongs to the space \( L^p' (dv_\alpha) \). But

\[
T \ast \varphi = \sum_{k=0}^{m} T f_k \ast \Delta_\alpha^k(\varphi)
\]

thus, for all \( T \in B' \),

\[
\| T \ast \varphi \|_{p', \alpha} \leq C(m + 1)\gamma_{m, p}(\varphi).
\]

This shows that the set \( \{ T \ast \varphi, T \in B' \} \) is bounded in \( L^p' (dv_\alpha) \) and therefore (ii) involves (iii).

(4) Suppose that (iii) holds and let \( T \in B' \). For all \( \varphi, \psi \in \mathcal{D}_*(\mathbb{R}^2) \), we have

\[
|\langle T T_\ast \varphi, \psi \rangle| = |\langle T T_\ast \psi, \varphi \rangle| \leq \| T \ast \psi \|_{p', \alpha} \| \varphi \|_{p, \alpha},
\]

from which, we deduce that the set

\[
\{ T T_\ast \varphi, T \in B', \varphi \in \mathcal{D}_*(\mathbb{R}^2); \| \varphi \|_{p, \alpha} \leq 1 \},
\]

is bounded in \( \mathcal{D}_*'(\mathbb{R}^2) \).

Now, using Theorem 4.7, it follows that for all \( \alpha > 0 \) there exists \( m \in \mathbb{N} \), such that for all \( \varphi \in \mathcal{D}_*(\mathbb{R}^2); \| \varphi \|_{p, \alpha} \leq 1 \), and \( T \in B' \), the mapping \( T T_\ast \varphi \) can be continuously extended on the space \( \mathcal{W}_0^m(\mathbb{R}^2) \) and the family of these extensions is equicontinuous. This means that there exists \( C > 0 \) such that for all \( T \in B' \), \( \varphi \in \mathcal{D}_*(\mathbb{R}^2) \), and \( \psi \in \mathcal{W}_0^m(\mathbb{R}^2) \), the inequality \( 4.14 \) holds. Using the relations \( 4.14 \) and \( 4.15 \), we deduce that the functional \( T T_\psi \) can be continuously extended on the space \( L^p(\sigma dv_\alpha) \) and from Riesz theorem there exists \( g_{T, \psi} \in L^p' (dv_\alpha) \), such that

\[
T T_\psi = T g_{T, \psi}.
\]

Applying again the relations \( 4.14 \) and \( 4.15 \), we deduce that for all \( T \in B' \),

\[
\| g_{T, \psi} \|_{p', \alpha} \leq C \mathcal{N}_{\infty, m}(\psi).
\]

Again by Proposition 4.5, it follows that there exist \( n \in \mathbb{N}, \psi_n \in \mathcal{W}_0^m(\mathbb{R}^2) \) and \( \varphi_n \in \mathcal{D}_*(\mathbb{R}^2) \) verifying for all \( T \in B' \),

\[
T = T \ast \delta = (I + \Delta_\alpha)^n(T \ast T_\psi) + T T_\varphi,
\]

and by the relation \( 4.18 \), we get

\[
T = (I + \Delta_\alpha)^n T g_{T, \psi} + T T_\varphi.
\]

On the other hand, from the hypothesis there exists \( C_1 > 0 \), such that

\[
\forall T \in B', \quad \| T \ast \varphi_n \|_{p', \alpha} \leq C_1,
\]
The relations (4.2), (4.20), (4.21) and (4.22) show that the set $B'$ is bounded in $\mathcal{M}_p'(\mathbb{R}^2)$.\qed

5. Convolution product on the space $\mathcal{M}_p'(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$

In this section, we define and study a convolution product on the space $\mathcal{M}_p'(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$, $1 \leq r \leq p < +\infty$, where $\mathcal{M}_r(\mathbb{R}^2)$ is the closure of the space $\mathcal{S}_r(\mathbb{R}^2)$ in $\mathcal{M}_r(\mathbb{R}^2)$.

