Some recurrence relations for the generalized basic hypergeometric functions *

S.D. Purohit

Abstract

In the present paper, we express the generalized basic hypergeometric function $r\Phi_s(·)$ for $r = s + 1$ in terms of an iterated $q$-integrals involving the basic analogue of the Gauss’s hypergeometric function. Further, using the relations between $q$-contiguous hypergeometric series, we obtain some recurrence relations for the generalized basic hypergeometric functions of one variable.

1 Introduction

The generalized basic hypergeometric series of Gasper and Rahman [4] is given by

$$r\Phi_s (a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) = r\Phi_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ q, x \\ b_1, \ldots, b_s \\ \end{array} \right]$$

$$= r\Phi_s [(a_r); (b_s); q, x] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} x^n \left\{ (-1)^n q^{n(n-1)/2} \right\}^{(1+s-r)} ,$$

where for real or complex $a$

$$(a; q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \in N, \end{cases}$$

is the $q$-shifted factorial, $r$ and $s$ are positive integers, and variable $x$, the numerator parameters $a_1, \ldots, a_r$, and the denominator parameters $b_1, \ldots, b_s$ being any complex quantities provided that

$b_j \neq q^{-m}, m = 0, 1, \ldots; j = 1, 2, \ldots, s.$

* Mathematics Subject Classifications: 33D15, 33D60.

Key words: Generalized basic hypergeometric functions, integral representations, recurrence relations.

©2009 Universiteti i Prishtines, Prishtine, Kosovë.

If $|q| < 1$, the series (1.1) converges absolutely for all $x$ if $r \leq s$ and for $|x| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|x| < |b_1 b_2 \cdots b_s|/|a_1 a_2 \cdots a_r|$. 

Further, in terms of the $q$-gamma function, (1.2) can be expressed as 

$$ (a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0, $$

where the $q$-gamma function (cf. Gasper and Rahman [4]) is given by 

$$ \Gamma_q(a) = \frac{(q; q)_\infty}{(q^a; q)_\infty(1-q)^{a-1}} = \frac{(q; q)_{a-1}}{(1-q)^{a-1}}, $$

where $a \neq 0, -1, -2, \cdots$. 

The theory of basic hypergeometric functions of one and more variables has a wide range of applications in various fields of Mathematical, Physical and Engineering Sciences, namely-Number theory, Partition theory, Combinatorial analysis, Lie theory, Fractional calculus, Integral transforms, Quantum theory etc. (see [1,2,4,5])

In the present work, we express the generalized basic hypergeometric function $s+1 \Phi_s(\cdot)$ in terms of an iterated $q$-integrals involving the $q$-Gauss hypergeometric function $2 \Phi_1(\cdot)$. Using $q$-contiguous relations for $2 \Phi_1(\cdot)$, we obtain some recurrence relations for the generalized basic hypergeometric functions of one variable. The above mentioned technique is a $q$-version of the technique used by Galuč and Kalla [3].

## 2 Integral representation

In this section, we express the generalized basic hypergeometric function $s+1 \Phi_s(\cdot)$ (for $r = s + 1$) in terms of an iterated integral involving the basic analogue of Gauss hypergeometric function.

**Theorem:** Let $\text{Re}(b_{k-i}) > 0$, for all $i = 0, 1, \cdots, s - 2$ and $|q| < 1$, then the iterated $q$-integral representation of $s+1 \Phi_s(\cdot)$ is given by

$$ s+1 \Phi_s \left[ \begin{array}{ccc} q^a, q^b, a_3, a_4, \cdots, a_s, a_{s+1} ; & q, x \\ q^\gamma, b_2, b_3, \cdots, b_s ; & \end{array} \right] = \prod_{i=0}^{s-2} \Gamma_q \left[ \begin{array}{c} b_{k-i} \\ a_{s+1-i}, b_{s-i} - a_{s+1-i} \end{array} \right] \times \int_0^1 \cdots \int_0^1 \prod_{i=0}^{s-2} t_{i+1}^{a_{s+1-i}-1} \cdot (t_{i+1}; q)_{b_{s-i} - a_{s+1-i}-1} $(s-1)\text{times}
\[
\times_2 \Phi_1 \left[ \begin{array}{c}
q^a, q^3 ; \\
q^\gamma, q, t_{s-1} \cdots t_{2} t_{1} x
\end{array} \right] d_q t_{s-1} \cdots d_q t_2 d_q t_1, (2.1)
\]

where \(|x| < 1\) and \(|t_{s-1} \cdots t_{2} t_{1} x| < 1\).

