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BOUNDEDNESS OF INTRINSIC LITTLEWOOD–PALEY FUNCTIONS ON MUSIELAK–ORLICZ MORREY AND CAMPANATO SPACES

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Abstract. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\varphi(x,\cdot)$ is nondecreasing, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ when $t > 0$, $\lim_{t\to\infty} \varphi(x,t) = \infty$ and $\varphi(\cdot,t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight uniformly in t. Let $\phi : [0,\infty) \to [0,\infty)$ be nondecreasing. In this article, the authors introduce the Musielak–Orlicz Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ and obtain the boundedness on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ of the intrinsic Lusin area function S_α, the intrinsic g-function g_α, the intrinsic g_λ^*-function $g^*_{\lambda,\alpha}$ and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions, where $\alpha \in (0,1]$, $\lambda \in (\min\{\max\{3, p_1\}, 3 + 2\alpha/n\}, \infty)$ and p_1 denotes the uniformly upper type index of φ. Let $\Phi : [0,\infty) \to [0,\infty)$ be nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ when $t > 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$, $w \in A_\infty(\mathbb{R}^n)$ and $\phi : (0,\infty) \to (0,\infty)$ be nonincreasing. The authors also introduce the weighted Orlicz–Morrey space $M_w^{\Phi,\phi}(\mathbb{R}^n)$ and obtain the boundedness on $M_w^{\Phi,\phi}(\mathbb{R}^n)$ of the aforementioned intrinsic Littlewood–Paley functions and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions. Finally, for $q \in [1,\infty)$, the boundedness of the aforementioned intrinsic Littlewood–Paley functions on the Musielak-Orlicz Campanato space $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ is also established.

1. Introduction

It is well known that the intrinsic Littlewood–Paley g-function and the intrinsic Lusin area function were first introduced by Wilson in [48] to answer a conjecture

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proposed by R. Fefferman and E. M. Stein on the boundedness of the Lusin area function S from the weighted Lebesgue space $L^2_{M(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$, where $0 \le v \in L^1_{\rm loc}(\mathbb{R}^n)$ and M denotes the Hardy-Littlewood maximal function. Observe that these intrinsic Littlewood–Paley functions can be thought of as "grand maximal" Littlewood–Paley functions in the style of the "grand maximal function" of C. Fefferman and Stein from [13]: they dominate all the Littlewood–Paley functions of the form $S(f)$ (and the classical ones as well), but are not essentially bigger than any one of them. Like the Fefferman-Stein and Hardy-Littlewood maximal functions, their generic natures make them pointwise equivalent to each other and extremely easy to work with. Moreover, the intrinsic Lusin area function has the distinct advantage of being pointwise comparable at different cone openings, which is a property long known not to hold true for the classical Lusin area function (see Wilson [48, 49]).

More applications of intrinsic Littlewood–Paley functions were given by Wilson [50, 51] and Lerner [28, 29]. In particular, Wilson [49] proved that these intrinsic Littlewood–Paley functions are bounded on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ when $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$ (the class of Muckenhoupt weights). Recently, Wang [47] and Justin [14] also obtained the boundedness of these intrinsic Littlewood–Paley functions on weighted Morrey spaces.

Recall that the classical Morrey space $\mathcal{M}^{p,\kappa}(\mathbb{R}^n)$ was first introduced by Morrey in [35] to investigate the local behavior of solutions to second order elliptic partial differential equations. For $p \in [1,\infty)$ and $\kappa \in [0,1)$, a function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ is said to belong to the *Morrey space* $\mathcal{M}^{p,\kappa}(\mathbb{R}^n)$, if

$$\|f\|_{\mathcal{M}^{p,\kappa}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \left[\frac{1}{|B|^\kappa}\int_B |f(y)|^p\,dy\right]^{1/p} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$. The boundedness, on the Morrey space, of classical operators, such as the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator, was studied in [1, 10]. In particular, Komori and Shirai [24] first introduced the weighted Morrey space and obtained the boundedness of the above these classical operators on this space.