Proposition 5.1. Let $p \in [1, +\infty]$. For every $(r, x) \in [0, +\infty) \times \mathbb{R}$, the translation operator $\tau_{(r, x)}$ given by Definition 2.1 (i), is a continuous mapping from $\mathcal{M}_p(\mathbb{R}^2)$ into itself. Moreover, for all $f \in \mathcal{M}_p(\mathbb{R}^2)$ and $k \in \mathbb{N}$, we have

$$\Delta^k_\alpha(\tau_{(r, x)}(f)) = \tau_{(r, x)}(\Delta^k_\alpha(f)), \quad (5.1)$$

where

$$\Delta^k_\alpha(T_f) = T_{\Delta^k_\alpha(f)}.$$ 

Proof. Let $f \in \mathcal{M}_p(\mathbb{R}^2)$. Since for all $(r, x) \in [0, +\infty) \times \mathbb{R}$, the translation operator $\tau_{(r, x)}$ is continuous from $L^p(\nu_\alpha)$ into itself, then the function $\tau_{(r, x)}(f)$ belongs to the space $L^p(\nu_\alpha)$. Moreover, for all $\varphi \in \mathcal{S}_r(\mathbb{R}^2)$ and $k \in \mathbb{N}$; we have

$$\langle \Delta^k_\alpha(T_{\tau_{(r, x)}(f)}), \varphi \rangle = \langle T_{\tau_{(r, x)}(f)}, \Delta^k_\alpha(\varphi) \rangle$$

$$= \int_0^{+\infty} \int_\mathbb{R} f(s, y) \tau_{(r, -x)}(\varphi)(s, y) dv_\alpha(s, y)$$

$$= \int_0^{+\infty} \int_\mathbb{R} \tau_{(r, -x)}(\alpha)(s, y) dv_\alpha(s, y)$$

$$= \langle T_f, \Delta^k_\alpha(\tau_{(r, -x)}(\varphi)) \rangle$$

$$= \langle \Delta^k_\alpha(T_f), \tau_{(r, -x)}(\varphi) \rangle$$

$$= \langle T_{\Delta^k_\alpha(f)}, \tau_{(r, -x)}(\varphi) \rangle$$

$$= \langle T_{\tau_{(r, x)}(\Delta^k_\alpha(f))}, \varphi \rangle.$$

Since the operator $\tau_{(r, x)}$ is continuous from $L^p(\nu_\alpha)$ into itself, we deduce that for all $f \in \mathcal{M}_p(\mathbb{R}^2)$ and $(r, x) \in [0, +\infty) \times \mathbb{R}$, the function $\tau_{(r, x)}(f)$ belongs to the space $\mathcal{M}_p(\mathbb{R}^2)$ and that for all $k \in \mathbb{N}$, $\Delta^k_\alpha(\tau_{(r, x)}(f)) = \tau_{(r, x)}(\Delta^k_\alpha(f))$. Moreover, from the relations (2.7) and (5.1), we have

$$\gamma_{m, p}(\tau_{(r, x)}(f)) = \max_{0 \leq k \leq m} \|\tau_{(r, x)}(\Delta^k_\alpha(f))\|_{p, \nu_\alpha} \leq \max_{0 \leq k \leq m} \|\Delta^k_\alpha(f)\|_{p, \nu_\alpha} = \gamma_{m, p}(f),$$

which shows that the operator $\tau_{(r, x)}$ is continuous from $\mathcal{M}_p(\mathbb{R}^2)$ into itself. \qed