**Proof:** To prove the theorem, we consider the well-known \(q\)-integral representation of \(\Phi_s(\cdot)\), namely
\[
\Phi_s \left[ \begin{array}{c}
a_1, \ldots, a_r ; \\
b_1, \ldots, b_s
\end{array} \right] = \Gamma_q \left[ \begin{array}{c}
b_s \\
q, x
\end{array} \right] \]
\[
\times \int_0^1 t^{a_r-1}(tq; q)_{b_r-a_r-1} \ r-1 \Phi_{s-1} \left[ \begin{array}{c}
a_1, \ldots, a_{r-1} ; \\
b_1, \ldots, b_{s-1}
\end{array} \right] q, tx \ d_q t, (2.2)
\]

which is the generalization of \(q\)-analogue of Euler’s integral representation, namely (cf. Gasper and Rahman [4, eqn.(1.11.9), p.19])
\[
2 \Phi_1 \left[ \begin{array}{c}
q^a, q^b ; \\
q^\gamma, q, x
\end{array} \right] = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1}(tq; q)_{c-b-1}
\]
\[
\times \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1}(tq; q)_{c-b-1}
\times_1 \Phi_0 \left[ \begin{array}{c}
q^a ; \\
q, tx
\end{array} \right] d_q t. (2.3)
\]

Therefore, relation (2.2) can also be written as
\[
s+1 \Phi_s \left[ \begin{array}{c}
q^a, q^3, a_3, a_4, \ldots, a_s, a_{s+1} ; \\
q^\gamma, b_2, b_3, \ldots, b_s
\end{array} \right] = \Gamma_q \left[ \begin{array}{c}
b_s \\
q, x
\end{array} \right] \]
\[
\times \int_0^1 t^{a_{s+1}-1}(tq; q)_{b_s-a_{s+1}-1} \ s \Phi_{s-1} \left[ \begin{array}{c}
q^a, q^3, a_3, a_4, \ldots, a_s ; \\
q^\gamma, b_2, b_3, \ldots, b_{s-1}
\end{array} \right] q, tx \ d_q t_{s+1}. (2.4)
\]

Repeating the process in the right-hand side of (2.4), we get
\[
s+1 \Phi_s \left[ \begin{array}{c}
q^a, q^3, a_3, a_4, \ldots, a_s, a_{s+1} ; \\
q^\gamma, b_2, b_3, \ldots, b_s
\end{array} \right] = \Gamma_q \left[ \begin{array}{c}
b_s \\
q, x
\end{array} \right]
\times \Gamma_q \left[ \begin{array}{c}
b_{s-1} \\
a, b_{s-1} - a_s
\end{array} \right] \int_0^1 \int_0^1 t_2^{a_{s+1}-1}(t_2q; q)_{b_{s-1}-a_s-1} t_1^{a_{s+1}-1}(t_1q; q)_{b_s-a_{s+1}-1}
\times \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1}(tq; q)_{c-b-1}
\times_1 \Phi_{s-2} \left[ \begin{array}{c}
q^a, q^3, a_3, a_4, \ldots, a_{s-1} ; \\
q^\gamma, b_2, b_3, \ldots, b_{s-2}
\end{array} \right] q, t_1 t_2 x \ d_q t_2 d_q t_1. (2.5)
\]

Successive operations \((s - 3)\) times in the right-hand side of (2.5) leads to the desired result (2.1).
3 Recurrence relations

In this section, as an application of the integral representation for $s+1\Phi_s(.)$, given by (2.1), we shall derive certain recurrence relation for the generalized basic hypergeometric series.

