On the Level Spaces of Fuzzy Topological Spaces

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Abstract

It is known that if \((X, T)\) is a fuzzy topological space and \(0 \leq \alpha < 1\) then the family \(T_\alpha = \{\alpha(G) : G \in T\}\) where \(\alpha(G) = \{x \in X : G(x) > \alpha\}\), forms a topology on \(X\). In the present paper some level properties have been modified and it is proved that a fuzzy topological space \((X, T)\) is \(\alpha\)-compact (resp. \(\alpha\)-Hausdorff, countably \(\alpha\)-compact, \(\alpha\)-Lindelöf, \(\alpha\)-connected) if and only if the corresponding \(\alpha\)-level topological space \((X, T_\alpha)\) is compact (resp. Hausdorff, countably compact, Lindelöf, connected, locally compact). Some basic properties of \(\alpha\)-level sets have also been obtained.

1 Introduction

The investigation of fuzzy topological spaces by considering the properties which a space may have to a certain degree or level was initiated by Gantner et. al [3]. This approach resulted into the investigation of \(\alpha\)-Hausdorff axiom [10], countable \(\alpha\)-compactness, \(\alpha\)-Lindelöf property [6], local \(\alpha\)-compactness [7], \(\alpha\)-closure [4] etc. in fuzzy topological spaces.

Throughout this paper Chang’s [1] definition of fuzzy topological space (abbreviated as fts) is used. If \(X\) is a set and \(T\) is a family of fuzzy subsets of \(X\) satisfying the following conditions (i) to (iii) then \(T\) is called a fuzzy topology on \(X\); (i) \(X, \phi \in T\) (ii) arbitrary union of members of \(T\) is again a member of \(T\) and (iii) intersection of finitely many members of \(T\) is again a member of \(T\). Further \((X, T)\) is called a fuzzy topological space (fts). If \((X, T)\) is a fts and \(0 \leq \alpha < 1\) then the family \(T_\alpha = \{\alpha(G) : G \in T\}\), of all subsets of \(X\) of the form \(\alpha(G) = \{x \in X : G(x) > \alpha\}\) called \(\alpha\)-level sets, forms a topology on \(X\) [4] and is called the \(\alpha\)-level topology on \(X\).

In this paper, some basic properties of \(\alpha\)-level sets have been obtained. The \(\alpha\)-Hausdorff axiom [10] and the local \(\alpha\)-compactness of [7] have been modified. The \(\alpha\)-connectedness has been proposed. It is proved that a fts \((X, T)\) is \(\alpha\)-compact (\(\alpha\)-Hausdorff, countably \(\alpha\)-compact, \(\alpha\)-Lindelöf, \(\alpha\)-connected, locally \(\alpha\)-compact) if and only if the corresponding \(\alpha\)-level topological space \((X, T_\alpha)\) is compact (resp. Hausdorff, countably compact, Lindelöf, connected, locally compact).

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2 \( \alpha \)-Level Sets and Their Basic Properties

If \( G \) is any fuzzy set in a set \( X \) and \( 0 \leq \alpha < 1 \) \((0 < \alpha \leq 1)\) then \( \alpha(G) = \{ x \in X : G(x) > \alpha \} \) \((\text{resp. } \alpha^*(G) = \{ x \in X : G(x) \geq \alpha \})\) is called an \( \alpha \)-level \((\text{resp. } \alpha^*\)-level\) set in \( X \).

The term crisp subset refers to an ordinary subset which is identified with its characteristic function as a fuzzy subset.

If \( f : X \to Y \) is a function and \( A \) is a fuzzy subset of \( X \) then \( f(A) \) is a fuzzy subset of \( Y \) defined by \( f(A)(y) = \sup \{ A(x) : x \in f^{-1}(y) \} \) for each \( y \in Y \). Further, if \( B \) is a fuzzy subset of \( Y \) then \( f^{-1}(B) \) is a fuzzy subset of \( X \) defined by \( f^{-1}(B)(x) = B(f(x)) \) for each \( x \in X \).

Some basic properties of \( \alpha \)-level sets are given in the following.