The precedent proposition allows us to define the coming convolution product

Definition 5.2. The convolution product of $T \in \mathcal{M}_p'(\mathbb{R}^2)$ and $f \in \mathcal{M}_p(\mathbb{R}^2)$ is defined by

$$\forall (r, x) \in [0, +\infty) \times \mathbb{R}; \quad T * f(r, x) = \langle T, \tau_{(r, -x)}(f) \rangle.$$
Let $T \in \mathcal{M}'_p(\mathbb{R}^2)$ and $\varphi \in \mathcal{M}_p(\mathbb{R}^2)$. From Theorem 4.1, there exist $m \in \mathbb{N}$ and 
$\{f_0, \ldots, f_m\} \subset L^{p'}(dv_\alpha)$, such that 
$$T = \sum_{k=0}^{m} \Delta^k_\alpha(T_{f_k}).$$
Thus, 
$$T * \varphi(r, x) = \sum_{k=0}^{m} \langle \Delta^k_\alpha(T_{f_k}), r(r,-x)(\varphi) \rangle$$
$$= \sum_{k=0}^{m} \langle T_{f_k}, r(r,-x)(\Delta^k_\alpha(\varphi)) \rangle$$
$$= \sum_{k=0}^{m} \langle f_k * \Delta^k_\alpha(\varphi), (r, x) \rangle$$
Using the relation (2.8) and the fact that $\varphi \in \mathcal{M}_p(\mathbb{R}^2)$ we deduce that the function $T * \varphi$ belongs to $L^\infty(dv_\alpha)$ and 
$$\|T * \varphi\|_{\infty,v_\alpha} \leq \gamma_{m,p}(\varphi) \left( \sum_{k=0}^{m} \|f_k\|_{p',v_\alpha} \right)$$
(5.2)
Let $T \in \mathcal{M}'_p(\mathbb{R}^2)$, $T = \sum_{k=0}^{m} \Delta^k_\alpha(T_{f_k})$ with $\{f_k\}_{0 \leq k \leq m} \subset L^{p'}(dv_\alpha)$ and $\phi \in \mathcal{M}_r(\mathbb{R}^2)$, $1 \leq r \leq p$. From the inequality (2.8), it follows that for $0 \leq k \leq m$ the function $f_k * \Delta^k_\alpha(\phi)$ belongs to the space $L^q(dv_\alpha)$ with $1/q = 1/r + 1/p' - 1 = 1/r - 1/p$ and by using the density of $\mathcal{S}_r(\mathbb{R}^2)$ in $\mathcal{M}_r(\mathbb{R}^2)$, we deduce that the expression 
$$\sum_{k=0}^{m} f_k * \Delta^k_\alpha(\phi)$$
is independent of the sequence $\{f_k\}_{0 \leq k \leq m}$. Then, we put 
$$T * \phi = \sum_{k=0}^{m} f_k * \Delta^k_\alpha(\phi).$$
Again, from the relation (2.8), we deduce that the function $T * \phi$ belongs to the space $L^q(dv_\alpha)$ and 
$$\|T * \phi\|_{q,v_\alpha} \leq \gamma_{m,r}(\phi) \left( \sum_{k=0}^{m} \|f_k\|_{p',v_\alpha} \right)$$
(5.3)
This allows us to say that 
$$\mathcal{M}'_p(\mathbb{R}^2) * \mathcal{M}_r(\mathbb{R}^2) \subset L^q(dv_\alpha).$$

Lemma 5.3. Let $1 \leq r \leq p < \infty$. Then 

i) The operator $\Delta_\alpha$ is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into itself. 

ii) For all $T \in \mathcal{M}'_p(\mathbb{R}^2)$ and $\phi \in \mathcal{M}_r(\mathbb{R}^2)$, the function $T * \phi$ belongs to the space $\mathcal{M}_q(\mathbb{R}^2)$ and we have 
$$\forall k \in \mathbb{N}, \quad \Delta^k_\alpha(T * \phi) = T * \Delta^k_\alpha(\phi).$$

Proof. i) Let $f \in \mathcal{M}_r(\mathbb{R}^2)$. There exists $(f_k)_k \subset \mathcal{S}_r(\mathbb{R}^2)$ such that 
$$\forall m \in \mathbb{N}, \quad \lim_{k \to +\infty} \gamma_{m,r}(f_k - f) = 0.$$
However, 
\[ \gamma_{m,r}(\Delta_n(f_k) - \Delta_n(f)) \leq \gamma_{m+1,r}(f_k - f), \]
thus, the sequence \((\Delta_n(f_k))\) of \(\mathcal{K}^r(\mathbb{R}^2)\) converges to \(\Delta_n(f)\) in \(\mathcal{M}_r(\mathbb{R}^2)\), which shows that the function \(\Delta_n(f)\) belongs to the space \(\mathcal{M}_r(\mathbb{R}^2)\).