Using the relation between $q$-contiguous basic hypergeometric functions [4, p.22]

\[
\begin{align*}
2\Phi_1 \left[ \begin{array}{c}
q^\alpha, q^\beta \\
q, x \\
q^\gamma 
\end{array} \right] &= 2\Phi_1 \left[ \begin{array}{c}
q^\alpha, q^\beta \\
q, x \\
q^\gamma 
\end{array} \right] - 2\Phi_1 \left[ \begin{array}{c}
q^\alpha+1, q^\beta+1 \\
q, x \\
q^\gamma+1 
\end{array} \right] \\
&= q^\gamma x \frac{(1 - q^\alpha)(1 - q^\beta)}{(q - q^\gamma)(1 - q^\gamma)} \\
&\times 2\Phi_1 \left[ \begin{array}{c}
q^\alpha+1, q^\beta+1 \\
q, x \\
q^\gamma+1 
\end{array} \right],
\end{align*}
\]

(3.1)

we get

\[
\begin{align*}
2\Phi_1 \left[ \begin{array}{c}
q^\alpha, q^\beta \\
q, ts_l \cdots t_2 t_1 x \\
q^\gamma 
\end{array} \right] &= 2\Phi_1 \left[ \begin{array}{c}
q^\alpha, q^\beta \\
q, ts_l \cdots t_2 t_1 x \\
q^\gamma 
\end{array} \right] \\
&\quad - q^\gamma t_s \cdots t_2 t_1 x \frac{(1 - q^\alpha)(1 - q^\beta)}{(q - q^\gamma)(1 - q^\gamma)} 2\Phi_1 \left[ \begin{array}{c}
q^\alpha+1, q^\beta+1 \\
q, ts_l \cdots t_2 t_1 x \\
q^\gamma+1 
\end{array} \right].
\end{align*}
\]

(3.2)

On substituting value from relation (3.2) in the right-hand side of the result (2.1), we have

\[
\begin{align*}
\begin{align*}
&\phantom{= s+1\Phi_s \left[ \begin{array}{c}
q^\alpha, q^\beta, a_3, a_4, \ldots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \ldots, b_s \\
q^\gamma-l 
\end{array} \right]} = \prod_{i=0}^{s-2} \frac{\Gamma_q \left( b_{s-i} \right)}{\Gamma_q \left( a_{s+1-i} - b_{s-i} \right)} \\
&\quad \times \int_0^1 \cdots \int_0^1 \prod_{i=0}^{s-2} \frac{1}{(i+1)q} \left( t_{i+1} q^\gamma \right) \left( b_{s-i} - a_{s+1-i} - 1 \right) \\
&\quad \times 2\Phi_1 \left[ \begin{array}{c}
q^\alpha, q^\beta \\
q, ts_l \cdots t_2 t_1 x \\
q^\gamma 
\end{array} \right] \\
&\quad \times \prod_{i=0}^{s-2} \Gamma_q \left( b_{s-i} \right) \left( a_{s+1-i} - b_{s-i} - a_{s+1-i} \right) \\
&\quad \times \int_0^1 \cdots \int_0^1 \prod_{i=0}^{s-2} \frac{1}{(i+1)q} \left( t_{i+1} q^\gamma \right) \left( b_{s-i} - a_{s+1-i} - 1 \right)
\end{align*}
\end{align*}
\]

\[
\begin{align*}
&\quad \times \prod_{i=0}^{s-2} \prod_{i=0}^{s-2} \frac{1}{(i+1)q} \left( t_{i+1} q^\gamma \right) \left( b_{s-i} - a_{s+1-i} - 1 \right)
\end{align*}
\]

Some recurrence relations

\[ \times 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^{\beta+1} \\ q^{\gamma+1} \\ q, t_{s-1} \cdots t_2 t_1 x \end{array} \right] d_q t_{s-1} \cdots d_q t_2 d_q t_1. \quad (3.3) \]