As a generalization of the space $\mathrm{BMO}(\mathbb{R}^n)$, the *Campanato space* $L^{p,\beta}(\mathbb{R}^n)$ for $\beta \in \mathbb{R}$ and $p \in [1,\infty)$, introduced by Campanato [9], was defined as the set of all locally integrable functions f such that

$$\|f\|_{L^{p,\beta}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} |B|^{-\beta}\left\{\frac{1}{|B|}\int_B |f(x)-f_B|^p\,dx\right\}^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and f_B denotes the average of f on B, namely,

$$f_B := \frac{1}{|B|}\int_B f(y)\,dy. \tag{1.1}$$

It is well known that, when $\kappa \in (0,1)$, $p \in [1,\infty)$ and $\beta = (\kappa-1)/p$, $\mathcal{M}^{p,\kappa}(\mathbb{R}^n)$ and $L^{p,\beta}(\mathbb{R}^n)$ coincide with equivalent norms (see, for example, [2]). Assuming the finiteness of the Littlewood–Paley functions on a positive measure set, Yabuta [52] first established the boundedness of the Littlewood–Paley functions on $L^{p,\beta}(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $\beta \in [-1/p, 1)$. Sun [45] further improved these results by assuming the finiteness of the Littlewood–Paley functions only on one point. Meng, Nakai and Yang [34] proved that some generalizations of the classical Littlewood–Paley functions, without assuming the regularity of their kernels, are bounded from $L^{p,\beta}(\mathbb{R}^n)$ to $L^{p,\beta}_*(\mathbb{R}^n)$ with $p \in [2,\infty)$ and $\beta \in [-1/p, 0]$, where $L^{p,\beta}_*(\mathbb{R}^n)$ is a proper subspace of $L^{p,\beta}(\mathbb{R}^n)$. This result, which was proved in [34] to be true even on spaces of homogeneous type in the sense of Coifman and Weiss (see [11]), further improves the result of Yabuta [52] and Sun [45].

On the other hand, Birnbaum-Orlicz [4] and Orlicz [39] introduced the Orlicz space, which is a natural generalization of $L^p(\mathbb{R}^n)$. Let φ be a growth function (see Definition 2.1 below for its definition). Recently, Ky [26] introduced a new *Musielak–Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$, which generalizes both the Orlicz-Hardy space (see, for example, [21, 46]) and the weighted Hardy space (see, for example, [16, 17, 25, 33, 44]). Moreover, characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of Littlewood–Paley functions (see [19, 30]) and the intrinsic ones (see [32]) were also obtained. As the dual space of $H^\varphi(\mathbb{R}^n)$, the *Musielak–Orlicz Campanato space* $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ with $q \in [1,\infty)$ was introduced in [31], in which some characterizations of $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ were also established. Recall that Musielak–Orlicz functions are the natural generalization of Orlicz functions that may vary in the spatial variables; see, for example, [36]. The motivation to study function spaces of Musielak–Orlicz type comes from their wide applications in physics and mathematics (see, for example, [6, 7, 8, 38, 26]). In particular, some special Musielak–Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [7, 8]), and the endpoint estimates for the div-curl lemma and the commutators of singular integral operators (see [5, 7, 27, 40]).

In this article, we introduce the Musielak–Orlicz Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ and the weighted Orlicz–Morrey space $M^{\Phi,\phi}_w(\mathbb{R}^n)$, and obtain the boundedness, respectively, on these spaces of intrinsic Littlewood–Paley functions and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions. Moreover, we also obtain the boundedness of intrinsic Littlewood–Paley functions on the Musielak–Orlicz Campanato space $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ which was introduced in [31].

To be precise, this article is organized as follows.