**Theorem 2.1** Let \( X, Y \) be any two sets and \( 0 \leq \alpha < 1 \). The following statements are true.

1. If \( G \) is any fuzzy set in \( X \) then \( G(x) \leq \alpha(G)(x) \) holds for all \( x \in X \) with \( G(x) > \alpha \).
2. If \( G \subseteq H \) then \( \alpha(G) \subseteq \alpha(H) \) for any two fuzzy sets \( G, H \) in \( X \).
3. \( \alpha(G) = G \) if and only if \( G \) is a crisp subset of \( X \).
4. \( \alpha(\alpha(G)) = \alpha(G) \) for any fuzzy set \( G \) in \( X \).
5. \( \alpha(\bigvee_{\lambda} G_{\lambda}) = \bigcup_{\lambda} \alpha(G_{\lambda}) \) for any family \( \{ G_{\lambda} : \lambda \in \Lambda \} \) of fuzzy sets in \( X \).
6. \( \alpha(\bigwedge_{\lambda} G_{\lambda}) = \bigcap_{\lambda} \alpha(G_{\lambda}) \) for any family \( \{ G_{\lambda} : \lambda \in \Lambda \} \) of fuzzy sets in \( X \).
7. If \( f : X \to Y \), then \( f(\alpha(G)) = \alpha(f(G)) \) for each fuzzy set \( G \) in \( X \).
8. If \( f : X \to Y \), then \( f^{-1}(\alpha(G)) = \alpha(f^{-1}(G)) \) for each fuzzy set \( G \) in \( Y \).
9. \( \alpha(G \times H) = \alpha(G) \times \alpha(H) \) for any two fuzzy sets \( G, H \) in \( X \) where \( G \times H \) is a fuzzy set in \( X \times Y \) given by \( (G \times H)(x, y) = G(x) \land H(y) \) for each \( (x, y) \in X \times Y \).

**Proof.** (1). Let \( x \in X \) with \( G(x) > \alpha \). Then \( x \in \alpha(G) \) so that \( (\alpha(G))(x) = 1 \geq G(x) > \alpha \) and therefore \( G(x) \leq (\alpha(G))(x) \).

(2) If \( x \in \alpha(G) \) then \( G(x) > \alpha \) and therefore \( H(x) \geq G(x) > \alpha \). Consequently \( x \in \alpha(H) \).

(3) If \( G \) is crisp and if \( x \in X \) then \( G(x) = 0 \) or \( 1 \). If \( G(x) = 0 \) then \( x \notin \alpha(G) \) and therefore \( (\alpha(G))(x) = 0 \) which proves \( G(x) = \alpha(G)(x) \). In case if \( G(x) = 1 \), then \( G(x) = 1 > \alpha \) and therefore \( x \in \alpha(G) \) which proves \( (\alpha(G))(x) = 1 = G(x) \). The converse part follows as \( \alpha(G) \) is crisp.
(4) Follows from (3) as $\alpha(G)$ is crisp.

(5) If $x \in \alpha(\bigvee_{\lambda} G_{\lambda})$ then $\text{Sup}(G_{\lambda}(x)) > \alpha$. Consequently there exists a $\lambda_{0}$ such that $G_{\lambda_{0}}(x) > \alpha$ which implies $x \in \alpha(G_{\lambda_{0}})$ and hence $x \in \bigcup_{\lambda} \alpha(G_{\lambda})$. Therefore $\alpha(\bigvee_{\lambda} G_{\lambda}) \subset \bigcup_{\lambda} \alpha(G_{\lambda})$. Similarly $\bigcup_{\lambda} \alpha(G_{\lambda}) \subset \alpha(\bigvee_{\lambda} G_{\lambda})$ and hence the equality.

(6) If $x \in \alpha(\bigwedge_{\lambda} G_{\lambda})$ then $(\bigwedge_{\lambda} G_{\lambda})(x) > \alpha$ and therefore $G_{\lambda}(x) > \alpha$ for each $\lambda$. This implies that $x \in \alpha(G_{\lambda})$ for each $\lambda$ and therefore $x \in \bigwedge_{\lambda} \alpha(G_{\lambda})$. Thus $\alpha(\bigwedge_{\lambda} G_{\lambda}) \subset \bigcap_{\lambda} \alpha(G_{\lambda})$. Similarly $\bigcap_{\lambda} \alpha(G_{\lambda}) \subset \alpha(\bigwedge_{\lambda} G_{\lambda})$.