Again, on making use of the result (2.1), the above result (3.3) leads to the following recurrence relation:

\[ s_{s+1} \Phi_s \left[ \begin{array}{c} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} \\ q, x \\ q^{\gamma}, b_2, b_3, \cdots, b_s \end{array} \right] \]

\[ = s_{s+1} \Phi_s \left[ \begin{array}{c} q^{\alpha}, q^{\beta}, a_3, a_4, \cdots, a_s, a_{s+1} \\ q^{\gamma-1}, b_2, b_3, \cdots, b_s \\ q, x \end{array} \right] - q^\gamma x \frac{(1-q^{\alpha})(1-q^{\beta})}{(q-q^{\gamma})(1-q^{\gamma})} \]

\[ \times \prod_{i=0}^{s-2} \frac{(1-q^{\alpha_i-i})}{(1-q^{\beta_i-i})} s_{s+1} \Phi_s \left[ \begin{array}{c} q^{\alpha+1}, q^{\beta+1}, q_3, q_4, \cdots, q_s, q_{s+1}q \\ q^{\gamma+1}, b_2q, b_3q, \cdots, b_sq \\ q, x \end{array} \right], \quad (3.4) \]

where \( \Re(b_{s-i}) > 0 \), for all \( i = 0, 1, \cdots, s - 2 \) and \( |x| < 1 \).

Similarly, if we consider the following \( q \)-contiguous relations (cf. Gasper and Rahman [4, p.22])

\[ 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^3 \\ q^{\gamma} \\ q, x \end{array} \right] - 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha}, q^3 \\ q^{\gamma} \\ q, x \end{array} \right] = q^\alpha x \frac{(1-q^3)}{(1-q^{\gamma})} \]

\[ \times 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^{\beta+1} \\ q^{\gamma+1} \\ q, x \end{array} \right], \quad (3.5) \]

\[ 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^3 \\ q^{\gamma+1} \\ q, x \end{array} \right] - 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha}, q^3 \\ q^{\gamma} \\ q, x \end{array} \right] = q^\alpha x \frac{(1-q^{\gamma-\alpha})(1-q^3)}{(1-q^{\gamma+1})(1-q^{\gamma})} \]

\[ \times 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^{\beta+1} \\ q^{\gamma+2} \\ q, x \end{array} \right], \quad (3.6) \]

and

\[ 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^{\beta-1} \\ q^{\gamma} \\ q, x \end{array} \right] - 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha}, q^{\beta} \\ q^{\gamma} \\ q, x \end{array} \right] = q^\alpha x \frac{(1-q^{\beta-\alpha+1})}{(1-q^{\gamma})} \]

\[ \times 2 \Phi_1 \left[ \begin{array}{c} q^{\alpha+1}, q^3 \\ q^{\gamma+1} \\ q, x \end{array} \right], \quad (3.7) \]
and make use of the result (2.1), we obtain the following respective recurrence relations for generalized basic hypergeometric functions, namely

\[
{s+1}_s \Phi_s \left[ \begin{array}{c}
q^\alpha, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right]
\]

\[
= s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right] - q^\alpha x \frac{(1 - q^\beta)(1 - q^\gamma)}{(1 - q)}
\]

\[
\times \prod_{i=0}^{s-2} \left[ \frac{1 - q^{a_{s+1-1}}}{1 - q^{b_{s-1}}} \right] s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q \\
q^\gamma, b_2 q, b_3 q, \cdots, b_s q
\end{array} ; q, x \right],
\]

where \( Re(b_{s-i}) > 0 \), for all \( i = 0, 1, \cdots, s - 2 \) and \( |x| < 1 \).

