On the local properties of factored Fourier series *

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Abstract

In the present paper, a theorem on local property of \(|\tilde{N}, p_n, \theta_n|_k\) summability of factored Fourier series which generalizes a result of Bor [3] has been proved.

1 Introduction

Let \(\sum a_n\) be a given infinite series with partial sums \((s_n)\). We denote by \(t_n\) the \(n\)-th \((C,1)\) mean of the sequence \((na_n)\). A series \(\sum a_n\) is said to be summable \(|C,1|_k, k \geq 1\), if (see [6],[8])

\[
\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.1}
\]

Let \((p_n)\) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.2}
\]

The sequence-to-sequence transformation

\[
\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}
\]

defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\tilde{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [7]). The series \(\sum a_n\) is said to be summable \(|\tilde{N}, p_n|_k, k \geq 1\), if (see [2])

\[
\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \tag{1.4}
\]

where

\[
\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.5}
\]

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In the special case $p_n = 1$ for all values of $n$, $|\hat{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability. Also, if we take $k = 1$ and $p_n = 1/(n + 1)$, then summability $|\hat{N}, p_n|_k$ is equivalent to the summability $|R, \log n, 1|$. Let $(\theta_n)$ be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|\hat{N}, p_n, \theta_n|_k$, $k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\Delta \sigma_{n-1}|^k < \infty.$$  \hfill (1.6)

If we take $\theta_n = \frac{p_n}{p_{n+1}}$, then $|\hat{N}, p_n, \theta_n|_k$ summability reduces to $|\hat{N}, p_n|_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of $n$, then we get $|C, 1|_k$ summability. Furthermore, if we take $\theta_n = n$, then $|\hat{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ (see [4]) summability. A sequence $(\lambda_n)$ is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer $n$, where $\Delta^2 \lambda_n = \Delta(\Delta \lambda_n)$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period $2\pi$, and integrable ($L$) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0$$ \hfill (1.7)

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$ \hfill (1.8)

\section{Known result}

Mohanty [11] has demonstrated that the summability $|R, \log n, 1|$ of

$$\sum A_n(t)/\log(n + 1),$$ \hfill (2.1)

at $t = x$, is a local property of the generating function of $\sum A_n(t)$. Later on Matsumoto [9] improved this result by replacing the series (2.1) by

$$\sum A_n(t)/\log \log(n + 1)^{1+\epsilon}, \epsilon > 0.$$ \hfill (2.2)

Generalizing the above result Bhatt [1] proved the following theorem.

\textbf{Theorem A.} If $(\lambda_n)$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.

Also, Mishra [10] has proved the following most general theorem on this matter.

\textbf{Theorem B.} If $(p_n)$ is a sequence such that

$$P_n = O(np_n)$$ \hfill (2.3)
\[ P_n \Delta p_n = O(p_n p_{n+1}), \quad (2.4) \]

then the summability \( |\bar{N}, p_n| \) of the series

\[ \sum_{n=1}^{\infty} A_n(t) \lambda_n p_n / np_n \quad (2.5) \]

at a point can be ensured by local property, where \( (\lambda_n) \) is as in Theorem A.

On the other hand Bor [3] has generalized Theorem B for \( |\bar{N}, p_n|_k \) summability in the following form.

**Theorem C.** Let \( k \geq 1 \) and \( (p_n) \) be a sequence such that the conditions (2.3) and (2.4) of Theorem B are satisfied. Then the summability \( |\bar{N}, p_n|_k \) of the series (2.5) at a point can be ensured by local property, where \( (\lambda_n) \) is as in Theorem A.

### 3 Main result

The aim of this paper is to generalize Theorem C for \( |\bar{N}, p_n, \theta_n|_k \) summability. We shall prove the following theorem.

**Theorem.** Let \( k \geq 1 \) and \( (p_n) \) be a sequence such that the conditions (2.3)-(2.4) of Theorem B are satisfied. If \( (\theta_n) \) is any sequence of positive constants such that

\[ \sum_{v=1}^{m} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{v} (\lambda_v)^k = O(1) \quad (3.1) \]

\[ \sum_{v=1}^{m} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \Delta \lambda_v = O(1) \quad (3.2) \]

\[ \sum_{v=1}^{m} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{v} (\lambda_{v+1})^k = O(1) \quad (3.3) \]

and

\[ \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O \left\{ \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{P_v} \right\} \quad (3.4) \]

then the summability \( |\bar{N}, p_n, \theta_n|_k \) of the series (2.5) at a point can be ensured by local property, where \( (\lambda_n) \) is as in Theorem A.

It should be noted that if we take \( \theta_n = \frac{p_n}{P_n} \), then we get Theorem C. In this case the conditions (3.1)-(3.3) are obvious and the condition (3.4) reduces to

\[ \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O \left( \frac{1}{P_v} \right), \]
which always holds.

We need the following lemmas for the proof of our theorem.

**Lemma 1 ([10]).** If the sequence \((p_n)\) is such that the conditions (2.3) and (2.4) of Theorem B are satisfied, then
\[
\Delta(P_n/np_n) = O(1/n). \tag{3.5}
\]

**Lemma 2 ([5]).** If \((\lambda_n)\) is a convex sequence such that \(\sum n^{-1}\lambda_n\) is convergent, then \((\lambda_n)\) is non-negative and decreasing, and \(n\Delta\lambda_n \to 0\) as \(n \to \infty\).