(7) If $y \in f(\alpha(G))$ then there is an element $x \in \alpha(G)$ such that $y = f(x)$. Now $G(x) > \alpha$ and therefore $\text{Sup}\{G(x) : x \in f^{-1}(y)\} > \alpha$ which implies $(f(G))(y) > \alpha$. Then $y \in \alpha(f(G))$. Thus $f(\alpha(G)) \subset \alpha(f(G))$. Similarly it can be shown that $\alpha(f(G)) \subset f(\alpha(G))$ and hence the result follows.

(8) Let $x \in f^{-1}(\alpha(G))$. Then $f(x) = y \in \alpha(G)$ so that $G(y) = G(f(x)) > \alpha$. Therefore $[f^{-1}(G)](x) > \alpha$ which implies $x \in \alpha[f^{-1}(G)]$ and hence it follows that $f^{-1}(\alpha(G)) \subset \alpha(f^{-1}(G))$. Similarly $\alpha(f^{-1}(G)) \subset f^{-1}(\alpha(G))$ and hence the equality.

(9) If $(x, y) \in \alpha(G \times H)$ then $(G \times H)(x, y) > \alpha$ and therefore $x \in \alpha(G)$ and $y \in \alpha(H)$. So $(x, y) \in \alpha(G) \times \alpha(H)$. Thus $\alpha(G \times H) \subset \alpha(G) \times \alpha(H)$. Similarly it can be shown that $\alpha(G) \times \alpha(H) \subset \alpha(G \times H)$ and hence the equality follows.

3 Level Spaces and Main Results

In the beginning of this section we deal with Rodabaugh’s [10] $\alpha$-Hausdorff fts.

**Definition 3.1** Let $0 \leq \alpha < 1$ $(0 < \alpha \leq 1)$. A fts $(X, T)$ is said to be $\alpha$-Hausdorff (resp. $\alpha^{*}$-Hausdorff) if for each $x, y$ in $X$ with $x \neq y$, there exist $G, H$ in $T$ such that $G(x) > \alpha$ (resp. $G(x) \geq \alpha$), $H(y) > \alpha$ (resp. $H(y) \geq \alpha$) and $G \wedge H = 0$.

We have the following

**Theorem 3.2** Let $0 \leq \alpha < 1$. If a fts $(X, T)$ is $\alpha$-Hausdorff, then $(X, T_{\alpha})$ is Hausdorff topological space.
**Proof.** Let \( x, y \in X \) with \( x \neq y \). Then there are \( G, H \) in \( T \) such that \( G(x) > \alpha \), \( H(y) > \alpha \) and \( G \cap H = 0 \). Then \( \alpha(G) \) and \( \alpha(H) \) are open sets in \((X, T_\alpha)\) and \( x \in \alpha(G), y \in \alpha(H) \). Also \( \alpha(G) \cap \alpha(H) = \phi \) since \( G \cap H = 0 \). Hence \((X, T_\alpha)\) is Hausdorff topological space.

The converse of the above theorem holds for the case of \( \alpha = 0 \), which is given in the following.

**Theorem 3.3** Let \((X, T)\) be a fts. If \((X, T_0)\) is Hausdorff topological space, then \((X, T)\) is 0-Hausdorff fts.

**Proof.** Let \( x, y \in X \) with \( x \neq y \). Then there are open sets \( U, V \) in \((X, T_0)\) such that \( x \in U, y \in V \) and \( U \cap V = \phi \). Let \( U = 0(G), V = 0(H) \) for some \( G, H \) in \( T \). Then it follows that \( G(x) > 0 \) and \( H(y) > 0 \). Further \( G \cap H = 0 \) as \( U \cap V = \phi \). Hence \((X, T)\) is 0-Hausdorff.

