LINEAR MAPS BETWEEN OPERATOR ALGEBRAS
PRESERVING CERTAIN SPECTRAL FUNCTIONS

XIAOHONG CAO* AND SHIZHAO CHEN

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ABSTRACT. Let $H$ be an infinite dimensional complex Hilbert space and let $\phi$ be a surjective linear map on $B(H)$ with $\phi(I) - I \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the closed ideal of all compact operators on $H$. If $\phi$ preserves the set of upper semi-Weyl operators and the set of all normal eigenvalues in both directions, then $\phi$ is an automorphism of the algebra $B(H)$. Also the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators is considered.

1. INTRODUCTION AND PRELIMINARIES

Let $H$ be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$ and $\mathcal{K}(H) \subseteq B(H)$ be the closed ideal of all compact operators. We write $T^*$ for the conjugate operator of $T \in B(H)$. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range $R(T)$ with finite dimensional null space $N(T)$ and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator $T \in B(H)$ (upper semi-Fredholm operator or lower semi-Fredholm operator), let $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T) = \text{codim} R(T)$. The index of a semi-Fredholm operator $T \in B(H)$ is given by $\text{ind}(T) = n(T) - d(T)$. The operator $T$ is Weyl if it is Fredholm of index

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*Corresponding author.

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Let $\sigma \in \mathcal{T}$ zero; $T$ is called Browder if $T$ is Fredholm with finite ascent and finite descent; $T \in B(H)$ is called upper semi-Weyl if $T$ is upper semi-Fredholm with $\text{ind}(T) \leq 0$. Let $SF^+_r(H)$ denote the set of all upper semi-Fredholm operators and let $\sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SF^+_r(H) \}$ be the essential approximate point spectrum of $T$. $\sigma(T)$, $\sigma_e(T)$, $\sigma_{SF^+}(T)$, $\sigma_{SF^-}(T)$, $\sigma_w(T)$ and $\sigma_b(T)$ denote the spectrum, the essential spectrum, the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the Weyl spectrum and the Browder spectrum respectively ([8, 9]). Let $\sigma_0(T) = \sigma(T) \setminus \sigma_b(T)$ denote the set of all normal eigenvalues.

Let $\Phi(H) \subseteq B(H)$ be the set of all Fredholm operators. We denote the Calkin algebra $B(H)/\mathcal{K}(H)$ by $\mathcal{C}(H)$. Let $\pi : B(H) \to \mathcal{C}(H)$ be the quotient map. A bijective linear map $\phi : B(H) \to B(H)$ is called a Jordan isomorphism if $\phi(A^2) = (\phi(A))^2$ for every $A \in B(H)$, or equivalently $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$ for all $A$ and $B$ in $B(H)$. It is obvious that every isomorphism and every anti-isomorphism is a Jordan isomorphism. For further properties of Jordan homomorphisms, we refer the reader to [10] and [11].

In the last two decades there has been considerable interest in the so-called linear preserver problems (see [1, 5, 16]). The goal of studying linear preservers is to give structural characterizations of linear maps on algebras having some special properties such as leaving invariant a certain subset of the algebra, or leaving invariant a certain function on the algebra. One of the most famous problem in this direction is Kaplansky’s problem([13]): Let $\phi$ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that $\phi$ is Jordan isomorphism? This problem was first solved in the finite dimensional case. J.Dieudonnê ([7]) and Marcus and Purves ([15]) proved that every unital invertibility preserving linear map on a complex matrix algebra is either an inner automorphism or a linear anti-automorphism. This result was later extended to the algebra of all bounded linear operators on a Banach space by A.R.Sourour([22]) and to von Neumann algebra by B.Aupetit([11]). Many other linear preserver problems have been extended to the infinite dimensional case. For the most significant partial obtained in this direction, we refer the reader to ([1, 18, 22, 23]). New contributions to the study of linear preserver problem in $B(H)$ have been recently made by Mbekhta in [17], Mbekhta, Rodman and Šemrl in [18], Mbekhta and Šemrl in [16] and Bendaoud, Bourhim and Sarih in [4].

In this article, we give the characterization of automorphism on $B(H)$. We get that: Let $\phi$ be a surjective linear maps on $B(H)$ with $\phi(I) - I \in \mathcal{K}(H)$ preserving the set of upper semi-Weyl operators and the set of all normal eigenvalues in both directions, then $\phi$ is an automorphism of the algebra $B(H)$. Also the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators is considered.