In Section 2, for a growth function φ and a nondecreasing function ϕ, we introduce the Musielak–Orlicz Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ and obtain the boundedness on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ of the intrinsic Lusin area function S_α, the intrinsic g-function g_α, the intrinsic g^*_λ-function $g^*_{\lambda,\alpha}$ with $\alpha \in (0,1]$ and $\lambda \in (\min\{\max\{3, p_1\}, 3+2\alpha/n\}, \infty)$ and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions. To this end, we first introduce an assistant function $\widetilde{\psi}$ and establish some estimates, respect to φ and $\widetilde{\psi}$, which play key roles in the proofs (see Lemma 2.8 below). Another key tool needed is a Musielak–Orlicz type interpolation theorem proved in [30]. We point out that, in [47], Wang established the boundedness of $g^*_{\lambda,\alpha}$ and $[b, g^*_{\lambda,\alpha}]$ on the weighted Morrey space $\mathcal{M}^{p,\kappa}_w(\mathbb{R}^n)$ with $\lambda > \max\{3, p\}$. This corresponds to the case when

$$\varphi(x,t) := w(x)t^p \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in [0,\infty) \tag{1.2}$$

with $w \in A_p(\mathbb{R}^n)$ and $p \in (1,\infty)$ of Theorem 2.15 and Proposition 2.20 below, in which, even for this special case, we also improve the range of $\lambda > p$ in [47] to a wider range $\lambda > 3 + 2\alpha/n$ when $p > 3 + 2\alpha/n$.

In Section 3, let $\Phi : [0,\infty) \to [0,\infty)$ be nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ when $t > 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$, $w \in A_\infty(\mathbb{R}^n)$ and $\phi : (0,\infty) \to (0,\infty)$ be nonincreasing. In this section, motivated by Nakai [37], we first introduce the weighted Orlicz–Morrey space $M_w^{\Phi,\phi}(\mathbb{R}^n)$ and obtain the boundedness on $M_w^{\Phi,\phi}(\mathbb{R}^n)$ of intrinsic Littlewood–Paley functions and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions.

In Section 4, for $q \in [1,\infty)$, the boundedness of the aforementioned intrinsic Littlewood–Paley functions on the Musielak–Orlicz Campanato space $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, which was introduced in [31], is also established. To be precise, following the ideas of [20] and [34], we first introduce a subspace $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)$ of $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ and prove that the intrinsic Littlewood–Paley functions are bounded from $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ to $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)$ which further implies that the intrinsic Littlewood–Paley functions are bounded on $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$. Even when

$$\varphi(x,t) := t^p \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in (0,\infty), \tag{1.3}$$

with $q \in (1,\infty)$ and $p \in (n/(n+1), q/(q-1)]$, these results are new.

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and by χ_E its *characteristic function*. For $p \in [1,\infty]$, we denote by p' its conjugate number, namely, $1/p + 1/p' = 1$. Also, let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

2. Boundedness of intrinsic Littlewood–Paley functions and their commutators on Musielak–Orlicz Morrey spaces

In this section, we introduce the Musielak–Orlicz Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ and establish the boundedness on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ of intrinsic Littlewood–Paley functions and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions. We begin with recalling the definition of growth functions which were first introduced by Ky [26].

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0,\infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$. We point out that, different from the classical Orlicz functions, the *Orlicz functions in this article may not be convex*. The function Φ is said to be of *upper type* p (resp. *lower type* p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $t \in [1,\infty)$ (resp. $t \in [0,1]$) and $s \in [0,\infty)$,

$$\Phi(st) \le Ct^p\Phi(s).$$

For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper type* p (resp. *uniformly lower type* p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0,1]$),

$$\varphi(x,st) \le Cs^p\varphi(x,t).$$

The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1, \infty)$,

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{ \int_B [\varphi(y,t)]^{-q'/q}\,dy \right\}^{q/q'} < \infty,$$

where $1/q + 1/q' = 1$, or, when $q = 1$,

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \left(\operatorname*{ess\,sup}_{y\in B} [\varphi(y,t)]^{-1} \right) < \infty.$$

Here the first supremums are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$. In particular, when $\varphi(x, t) := w(x)$ for all $x \in \mathbb{R}^n$, where w is a weight function, $\mathbb{A}_q(\mathbb{R}^n)$ is just the classical $A_q(\mathbb{R}^n)$ weight class of Muckenhoupt.

Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

Now we recall the notion of growth functions.

Definition 2.1. A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a *growth function*, if the following conditions are satisfied:

- (i) φ is a *Musielak–Orlicz function*, namely,
  - (i)$_1$ the function $\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  - (i)$_2$ the function $\varphi(\cdot, t)$ is a measurable function for all $t \in [0, \infty)$.
- (ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.
- (iii) φ is of uniformly lower type p_0 and of uniformly upper type p_1, where $0 < p_0 \le p_1 < \infty$.

Remark 2.2. (i) The notion of growth functions here is slightly different from [26]. We only need $0 < p_0 \le p_1 < \infty$ here, however, in [26], $p_0 \in (0, 1]$ and $p_1 = 1$.

(ii) By ii) of [26, Lemma 4.1], without loss of generality, we may assume that, for all $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is continuous and strictly increasing. Otherwise, we may replace φ by another equivalent growth function $\widetilde{\varphi}$ which is continuous and strictly increasing.

Throughout the whole paper, we *always assume that* φ *is a growth function* as in Definition 2.1 and, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0, \infty)$, we *denote* $\int_E \varphi(x, t)\,dx$ *by* $\varphi(E, t)$.

The *Musielak–Orlicz space* $L^\varphi(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx < \infty$ with the *Luxembourg norm* (or *Luxembourg-Nakano norm*)

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\mu \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\mu}\right) dx \le 1\right\}.$$

If φ is as in (1.2) with $p \in (0, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then $L^\varphi(\mathbb{R}^n)$ coincides with the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$.

Now, we introduce the Musielak–Orlicz Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$.

Definition 2.3. Let φ be a growth function and $\phi : [0,\infty) \to [0,\infty)$ be nondecreasing. A locally integrable function f on $\mathbb{R}^n$ is said to belong to the *Musielak–Orlicz Morrey space* $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$, if

$$\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \phi(\varphi(B,1))\|f\|_{\varphi,B} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and

$$\|f\|_{\varphi,B} := \inf\left\{\mu \in (0,\infty) : \ \frac{1}{\varphi(B,1)}\int_B \varphi\left(x, \frac{|f(x)|}{\mu}\right) dx \le 1\right\}.$$

Remark 2.4. (i) We first claim that $\|\cdot\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}$ is a quasi-norm. Indeed, since φ is of uniformly lower type p_0 and of uniformly upper type p_1 with $0 < p_0 \le p_1 < \infty$, we see that, for any $x \in \mathbb{R}^n$ and $0 < a \le b$,

$$\varphi(x, a+b) \lesssim \left(\frac{a+b}{2b}\right)^{p_0} \varphi(x, 2b) \lesssim 2^{p_1}\varphi(x,b) \lesssim \varphi(x,a) + \varphi(x,b),$$

which further implies that, for any ball $B \subset \mathbb{R}^n$ and $f,\ g \in L^1_{\rm loc}(\mathbb{R}^n)$ with $\|f\|_{\varphi,B}+\|g\|_{\varphi,B} \neq 0$,

$$\begin{aligned}&\frac{1}{\varphi(B,1)}\int_B \varphi\left(x, \frac{|f(x)+g(x)|}{\|f\|_{\varphi,B}+\|g\|_{\varphi,B}}\right) dx\\ &\quad\lesssim \frac{1}{\varphi(B,1)}\int_B \left[\varphi\left(x, \frac{|f(x)|}{\|f\|_{\varphi,B}+\|g\|_{\varphi,B}}\right) + \varphi\left(x, \frac{|g(x)|}{\|f\|_{\varphi,B}+\|g\|_{\varphi,B}}\right)\right] dx \lesssim 1\end{aligned}$$