\[
{s+1}_s \Phi_s \left[ \begin{array}{c}
q^\alpha, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right]
\]

\[
= s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma+1, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right] - q^\alpha x \frac{(1 - q^\gamma - q)(1 - q^\beta)}{(1 - q^\gamma+1)(1 - q^\gamma)}
\]

\[
\times \prod_{i=0}^{s-2} \left[ \frac{1 - q^{a_{s+1-1}}}{1 - q^{b_{s-1}}} \right] s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta+1, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q \\
q^\gamma+2, b_2 q, b_3 q, \cdots, b_s q
\end{array} ; q, x \right],
\]

where \( Re(b_{s-i}) > 0 \), for all \( i = 0, 1, \cdots, s - 2 \) and \( |x| < 1 \).

\[
{s+1}_s \Phi_s \left[ \begin{array}{c}
q^\alpha, q^\beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right]
\]

\[
= s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta-1, a_3, a_4, \cdots, a_s, a_{s+1} \\
q^\gamma, b_2, b_3, \cdots, b_s
\end{array} ; q, x \right] - q^\alpha x \frac{(1 - q^\beta-1)(1 - q^\beta-\alpha+1)}{(1 - q^\gamma)}
\]

\[
\times \prod_{i=0}^{s-2} \left[ \frac{1 - q^{a_{s+1-1}}}{1 - q^{b_{s-1}}} \right] s+1 \Phi_s \left[ \begin{array}{c}
q^{\alpha+1}, q^\beta, a_3 q, a_4 q, \cdots, a_s q, a_{s+1} q \\
q^\gamma+1, b_2 q, b_3 q, \cdots, b_s q
\end{array} ; q, x \right],
\]

where \( Re(b_{s-i}) > 0 \), for all \( i = 0, 1, \cdots, s - 2 \) and \( |x| < 1 \).
4 Special cases

In view of the limit formulae

\[ \lim_{q \to 1^-} \Gamma_q(a) = \Gamma(a) \quad \text{and} \quad \lim_{q \to 1^-} \left( \frac{q^a}{q} \right)_n = (a)_n, \]  

(4.1)

where

\[ (a)_n = a(a+1) \cdots (a+n-1), \]  

(4.2)

one can note that the result (2.1) is the \( q \)-extension of the known result due to Galu\'e and Kalla [3, eqn.(4), p.52], namely

\[
\begin{align*}
{}_{s+1}F_s & \left[ \begin{array}{c}
\alpha, \beta, a_3, a_4, \cdots, a_s, a_{s+1} \\
\gamma, b_2, b_3, \cdots, b_s
\end{array} ; \right. \\
& \left. x \right] = \prod_{i=0}^{s-2} \Gamma \left[ b_{s-i} \right] \prod_{i=0}^{s-1} \Gamma \left[ a_{s+1-i} \right. \\
& \left. b_{s-i} - a_{s+1-i} \right] \\
& \times \int_0^1 \cdots \int_0^1 \prod_{i=0}^{s-2} (1-t_{i+1})^{b_{s-i}-a_{s+1-i}-1} \\
& \times {}_{2}F_1 \left[ \begin{array}{c}
\alpha, \beta \\
\gamma
\end{array} ; \\
t_{s-1} \cdots t_2 t_1 x \right] dt_{s-1} \cdots dt_2 dt_1,
\end{align*}
\]

(4.3)

where \( \text{Re}(b_{s-i}) > 0 \), for all \( i = 0, 1, \cdots, s - 2, |x| < 1 \) and \( |t_{s-1} \cdots t_2 t_1 x| < 1 \).

Similarly, if we let \( q \to 1^- \) and use the formula (1.4), then the results (3.4) and (3.8)-(3.10) correspond to the recurrence relations for generalized hypergeometric functions.

We conclude with the remark that the results deduced in the present article appears to be a new contribution to the theory of basic hypergeometric series. Secondly, one can easily obtain number of recurrence relations for the basic hypergeometric functions by the applications of iterated \( q \)-integral representation for \( \Phi_s(\cdot) \).

Acknowledgements  The author express his sincerest thanks to the referees for some valuable suggestions.

References


S.D. Purohit  
Department of Basic-Sciences (Mathematics),  
College of Technology and Engineering,  
M.P. University of Agriculture and Technology, Udaipur-313001, India  
e-mail: sunila_purohit@yahoo.com