2. Main results

An operator is left invertible if it has a left inverse. It turns out that an operator $T \in B(H)$ is left invertible if and only if it is bounded below, or equivalently, it is upper semi-Fredholm with $n(T) = 0$. Let $\sigma_a(T) = \{ \lambda \in \mathbb{C} : T - \lambda I$ is
not left invertible). We say that a linear map $\phi : B(H) \rightarrow B(H)$ preserves the set of upper semi-Weyl operators (left invertible operators) in both directions if $T \in SF^+_-(H)$ ($T$ is left invertible) $\Rightarrow \phi(T) \in SF^+_-(H)$ ($\phi(T)$ is left invertible).

A linear map $\phi : B(H) \rightarrow B(H)$ is said to be surjective up to compact operators if for every $T \in B(H)$ there exists $T' \in B(H)$ such that $T - \phi(T') \in \mathcal{K}(H)$. It is clear that if $\phi$ is surjective, then it is surjective up to compact operators.

Remark 2.1. (1) If a linear map $\phi : B(H) \rightarrow B(H)$ preserves the set of upper semi-Weyl operators in both directions, we can not induce that $\phi$ preserves the set of left invertible operators in both directions. For example, let $A, B \in B(\ell_2)$ be defined by:

\[
A(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots),
B(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),
\]

and let $\phi(T) = ATB$, $T \in B(\ell_2)$. We can see that both $A$ and $B$ are Fredholm operators, and $\text{ind}(A) + \text{ind}(B) = 0$. By the properties of the index it follows that $T \in SF^+_-(B(\ell_2))$ if and only if $\phi(T) \in SF^+_-(B(\ell_2))$. For any $T \in B(\ell_2)$, let $T_1 = BTA$, then $\phi(T_1) = T$. Thus $\phi : B(\ell_2) \rightarrow B(\ell_2)$ is surjective and $\phi$ preserves the set of upper semi-Weyl operators in both directions. But $\phi$ does not preserve the set of left invertible operators in both directions. In fact, for an operator $T \in B(\ell_2)$ defined by:

\[
T(x_1, x_2, x_3, \cdots) = (x_2 - x_1, x_2 - x_1, x_3, x_4, \cdots),
\]

we can find that $\phi(T) = I$ is left invertible but $T$ is not left invertible.

(2) If a linear map $\phi : B(H) \rightarrow B(H)$ preserves the set of left invertible operators in both directions, we can not induce that $\phi$ preserves the set of upper semi-Weyl operators in both directions. For example, let $A \in B(\ell_2)$ be defined by:

\[
A(x_1, x_2, x_3, \cdots) = (0, 0, x_1, x_2, \cdots),
\]

$B \in B(\ell_2)$ is invertible and let $\phi(T) = ATB$, $T \in B(\ell_2)$. We can see that $A$ is left invertible, there exists $A_1 \in B(\ell_2)$ such that $A_1A = I$. Since $A \in B(\ell_2)$ is Fredholm, there are $A_2 \in B(\ell_2)$ and a compact operator $K_0$ satisfying $AA_2 = I + K_0$. For any $T \in B(\ell_2)$, let $T_0 = A_2TB^{-1}$ and $K = -K_0T$. Then $K$ is compact and $T = \phi(T_0) + K$, which means that $\phi$ is surjective up to compact operators. For any left invertible operator $T \in B(\ell_2)$, suppose that $T_1T = I$. Then $B^{-1}T_1A_1\phi(T) = I$, this shows that $\phi(T)$ is left invertible. For the converse, if $\phi(T)$ is left invertible and suppose $D\phi(T) = I$. Then $BDAT = BDATBB^{-1} = BD\phi(T)BB^{-1} = BB^{-1} = I$, thus $T \in B(\ell_2)$ is left invertible. It follows that $\phi$ preserves the set of left invertible operators in both directions. But $\phi$ does not preserve the set of upper semi-Weyl operators in both directions. In fact, let $T \in B(\ell_2)$ be defined as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$, then $\phi(T)$ is upper semi-Weyl with $\text{ind}(\phi(T)) = \text{ind}(A) + \text{ind}(T) + \text{ind}(B) = -2 + 1 + 0 = -2$ but $T$ is not upper semi-Weyl.