and hence, by $p_0 \in (0,\infty)$,

$$\|f+g\|_{\varphi,B} \lesssim \|f\|_{\varphi,B} + \|g\|_{\varphi,B},$$

where the implicit positive constant is independent of B. This further implies that $\|\cdot\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}$ is a quasi-norm, namely, for any $f,\ g \in \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$, there exists a constant $\kappa \in [1,\infty)$ such that

$$\|f+g\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \le \kappa\left[\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} + \|g\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}\right].$$

Thus, the claim holds true.

Moreover, from the claim and the Aoki–Rolewicz theorem in [3, 42], it follows that there exists a quasi-norm $|||\cdot|||$ on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ and $\gamma \in (0,1]$ such that, for all $f \in \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$, $|||f||| \sim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}$ and, for any sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\left|\left|\left|\sum_{j\in\mathbb{N}} f_j\right|\right|\right|^\gamma \le \sum_{j\in\mathbb{N}} |||f_j|||^\gamma,$$

which is needed later.

(ii) If φ is as in (1.3) with $p \in (1,\infty)$ and $\phi(t) := t^s$ for all $t \in [0,\infty)$ with $s \in (0, 1/p)$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ coincides with the classical Morrey space $\mathcal{M}^{p,1-sp}(\mathbb{R}^n)$.

(iii) If $\varphi(x,t) := \Phi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$ with Φ being an Orlicz function, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ coincides with the Orlicz–Morrey space in [43].

(iv) If φ is as in (1.2) with $p \in (1,\infty), w \in A_p(\mathbb{R}^n)$ and $\phi(t)$ is as in (ii), then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ coincides with the weighted Morrey space $\mathcal{M}^{p,1-sp}_w(\mathbb{R}^n)$ in [47] (Observe that the weighted Morrey space $\mathcal{M}^{p,1-sp}_w(\mathbb{R}^n)$ was denoted by another notation in [47]).

Now we recall the notions of intrinsic Littlewood–Paley functions introduced by Wilson [48].

For $\alpha \in (0,1]$, let $\mathcal{C}_\alpha(\mathbb{R}^n)$ be the family of functions θ, defined on $\mathbb{R}^n$, such that $\operatorname{supp} \theta \subset \{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \theta(x)\,dx = 0$ and, for all $x_1, x_2 \in \mathbb{R}^n$,

$$|\theta(x_1) - \theta(x_2)| \le |x_1 - x_2|^\alpha.$$

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, let

$$A_\alpha(f)(y,t) := \sup_{\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)} |f * \theta_t(y)| = \sup_{\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} \theta_t(y-z) f(z)\,dz \right|.$$

For all $\alpha \in (0,1]$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the *intrinsic Littlewood–Paley g-function* $g_\alpha(f)$, the *intrinsic Lusin area function* $S_\alpha(f)$ and the *intrinsic Littlewood–Paley g^*_λ-function* $g^*_{\lambda,\alpha}(f)$ of f are, respectively, defined by setting, for all $x \in \mathbb{R}^n$,

$$g_\alpha(f)(x) := \left\{ \int_0^\infty [A_\alpha(f)(x,t)]^2 \frac{dt}{t} \right\}^{1/2},$$

$$S_\alpha(f)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n:\ |y-x|<t\}} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}$$

and

$$g^*_{\lambda,\alpha}(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

Let $\beta \in (0,\infty)$. We also introduce the varying-aperture version $S_{\alpha,\beta}(f)$ of $S_\alpha(f)$ by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$S_{\alpha,\beta}(f)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n:\ |y-x|<\beta t\}} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

To obtain the boundedness of all the intrinsic Littlewood–Paley functions on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$, we need to introduce an auxiliary function $\widetilde{\psi}$ and establish some technical lemmas first.