**Definition 3.4** Let \( X \) be a set and \( 0 \leq \alpha < 1 \) (\( 0 < \alpha \leq 1 \)). A family \( \{ G_\lambda \}_{\lambda} \) of fuzzy sets in \( X \) is said to be \( \alpha \)-disjoint (resp. \( \alpha^* \)-disjoint) if \( \bigwedge_\lambda G_\lambda \leq \alpha \) (resp. \( \bigwedge_\lambda G_\lambda < \alpha \)).

It is evident that two fuzzy sets \( G, H \) in \( X \) are \( \alpha \)-disjoint (\( \alpha^* \)-disjoint) if and only if for each \( x \) in \( X \) either \( G(x) \leq \alpha \) (resp. \( G(x) < \alpha \)) or \( H(x) \leq \alpha \) (resp. \( H(x) < \alpha \)).

Rodabaugh’s definition is suitably modified in the following.

**Definition 3.5** Let \( 0 \leq \alpha < 1 \) (\( 0 < \alpha \leq 1 \)). A fts \((X, T)\) is said to be \( \alpha \)-Hausdorff (resp. \( \alpha^* \)-Hausdorff) if for each \( x, y \) in \( X \) with \( x \neq y \), there exist \( G, H \) in \( T \) such that \( G(x) > \alpha \) (resp. \( G(x) \geq \alpha \)), \( H(y) > \alpha \) (resp. \( H(y) \geq \alpha \)) and \( G, H \) are \( \alpha \)-disjoint (resp. \( \alpha^* \)-disjoint).

For the modified class of \( \alpha \)-Hausdorff fuzzy topological spaces we have the following.

**Theorem 3.6** Let \( 0 \leq \alpha < 1 \). A fts \((X, T)\) is a \( \alpha \)-Hausdorff if and only if \((X, T_\alpha)\) is Hausdorff topological space.

**Proof.** Let \((X, T)\) be \( \alpha \)-Hausdorff. Let \( x, y \in X \) with \( x \neq y \). Then there exist \( G, H \) in \( T \) with \( G(x) > \alpha \), \( H(y) > \alpha \) and \( G \cap H \leq \alpha \). Then \( \alpha(G), \alpha(H) \) are open sets in \((X, T_\alpha)\) such that \( x \in \alpha(G), y \in \alpha(H) \) and \( \alpha(G) \cap \alpha(H) = \alpha(G \cap H) = \{ x \in X : (G \cap H)(x) > \alpha \} = \phi \) as \( G \cap H \leq \alpha \). Therefore \((X, T_\alpha)\) is \( \alpha \)-Hausdorff.

Conversely, suppose \((X, T_\alpha)\) is \( \alpha \)-Hausdorff. Let \( x, y \in X \) with \( x \neq y \). Then there exist open sets \( U, V \) in \((X, T_\alpha)\) such that \( x \in U, y \in V \) and \( U \cap V = \phi \). Let \( U = \alpha(G) \) and \( V = \alpha(H) \) for some \( G, H \in T \). Then \( x \in \alpha(G) \) and \( y \in \alpha(H) \). Therefore \( G(x) > \alpha \) and \( H(y) > \alpha \). Further \( G \cap H \leq \alpha \) as \( U \cap V = \phi \). Hence \((X, T)\) is \( \alpha \)-Hausdorff.
Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A family $\{G_{\lambda} : \lambda \in \Lambda\}$ of fuzzy subsets of a fts $(X, T)$ is said to be an $\alpha$-shading ($\alpha^*$-shading) of X if for each $x \in X$, there exists a $G_{\lambda_x}$ in $\{G_{\lambda} : \lambda \in \Lambda\}$ such that $G_{\lambda_x}(x) > \alpha$ ($\geq \alpha$).

The following definition is due to Gantner et. al [3].

**Definition 3.7** Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-compact (resp. $\alpha^*$-compact) if each $\alpha$-shading (resp. $\alpha^*$-shading) of $X$ by open fuzzy sets has a finite $\alpha$-subshading (resp. $\alpha^*$-subshading).

We have the following

**Theorem 3.8** Let $0 \leq \alpha < 1$. A fts $(X, T)$ is $\alpha$-compact if and only if $(X, T_{\alpha})$ is compact topological space.