It is well known that the set of left invertible operators is a subset of $SF^+_-(H)$, we need to study the relation between the linear maps preserving the set of upper semi-Weyl operators and the linear maps preserving the set of left invertible operators. Let’s begin with a Theorem.
Theorem 2.2. Let $\phi : B(H) \to B(H)$ be a surjective linear map preserving upper semi-Weyl operators in both directions and $\phi(I) - I \in \mathcal{K}(H)$. If $\sigma_0(K) = \sigma_0(\phi(K))$ for any Riesz operator $K$, then there is an invertible linear operator $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for any $T \in B(H)$.

Proof. We will prove the Theorem by seven steps:

(i) For any $T \in B(H)$, $\sigma_ea(T) = \sigma_ea(\phi(T))$.

Let $\phi(I) = I + K$, where $K \in \mathcal{K}(H)$. Since $T - \lambda I \in SF_+^-(H) \iff \phi(T) - \lambda I = \phi(T) - \lambda I = \phi(T) - \lambda \phi(I) = \phi(T) - \lambda I - \lambda K \in SF_+^-(H) \iff \phi(T) - \lambda I \in SF_+^-(H)$, it follows that $\sigma_ea(T) = \sigma_ea(\phi(T))$ for any $T \in B(H)$.

(ii) $\phi$ preserves compact operators in both directions.

First we claim that

$$\mathcal{K}(H) = \{ K \in B(H) : K + SF_+^-(H) \in SF_+^-(H) \}$$

$$= \{ K \in B(H) : \sigma_ea(T + K) = \sigma_ea(T) \text{ for all } T \in B(H) \}.$$ 

From the stability properties of index function, it is clear that $\mathcal{K}(H) \subseteq \{ K \in B(H) : K + SF_+^-(H) \in SF_+^-(H) \} = \{ K \in B(H) : \sigma_ea(T + K) = \sigma_ea(T) \text{ for all } T \in B(H) \}$.

Let $\partial E$ and $\eta E$ denote the boundary and the polynomial convex hull of a compact subset $E$ of $\mathbb{C}$ respectively. For any $T \in B(H)$, since

$$\partial \sigma_w(T) \subseteq \partial \sigma_e(T) \subseteq \sigma_e(T) \subseteq \sigma_w(T) \text{ and } \partial \sigma_u(T) \subseteq \partial \sigma_e(T) \subseteq \sigma_e(T) \subseteq \sigma_u(T),$$

it follows that $\eta \sigma_ea(T) = \eta \sigma_w(T) = \eta \sigma_e(T)$.

Now, let $K \in B(H)$ such that $\sigma_ea(T + K) = \sigma_ea(T)$ for all $T \in B(H)$. Then by Theorem 5.3.1 in [2], $\eta \sigma_ea(T + K) = \eta \sigma_e(T) \text{ for all } T \in B(H)$. Taking into account the semisimplicity of $\mathcal{C}(H)$ and the spectral characterization of the radical, it is not difficult to prove that the $\mathcal{K}(H) = \{ K \in B(H) : K + SF_+^-(H) \in SF_+^-(H) \} = \{ K \in B(H) : \sigma_ea(T + K) = \sigma_ea(T) \text{ for all } T \in B(H) \}$.

Let $K \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$, since $\phi$ preserves upper semi-Weyl operators in both directions, there exists $T' \in SF_+^-(H)$ for which $T = \phi(T')$. Hence $T + \phi(K) = \phi(T') + \phi(K) = \phi(T' + K) \in SF_+^-(H)$. Then $\phi(K) \in \mathcal{K}(H)$.

For the converse, let $\phi(K) \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$, $\phi(T + K) = \phi(T) + \phi(K) \in SF_+^-(H)$, then $T + K \in SF_+^-(H)$. It follows that $K \in \mathcal{K}(H)$. Now we prove that $\phi$ preserves compact operators in both directions.

Since $\phi$ preserves compact operators in both directions, it follows that $\sigma(K) = \{ 0 \} \cup \sigma_0(K) = \{ 0 \} \cup \sigma_0(\phi(K)) = \sigma(\phi(K))$ for any compact operator $K$.

(iii) $\mathcal{N}(\phi) \subseteq \mathcal{K}(H)$.

If $K \in \mathcal{N}(\phi)$ and $T \in SF_+^-(H)$, then $\phi(T + K) = \phi(T) \in SF_+^-(H)$. Thus for all $T \in SF_+^-(H)$, $T + K \in SF_+^-(H)$. Thus $K \in \mathcal{K}(H)$.