Let φ be a growth function with $1 \le p_0 \le p_1 < \infty$. For all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, let

$$\psi(x,t) := \varphi(x,t)/\varphi(x,1).$$

Obviously, for all $x \in \mathbb{R}^n$, $\psi(x,\cdot)$ is an Orlicz function and, for all $t \in [0,\infty)$, $\psi(\cdot,t)$ is measurable. For all $x \in \mathbb{R}^n$ and $s \in [0,\infty)$, the *complementary function of ψ* is defined by

$$\widetilde{\psi}(x,s) := \sup_{t>0} \{st - \psi(x,t)\} \tag{2.1}$$

(see [36, Definition 13.7]). On the complementary function $\widetilde{\psi}$, we have the following properties.

Lemma 2.5. *Let φ be as in Definition 2.1 with $1 \le p_0 \le p_1 < \infty$ and $\widetilde{\psi}$ as in (2.1).*

(i) *If* $1 \le p_0 \le p_1 < \infty$, *then there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$,

$$0 \le \widetilde{\psi}(x,1) \le C.$$

(ii) *If* $1 < p_0 \le p_1 < \infty$, *then* $\widetilde{\psi}$ *is a growth function of uniformly lower type* p_1' *and uniformly upper type* p_0', *where* $1/p_0 + 1/p_0' = 1 = 1/p_1 + 1/p_1'$.

Proof. To show (i), for all $x \in \mathbb{R}^n$, since there exist positive constants C_0, C_1 such that, for any $t \in (0,1]$, $\varphi(x,1) \le C_1\varphi(x,t)/t^{p_1}$ and, for any $t \in (1,\infty)$, $\varphi(x,1) \le C_0\varphi(x,t)/t^{p_0}$, it follows that

$$\begin{aligned}\widetilde{\psi}(x,1) &= \sup_{t\in(0,\infty)}\{t-\psi(x,t)\} = \sup_{t\in(0,\infty)}\left\{t-\frac{\varphi(x,t)}{\varphi(x,1)}\right\}\\ &\le \sup_{t\in(0,1]}\{t-t^{p_1}/C_1\} + \sup_{t\in(1,\infty)}\{t-t^{p_0}/C_0\} \lesssim 1.\end{aligned}$$

Thus, (i) holds true.

To show (ii), for any $\lambda \in [1,\infty)$, C_0 as in the proof of (i) and $l \in (0,\infty)$, let $m := (\frac{1}{\lambda C_0})^{\frac{1}{p_0-1}}$ and $s := \frac{l}{C_0 m^{p_0}}$. Without loss of generality, we may assume $C_0 \ge 1$. Then, we have $m \in (0,1]$, $s \in (0,\infty)$ and

$$\begin{aligned}\widetilde{\psi}(x,\lambda l) = \widetilde{\psi}(x,ms) &= \sup_{t>0}\{mst-\psi(x,t)\} \le \sup_{t>0}\left\{smt-\frac{\psi(x,mt)}{C_0m^{p_0}}\right\}\\ &= \frac{1}{C_0m^{p_0}}\sup_{t>0}\{C_0m^{p_0}smt-\psi(x,mt)\} = \frac{1}{C_0m^{p_0}}\widetilde{\psi}(x,C_0m^{p_0}s)\\ &= \frac{\lambda^{p_0'}}{C_0^{1-p_0'}}\widetilde{\psi}(x,l),\end{aligned}$$

which implies that $\widetilde{\psi}$ is of uniformly upper type p_0'. By a similar argument, we also see that $\widetilde{\psi}$ is of uniformly lower type p_1', which completes the proof of (ii) and hence Lemma 2.5. □

For any ball $B \subset \mathbb{R}^n$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|g\|_{\widetilde{\psi},B} := \inf\left\{\mu \in (0,\infty):\ \frac{1}{\varphi(B,1)}\int_B \widetilde{\psi}\left(x,\frac{|g(x)|}{\mu}\right)\varphi(x,1)\,dx \le 1\right\}.$$

For φ and $\widetilde{\psi}$, we also have the following properties.