ii) If \(\phi \in \mathcal{K}^r(\mathbb{R}^2)\), then the function \(T \ast \phi\) is infinitely differentiable, and we have 
\[ \Delta_n^k(T \ast \phi) = T \Delta_n^k(\phi), \]
therefore, the result follows from the density of \(\mathcal{K}^r(\mathbb{R}^2)\) in \(\mathcal{M}_r(\mathbb{R}^2)\), the relation (5.3) and the fact that the operator \(\Delta_n\) is continuous from \(\mathcal{M}_r(\mathbb{R}^2)\) into itself. □

**Proposition 5.4.** Let \(1 \leq r \leq p < \infty\) and \(q \in [1, +\infty]\), such that
\[ \frac{1}{q} = \frac{1}{r} - \frac{1}{p}. \] (5.4)
Then for every \(T \in \mathcal{M}'_p(\mathbb{R}^2)\), the mapping
\[ \phi \rightarrow T \ast \phi \]
is continuous from \(\mathcal{M}_r(\mathbb{R}^2)\) into \(\mathcal{M}_q(\mathbb{R}^2)\).

**Proof.** Let \(T \in \mathcal{M}'_p(\mathbb{R}^2); T = \sum_{k=0}^{m} \Delta_n^k(T_{f_k})\) and \(\phi \in \mathcal{M}_r(\mathbb{R}^2)\). From Lemma 5.3 the function \(T \ast \phi\) belongs to the space \(\mathcal{M}_q(\mathbb{R}^2)\) and for all \(l \in \mathbb{N}\)
\[ \gamma_{l,q}(T \ast \phi) = \max_{0 \leq k \leq l} \|\Delta_n^k(T \ast \phi)\|_{q,v_\alpha} = \max_{0 \leq k \leq l} \|T \ast \Delta_n^k(\phi)\|_{q,v_\alpha}. \]
According to the relation (5.3), it follows that
\[ \gamma_{l,q}(T \ast \phi) \leq \left( \sum_{k=0}^{m} \|f_k\|_{p',v_\alpha} \right) \max_{0 \leq k \leq l} \gamma_{m,r}(\Delta_n^k(\phi)) \]
\[ \leq \left( \sum_{k=0}^{m} \|f_k\|_{p',v_\alpha} \right) \gamma_{m+1,r}(\phi). \]

**Definition 5.5.** Let \(1 \leq p, q, r < +\infty\), such that (5.4) holds. The convolution product of \(T \in \mathcal{M}'_p(\mathbb{R}^2)\) and \(S \in \mathcal{M}'_q(\mathbb{R}^2)\) is defined for all \(\phi \in \mathcal{M}_r(\mathbb{R}^2)\), by
\[ \langle S \ast T, \phi \rangle = \langle S, T \ast \phi \rangle. \]

From this Definition and Proposition 5.4 we deduce the following result

**Proposition 5.6.** Let \(1 \leq p, q, r < +\infty\) such that (5.4) holds. Then, for all \(T \in \mathcal{M}'_p(\mathbb{R}^2)\) and \(S \in \mathcal{M}'_q(\mathbb{R}^2)\), the functional \(S \ast T\) is continuous on \(\mathcal{M}_r(\mathbb{R}^2)\).

**References**


LAKHDAR TANNECH RACHDI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF TUNIS, 2092 ELMANAR H, TUNISIA
E-mail address: lakhdartannech.rachdi@fst.rnu.tn
Cyrine Baccar
Département de mathématiques appliquées, ISI, 2, rue Abourraihan Al Bayrouni 2080,
Ariana, Tunis,
E-mail address: cyrine.baccar@isi.rnu.tn