(iv) Let $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ be an induced linear map such that $\phi \circ \pi = \pi \circ \phi$, then $\varphi$ is isomorphism.

$\varphi$ induces a linear map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ such that $\varphi \circ \pi = \pi \circ \phi$. Clearly, $\varphi$ is surjective since $\phi$ is surjective. By hypothesis and (ii), $\varphi$ is $\eta \sigma$-preserving. From Corollary 2.3 in [5], $\varphi$ is injective, and by Theorem 3.1 in [5], $\varphi$ is either a homomorphism or an anti-homomorphism.
First we will prove that \( \phi \) preserves upper semi-Fredholm operators in both directions. By Theorem 2.1 in [17], we know that \( \phi \) preserves Fredholm operators in both directions. Let \( T \in \mathcal{B}(H) \) be an upper semi-Fredholm, there are two cases to consider: \( d(T) = \infty \) and \( d(T) < \infty \). If \( d(T) = \infty \), using the fact that \( \phi \) is a linear map preserving upper semi-Weyl operators in both directions, we know that \( \phi(T) \) is upper semi-Fredholm. If \( d(T) < \infty \), then \( T \) is Fredholm, thus \( \phi(T) \) is Fredholm since \( \phi \) preserves Fredholm operators in both directions. Using the same way, we can prove that \( T \) is upper semi-Fredholm if \( \phi(T) \) is upper semi-Fredholm. By Corollary 3.6 in [3], \( \varphi \) is an isomorphism.

As \( \phi \) preserves the essential spectrum, from Theorem 3.3 in [17] we deduce that \( \text{ind}(\phi(T)) = \text{ind}(T) \) or \( \text{ind}(\phi(T)) = -\text{ind}(T) \) for every Fredholm operator \( T \in \mathcal{B}(H) \). Since \( \phi \) preserves upper semi-Weyl operators in both directions, it follows that \( \text{ind}(\phi(T)) \cdot \text{ind}(T) \geq 0 \) for any \( T \in \mathcal{B}(H) \). Thus \( \text{ind}(\phi(T)) = \text{ind}(T) \) for any \( T \in \Phi(H) \). Also we can prove that \( \text{ind}(\phi(T)) = \text{ind}(T) \) for any upper semi-Fredholm operator \( T \in \mathcal{B}(H) \). For lower semi-Fredholm operator \( T \in \mathcal{B}(H) \), we also have \( \text{ind}(\phi(T)) = \text{ind}(T) \). In fact, since \( \varphi \) is an isomorphism, by Corollary 3.6 in [3], \( \phi \) preserves lower semi-Fredholm operators in both directions. Let \( T \in \mathcal{B}(H) \) be a lower semi-Fredholm operator, then \( \phi(T) \) is a lower semi-Fredholm operator. There are also two cases to consider: \( n(T) = \infty \) and \( n(T) < \infty \). If \( n(T) = \infty \), using the fact that \( \phi \) is a linear map preserving Fredholm operators in both directions, we know that \( n(\phi(T)) = \infty \), then \( \text{ind}(\phi(T)) = \text{ind}(T) = \infty \). If \( n(T) < \infty \), then \( T \) is Fredholm, thus \( \phi(T) \) is Fredholm since \( \phi \) preserves Fredholm operators in both directions. Then \( \text{ind}(\phi(T)) = \text{ind}(T) \) again.

(v) \( \phi \) is injective.

If \( \phi(T) = 0 \), then \( T \) is compact and hence \( \sigma(T) = \{0\} \cup \sigma_0(T) = \{0\} \cup \sigma_0(\phi(T)) = \{0\} \) since \( \sigma_0(\phi(T)) = \emptyset \). This means that \( T \) is quasinilpotent. Assume that \( T \neq 0 \), we can find \( x \in H \) such that \( Tx \neq y \). Clearly, \( x \) and \( y \) are linear independent. Define a nilpotent operator \( N \in \mathcal{B}(H) \) by:

\[
N x = x - y, \quad N y = x - y, \quad N z = 0, \quad \text{for } z \in \{x, y\},
\]

Then both \( N \) and \( N + T \) are compact, thus \( \phi(N + T) = \phi(N) \) is compact. From the condition we can find \( \sigma(T + N) = \sigma(\phi(T + N)) \), then \( \sigma(T + N) = \sigma(\phi(T + N)) = \sigma(\phi(N)) = \sigma(N) = \{0\} \), which means that \( T + N \) is quasinilpotent. This is in contradiction to the fact that \( 1 \in \sigma(T + N) \).