**Lemma 3.** Let \(k \geq 1\). If \((s_n)\) is bounded and all conditions of the Theorem are satisfied, then the series
\[
\sum_{n=1}^{\infty} A_n\lambda_n P_n/np_n \tag{3.6}
\]
is summable \(|\bar{N}, p_n, \theta_n |_k\), where \((\lambda_n)\) is as in Theorem A.

**Remark.** Since \((\lambda_n)\) is a convex sequence, therefore \((\lambda_n)^k\) is also convex sequence and \(\sum (1/n)(\lambda_n)^k < \infty\).

**Proof of Lemma 3.** Let \((T_n)\) denotes the \((\bar{N}, p_n)\) mean of the series (3.6). Then, by definition, we have
\[
T_n = \frac{1}{P_n} \sum_{v=0}^{n} P_v \sum_{r=0}^{v} a_r \lambda_r P_r/np_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1})a_v \lambda_v P_v/np_v.
\]

Then
\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} P_v \frac{a_v \lambda_v}{vp_v}, \quad n \geq 1, \quad (P_{-1} = 0).
\]

By Abel’s transformation, we have
\[
T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v P_v s_v \lambda_v \frac{1}{vp_v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v P_v \Delta \lambda_v \frac{1}{vp_v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_{v+1} \Delta(P_v/np_v)s_v + s_{n} \lambda_n \frac{1}{n}
\]
\[
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say}.
\]

To prove the lemma, by Minkowski’s inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{3.7}
\]
Now, applying Hölder’s inequality, we have that

\[
\sum_{n=2}^{m+1} g_n^{k-1} | T_{n,1} |^k \leq \sum_{n=2}^{m+1} g_n^{k-1} \left( \frac{p_n}{p_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \theta_v | s_v |^k \left( \frac{\lambda_v P_v}{v p_v} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \theta_v \left( \frac{P_v}{v^k} \right)^{k-1} \frac{p_n}{P_n P_{n-1}} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \left( \frac{P_v}{v^k} \right)^{k-1} \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left( \theta_v \frac{p_n}{P_v} \right)^{k-1} \frac{p_n}{P_n P_{n-1}}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \left( \frac{P_v}{v^k} \right)^{k-1} \frac{1}{v^k} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{v} (\lambda_v)^k = O(1) \quad \text{as} \quad m \to \infty,
\]

by virtue of the hypotheses of the Theorem. Since

\[
\sum_{v=1}^{n-1} P_v \Delta \lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} \Delta \lambda_v \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v \leq \sum_{v=1}^{n-1} \Delta \lambda_v = O(1),
\]

by Lemma 2, we have that

\[
\sum_{n=2}^{m+1} g_n^{k-1} | T_{n,2} |^k \leq \sum_{n=2}^{m+1} g_n^{k-1} \left( \frac{p_n}{p_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{v p_v} \right)^k P_v \Delta \lambda_v | s_v |^k \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{v p_v} \right)^k \frac{1}{v^k} P_v \Delta \lambda_v \sum_{n=v+1}^{m+1} \left( \frac{\theta_v p_n}{P_v} \right)^{k-1} \frac{p_n}{P_n P_{n-1}}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{v p_v} \right)^k \frac{1}{v^k} \Delta \lambda_v \left( \frac{\theta_v p_v}{P_v} \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \Delta \lambda_v
\]

\[
= O(1) \quad \text{as} \quad m \to \infty,
\]

in view of the hypotheses of the Theorem and Lemma 2.
Using the fact that $\Delta(P_v/p_v) = O(1/v)$ by Lemma 1, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,3}|^k = \sum_{n=1}^{m} \theta_n^{k-1} \left( \frac{P_n}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}} \left| \sum_{v=1}^{n-1} P_v \lambda_{v+1} \Delta(P_v/p_v) s_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{P_n}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}} \left\{ \sum_{v=1}^{n-1} P_v \lambda_{v+1} \frac{1}{v} \right\}$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{P_n}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{P^k_{n-1}} \right)^k p_v \lambda_{v+1} \frac{1}{v^k}$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{P_n}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}}$$

by virtue of the hypotheses of the Theorem. Finally, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} |T_{n,4}|^k = \sum_{n=1}^{m} \theta_n^{k-1} (\lambda_n)^k \frac{1}{v^k}$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} (\lambda_n)^k |s_n|^k \frac{1}{v^{k-1}} \frac{1}{n}$$

$$= O(1) \sum_{n=1}^{m} \left( \frac{\theta_n P_n}{P^k_{n-1}} \right)^k \frac{1}{v^{k-1}} (\lambda_n)^k$$

$$= O(1) \text{ as } m \to \infty,$$

in view of the hypotheses of the Theorem. Therefore we get that

$$\sum_{n=1}^{m} \theta_n^{k-1} |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3, 4.$$

which completes the proof of the Lemma 3.

Remark. If we take $k = 1$, then we get a result due to Mishra [10].
4 Proof of the Theorem

Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of $x$ depends on the behaviour of the function in the immediate neighborhood of this point only, hence the truth of the Theorem is necessary consequence of Lemma 3.

References


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