**Proof.** Let $(X, T)$ be $\alpha$-compact. Let $U = \{U_{\lambda} : \lambda \in \Lambda\}$ be an open cover of $(X, T_{\alpha})$. Then, since for each $U_{\lambda}$, there exists a $G_{\lambda}$ in $T$ such that $U_{\lambda} = \alpha(G_{\lambda})$, we have $U = \{\alpha(G_{\lambda}) : \lambda \in \Lambda\}$. Then the family $V = \{G_{\lambda} : \lambda \in \Lambda\}$ is an $\alpha$-shading of $(X, T)$. To see this, let $x \in X$. Since $U$ is an open cover of $(X, T_{\alpha})$, there is an $U_{\lambda_x} \in U$ such that $x \in U_{\lambda_x}$. But $U_{\lambda_x} = \alpha(G_{\lambda_x})$, for some $G_{\lambda_x} \in T$. Therefore $x \in \alpha(G_{\lambda_x})$ which implies that $G_{\lambda_x}(x) > \alpha$. By $\alpha$-compactness of $(X, T)$, $V$ has a finite $\alpha$-subshading say $\{G_{\lambda_i}\}_{i=1}^k$. Then $\{\alpha(G_{\lambda_i})\}_{i=1}^k$ forms a finite subcover of $U$ and thus $(X, T_{\alpha})$ is compact.

Conversely, let $(X, T_{\alpha})$ be compact and $U = \{G_{\lambda} : \lambda \in \Lambda\}$ be an open $\alpha$-shading of $(X, T)$. Then the family $V = \{\alpha(G_{\lambda}) : \lambda \in \Lambda\}$ is an open cover of $(X, T_{\alpha})$. For, let $x \in X$. Then there exists a $G_{\lambda_x}$ in $U$ such that $G_{\lambda_x}(x) > \alpha$. Therefore $x \in \alpha(G_{\lambda_x})$ and $(G_{\lambda_x}) \in V$. By compactness of $(X, T_{\alpha})$, $V$ has a finite subcover say $\{\alpha(G_{\lambda_i})\}_{i=1}^n$. Then the family $\{G_{\lambda_i}\}_{i=1}^n$ forms a finite $\alpha$-subshading of $U$ and hence $(X, T)$ is $\alpha$-compact.

Countable compact fts have been studied in [6, 11-13].

**Definition 3.9** Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X, T)$ is said to be countably $\alpha$-compact (resp. countably $\alpha^*$-compact) if every countable open $\alpha$-shading (resp. countable open $\alpha^*$-shading) of $X$ has a finite $\alpha$-subshading (resp. finite $\alpha^*$-subshading).

It is easy to verify the following

**Theorem 3.10** Let $0 \leq \alpha < 1$. A fts $(X, T)$ is countably $\alpha$-compact if and only if $(X, T_{\alpha})$ is countably compact topological space.

Lindelöf fuzzy topological spaces were studied in [6, 8, and 12]. Lindelöf fuzzy topological spaces, using shading families, were introduced in [16].

**Definition 3.11** Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-Lindelöf (resp. $\alpha^*$-Lindelöf) if and only if every open $\alpha$-shading (resp. open $\alpha^*$-shading) of $X$ has a countable $\alpha$-subshading (resp. countable $\alpha^*$-subshading).

Again it is easy to verify the following.
Theorem 3.12 Let $0 \leq \alpha < 1$. A fts $(X, T)$ is $\alpha$-Lindelöf if and only if $(X, T_{o})$ is Lindelöf topological space.

Definition 3.13 Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). Let $X$ be a non-empty set. A fuzzy set $A$ in $X$ is said to be an empty fuzzy set of order $\alpha$ (resp. order $\alpha^{*}$) if $A(x) \leq \alpha$ (resp. $A(x) < \alpha$) for each $x \in X$.

A fuzzy set $A$ in $X$ is said to be non-empty of order $\alpha$ (resp. order $\alpha^{*}$) if there exists $x_{o} \in X$ such that $A(x_{o}) > \alpha$ (resp. $A(x_{o}) \geq \alpha$).