(vi) \( \phi(T) \) is an idempotent of rank one if and only if \( T \) is an idempotent of rank one.

Let \( P \in \mathcal{B}(H) \) be an idempotent of rank one and let \( \phi(P) = Q \). Since both \( P \) and \( Q \) are compact operators, \( \sigma(Q) = \sigma(P) = \{0, 1\} \). For any \( K \in \mathcal{F}_2(H) \), where \( \mathcal{F}_2(H) \) denotes the set of all operators in \( \mathcal{B}(H) \) with rank not greater than 2, there is \( S \in \mathcal{B}(H) \) such that \( K = \phi(S) \) as \( \phi \) is surjective. Thus by Theorem 1 in [12] we must have that \( \sigma(S + P) \cap \sigma(S + 2P) \subseteq \sigma(S) \). Since \( S + P, S + 2P \) and \( S \) are all compact operators, it follows that \( \sigma(S + P) = \sigma(\phi(S + P)) = \sigma(K + Q), \sigma(S + 2P) = \sigma(\phi(S + 2P)) = \sigma(K + 2Q) \) and \( \sigma(S) = \sigma(\phi(S)) = \sigma(K) \). Then \( \sigma(K + Q) \cap \sigma(K + 2Q) \subseteq \sigma(K) \). By Lemma 2.2 in [6], we know that \( \text{rank} Q = 1 \). This implies that \( Q \) satisfies a quadratic polynomial equation \( p(Q) = 0 \) ([14]).
Using the fact that $\sigma(Q) = \{0, 1\}$, we know that $p$ is of the form $p(\lambda) = \lambda(\lambda - 1)$. Then $Q^2 = Q$.

We get that $\phi$ preserves idempotent of rank one. The same must be true for $\phi^{-1}$, and consequently, $\phi$ preserves idempotents of rank one in both directions. According to Proposition 2.6 in [19] there exists either an invertible $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for all finite rank operators $T \in B(H)$, or a bounded invertible conjugate-linear operator $C$ on $H$ such that $\phi(T) = CT^*C^{-1}$ for every $T \in B(H)$ of finite rank.

(vii) There is an invertible linear operator $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for any $T \in B(H)$.

Let $T \in B(H)$ such that $T^2 = 0$. Then $\sigma(T) = \{0\}$ and $\sigma_0(T) = \emptyset$. Since $T - \lambda I$ is Weyl for any $\lambda \neq 0$ and $\phi$ is a linear map preserving upper semi-Weyl operators in both directions, it follows that $\phi(T) - \lambda I$ is Weyl for any $\lambda \neq 0$. This implies that $\phi(T)$ is a Riesz operator. For every operator $U$ of rank one, we know that both $T + U$ and $\phi(T) + \phi(U)$ are Riesz operators. Then $\sigma(T + U) = \sigma(\phi(T) + \phi(U))$. By assuming that $\phi(U) = AUA^{-1}$, this can be rewritten as $\sigma(T + U) = \sigma(A^{-1}\phi(T)A + U)$ for each rank one operator $U$. This gives directly that $T = A^{-1}\phi(T)A$, and hence $\phi(T) = ATA^{-1}$. Then $\phi(T) = ATA^{-1}$ for every $T \in B(H)$ by Theorem 2 in [20].

In the second case we show that similarly that $\phi(T) = CT^*C^{-1}$ for all $T \in B(H)$. It follows from that $\text{ind}(T) = \text{ind}(\phi(T))$ if $T$ is Fredholm, we know that the second case cannot occur. The proof of the Theorem is complete. 

In the proof of Theorem 2.2, we use P.Šemrl’s method in Theorem 4 in [21], but there are many differences in two proofs.

Similar to the proof of Lemma 1 in [12], we can get that: Let $A \in B(H)$. If $\sigma_a(T + A) \subseteq \sigma_a(T)$ for every rank one operator $T$, then $A = 0$.

For surjective linear map $\phi : B(H) \to B(H)$, if $\sigma_a(T) \subseteq \sigma_a(\phi(T))$ for any $T \in B(H)$ and $\sigma_a(T) = \sigma_a(\phi(T))$ for any Riesz operator $T$, then $\phi(I) = I$. In fact, suppose that $\phi(S) = I$. For any rank one operator $F$, since $\sigma_a(F + S - I) = \sigma_a(F + S) - 1 \subseteq \sigma_a(\phi(F) + \phi(S)) - 1 = \sigma_a(\phi(F) + I) - 1 = \sigma_a(\phi(F)) = \sigma_a(F)$, we know that $S - I = 0$, then $S = I$, which means that $\phi(I) = I$. In the proof of Theorem 2.2, we can see that if $\phi$ preserves Riesz operators in both directions and if $\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T$, then there exists either an invertible $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for every $T \in B(H)$, or a bounded invertible conjugate-linear operator $C$ on $H$ such that $\phi(T) = CT^*C^{-1}$ for every $T \in B(H)$.

**Corollary 2.3.** Let $\phi : B(H) \to B(H)$ be a surjective linear map preserving upper semi-Weyl operators in both directions. If $\sigma_a(T) \subseteq \sigma_a(\phi(T))$ for any $T \in B(H)$ and $\sigma_a(T) = \sigma_a(\phi(T))$ for any Riesz operator $T$, then there is an invertible linear operator $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for any $T \in B(H)$.

**Proof.** Since $\phi(I) = I$ and $\phi : B(H) \to B(H)$ preserves upper semi-Weyl operators in both directions, we can prove that $\phi$ preserves Riesz operators in both directions. Then $\sigma(T) = \sigma_a(T) = \sigma_a(\phi(T)) = \sigma(\phi(T))$ for any Riesz operator $T$. 
Thus $\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T$. By Theorem 2.2, the result is true. \hfill \Box

**Corollary 2.4.** Let $\phi : B(H) \to B(H)$ be a surjective linear map. If $\phi(I) - I \in \mathcal{K}(H)$ and $\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T \in B(H)$, then the following statements are equivalent:

\begin{enumerate}
\item $\sigma_a(T) = \sigma_a(\phi(T))$ for any $T \in B(H)$;
\item $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ for any $T \in B(H)$;
\item $\sigma_e(T) = \sigma_e(\phi(T))$ and $\text{ind}(T) = \text{ind}(\phi(T))$ if $T$ is a Fredholm operator;
\item $\sigma_{SF+}(T) = \sigma_{SF+}(\phi(T))$ and $\text{ind}(T) = \text{ind}(\phi(T))$ if $T$ is an upper semi-Fredholm operator;
\item $\sigma_{SF-}(T) = \sigma_{SF-}(\phi(T))$ and $\text{ind}(T) = \text{ind}(\phi(T))$ if $T$ is a lower semi-Fredholm operator;
\item There exists an invertible operator $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for every $T \in B(H)$.
\end{enumerate}

**Proof.** It follows from Theorem 2.2, Theorem 2.1 in [17], Theorem 4.8 in [3] and Corollary 3.6 in [3], that (2), (3), (4), (5) and (6) are equivalent. The implication (6) $\Rightarrow$ (1) is clear, and the converse can be argued as in Theorem 4 in [21]. \hfill \Box

From the proof of Theorem 4 in [21], we know that if $\phi : B(H) \to B(H)$ be a surjective linear map and $\sigma_a(T) = \sigma_a(\phi(T))$ for any $T \in B(H)$, then (2), (3), (4) and (5) in Corollary 2.4 are true.

**Remark 2.5.** In Corollary 2.4, the condition “$\sigma_0(T) = \sigma_0(\phi(T))$ for any Riesz operator $T \in B(H)$” is essential. For example, let $A, B \in B(\ell_2)$ and $\phi$ be defined as in (1) in Remark 2.1. Then $\phi : B(H) \to B(H)$ is a surjective linear map preserving upper semi-Weyl operators in both directions and $\phi(I) = I$, which means that $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ for any $T \in B(H)$ (from the proof of Theorem 2.2). Let $T_0 = BA$, then $T_0(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, x_4, \cdots)$ and $\phi(T_0) = I$. Since $T_0 = T_0^0$ and $\phi(T_0)$ is invertible, we can see that $0 \notin \sigma_0(T_0)$ but $0 \notin \sigma_0(\phi(T_0))$. Then we can not induce that $\phi$ preserves the set of left invertible operators in both directions from (1) in Remark 2.1.

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**References**


College of Mathematics and Information Science, Shaanxi Normal University, Xi’an, 710062, People’s Republic of China.

E-mail address: xiaohongcao@snnu.edu.cn
E-mail address: cshw4563876@yahoo.cn