Connectedness in fuzzy topological spaces was studied in [5, 9].

Definition 3.14 Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-disconnected (resp. $\alpha^{*}$-disconnected) if there exists an $\alpha$-shading (resp. $\alpha^{*}$-shading) family of two open fuzzy sets in $X$ which are non-empty of order $\alpha$ (resp. order $\alpha^{*}$) and $\alpha$-disjoint (resp. $\alpha^{*}$-disjoint).

Definition 3.15 Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-connected (resp. $\alpha^{*}$-connected) if there does not exist an $\alpha$-shading (resp. $\alpha^{*}$-shading) family of two open fuzzy sets in $X$ which are non-empty of order $\alpha$ (resp. order $\alpha^{*}$) and $\alpha$-disjoint (resp. $\alpha^{*}$-disjoint).

We now prove the following

Theorem 3.16 Let $0 \leq \alpha < 1$. A fts $(X, T)$ is $\alpha$-connected if and only if $(X, T_{o})$ is connected topological space.

Proof. Let $(X, T)$ be $\alpha$-connected. Suppose $(X, T_{o})$ is disconnected. Then there exist non-empty disjoint open sets $U, V$ in $(X, T_{o})$ such that $U \cup V = X$. Let $U = \alpha(G), V = \alpha(H)$ for some $G, H \in T$. Since $U, V$ are non-empty sets it follows that $G$ and $H$ are non-empty fuzzy sets of order $\alpha$. Further $\{G, H\}$ is an $\alpha$-shading of $X$. For if $x \in X$ then $x \in U$ or $x \in V$ and therefore $x \in \alpha(G)$ or $x \in \alpha(H)$ which implies that $G(x) > \alpha$ or $H(x) > \alpha$. Also $G, H$ are $\alpha$-disjoint: For, $U \cap V = \phi$ implies that $\alpha(G) \cap \alpha(H) = \phi$. Therefore $\alpha(G \cap H) = \phi$. That is $\{x \in X : (G \cap H)(x) > \alpha\} = \phi$. Therefore for each $x \in X, (G \cap H)(x) \leq \alpha$ and so $G, H$ are $\alpha$-disjoint. Thus it follows that $\{G, H\}$ is an $\alpha$-shading of open fuzzy sets which are non-empty of order $\alpha$ and are $\alpha$-disjoint. Therefore $(X, T)$ is $\alpha$-disconnected, which contradicts the hypothesis. Hence $(X, T_{o})$ is connected topological space.

Conversely, suppose $(X, T_{o})$ is connected. Let $(X, T)$ be $\alpha$-disconnected. Then there exist an $\alpha$-shading $\{G, H\}$ of two open fuzzy sets in $X$ which are non-empty of order $\alpha$ and $\alpha$-disjoint. Clearly $\alpha(G), \alpha(H)$ are open sets in $(X, T_{o})$. Further $\alpha(G), \alpha(H)$ are non-empty as $G, H$ are non-empty of order $\alpha$. Also $\alpha(G) \cap \alpha(H) = \alpha(G \cap H) = \{x \in X : (G \cap H)(x) > \alpha\} = \phi$ since $(G \cap H)(x) \leq \alpha$ as $G, H$ are $\alpha$-disjoint. Finally $\alpha(G) \cup \alpha(H) = X$: For if $x \in X$ then either $G(x) > \alpha$ or $H(x) > \alpha$ as $\{G, H\}$ is an $\alpha$-shading of $X$. Therefore $x \in \alpha(G)$ or $x \in \alpha(H)$ and therefore $x \in \alpha(G) \cup \alpha(H)$. Thus $X \subset \alpha(G) \cup \alpha(H)$. Also $\alpha(G) \cup \alpha(H) \subset X$ is obvious. Therefore $\alpha(G) \cup \alpha(H) = X$. Hence it
follows that $X$ is the union of two non-empty disjoint open sets in $(X,T_{a})$ and therefore $(X,T_{a})$ is disconnected, which contradicts the hypothesis. Hence $(X,T)$ is $a$-connected fts.

Local compactness in fuzzy topological spaces was studied in [2, 3, 7, 14]. The definition of local compactness in [7] is modified in the following.

**Definition 3.17** Let $0 \leq \alpha < 1$ ($0 < \alpha \leq 1$). A fts $(X,T)$ is said to be locally $\alpha$-compact (resp. locally $\alpha^{*}$-compact) if for each $p \in X$ there exists an open fuzzy set $N$ such that $N(p) > \alpha$ (resp. $N(p) \geq \alpha$) and $\overline{\alpha(N)}$ (resp. $\overline{\alpha^{*}(N)}$) is $\alpha$-compact (resp. $\alpha^{*}$-compact).

We prove the following

**Theorem 3.18** Let $0 \leq \alpha < 1$. A fts $(X,T)$ is locally $\alpha$-compact if and only if $(X,T_{a})$ is locally compact topological space.

**Proof.** Let $(X,T)$ be locally $\alpha$-compact. Let $x \in X$. There exists an open fuzzy set $N$ in $(X,T)$ such that $N(x) > \alpha$ and $\alpha(N)$ is $\alpha$-compact. Therefore $\alpha(N)$ is an open set in $(X,T_{a})$ containing $x$ such that $\alpha(N)$ is compact subset in $(X,T_{a})$: For if $\{U_{\lambda} = \alpha(G_{\lambda}) : \lambda \in \Lambda, G_{\lambda} \in T\}$ is an open cover of $\alpha(N)$ in $(X,T_{a})$ then the family $\{G_{\lambda} : \lambda \in \Lambda\}$ is an open $\alpha$-shading of $\alpha(N)$ in $(X,T)$. Since $\alpha(N)$ is $\alpha$-compact $\{G_{\lambda} : \lambda \in \Lambda\}$ has a finite $\alpha$-subshading say $\{G_{\lambda_{i}}\}_{i=1}^{k}$. Then $\{\alpha(G_{\lambda_{i}}) = U_{\lambda_{i}} : i = 1,2,\ldots,k\}$ is a finite subcover of $\{U_{\lambda} : \lambda \in \Lambda\}$ for $\alpha(N)$. So $\alpha(N)$ is a compact subset of $(X,T_{a})$. Thus for each $x \in X$, there exists an open set $\alpha(N)$ in $(X,T_{a})$ such that $x \in \alpha(N)$ and $\alpha(N)$ is compact. Hence $(X,T_{a})$ is locally compact topological space.

Conversely, suppose $(X,T_{a})$ is locally compact. Let $p \in X$. Then there exists an open set $\alpha(G)$ in $(X,T_{a})$, where $G \in T$, such that $p \in \alpha(G)$ and $\alpha(G)$ is compact set in $(X,T_{a})$. Now $G \in T$ and $G(p) > \alpha$. Further $\overline{\alpha(G)}$ is $\alpha$-compact in $(X,T)$: For if $\{H_{\lambda} : \lambda \in \Lambda \}$ is an open $\alpha$-shading of $\overline{\alpha(G)}$ in $(X,T)$, then $\{\alpha(H_{\lambda}) : \lambda \in \Lambda\}$ is an open cover of $\overline{\alpha(G)}$. Since $\overline{\alpha(G)}$ is compact in $(X,T_{a})$, $\{\alpha(H_{\lambda}) : \lambda \in \Lambda\}$ has a finite $\alpha$-subshading say $\{\alpha(H_{\lambda_{i}}) : i = 1,2,\ldots,k\}$. Then $\{H_{\lambda_{i}} : i = 1,2,\ldots,k\}$ is a finite $\alpha$-subshading of $\{H_{\lambda} : \lambda \in \Lambda\}$ for $\overline{\alpha(G)}$. Therefore every open $\alpha$-shading for $\overline{\alpha(G)}$ has a finite $\alpha$-subshading and therefore $\overline{\alpha(G)}$ is $\alpha$-compact. Thus for each $p \in X$ there exists an open fuzzy set $G$ in $(X,T)$ such that $G(p) > \alpha$ and $\overline{\alpha(G)}$ is $\alpha$-compact in $(X,T)$. Hence $(X,T)$ is locally $\alpha$-compact.

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