Lemma 2.6. *Let* $\widetilde{C}$ *be a positive constant. Then there exists a positive constant* C *such that*

(i) *for any ball* $B \subset \mathbb{R}^n$ *and* $\mu \in (0,\infty)$,

$$\frac{1}{\varphi(B,1)}\int_B \varphi\left(x,\frac{|f(x)|}{\mu}\right)dx \le \widetilde{C}$$

implies that $\|f\|_{\varphi,B} \le C\mu$;

(ii) *for any ball* $B \subset \mathbb{R}^n$ *and* $\mu \in (0,\infty)$,

$$\frac{1}{\varphi(B,1)}\int_B \widetilde{\psi}\left(x,\frac{|f(x)|}{\mu}\right)\varphi(x,1)\,dx \le \widetilde{C}$$

implies that $\|f\|_{\widetilde{\psi},B} \le C\mu$.

The proof of Lemma 2.6 is similar to that of [26, Lemma 4.3], the details being omitted.

Lemma 2.7. *Let* φ *be a growth function with* $1 < p_0 \le p_1 < \infty$*. Then, for any ball* $B \subset \mathbb{R}^n$ *and* $\|f\|_{\varphi,B} \neq 0$*, it holds true that*

$$\frac{1}{\varphi(B,1)} \int_B \varphi\left(x, \frac{|f(x)|}{\|f\|_{\varphi,B}}\right) dx = 1$$

and, for all $\|f\|_{\widetilde{\psi},B} \neq 0$*, it holds true that*

$$\frac{1}{\varphi(B,1)} \int_B \widetilde{\psi}\left(x, \frac{|f(x)|}{\|f\|_{\widetilde{\psi},B}}\right) \varphi(x,1)\, dx = 1.$$

The proof of Lemma 2.7 is similar to that of [26, Lemma 4.2], the details being omitted.

The following lemma is a generalized Hölder inequality with respect to φ.

Lemma 2.8. *If* φ *is a growth function as in Definition 2.1, then, for any ball* $B \subset \mathbb{R}^n$ *and* $f,\, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*,*

$$\frac{1}{\varphi(B,1)} \int_B |f(x)||g(x)|\varphi(x,1)\, dx \le 2\|f\|_{\varphi,B}\|g\|_{\widetilde{\psi},B}.$$

Proof. By (2.1), we know that, for any $x \in \mathbb{R}^n$ and ball $B \subset \mathbb{R}^n$,

$$\frac{|f(x)|}{\|f\|_{\varphi,B}} \frac{|g(x)|}{\|g\|_{\widetilde{\psi},B}} \le \varphi\left(x, \frac{|f(x)|}{\|f\|_{\varphi,B}}\right) + \widetilde{\psi}\left(x, \frac{|g(x)|}{\|g\|_{\widetilde{\psi},B}}\right) \varphi(x,1),$$

which, together with Lemma 2.7, implies that

$$\begin{aligned}\frac{1}{\varphi(B,1)} \int_B \frac{|f(x)|}{\|f\|_{\varphi,B}} \frac{|g(x)|}{\|g\|_{\widetilde{\psi},B}} \varphi(x,1)\, dx &\le \frac{1}{\varphi(B,1)} \int_B \varphi\left(x, \frac{|f(x)|}{\|f\|_{\varphi,B}}\right) dx \\ &\quad + \frac{1}{\varphi(B,1)} \int_B \widetilde{\psi}\left(x, \frac{|g(x)|}{\|g\|_{\widetilde{\psi},B}}\right) \varphi(x,1)\, dx \\ &\le 2.\end{aligned}$$

Thus,

$$\frac{1}{\varphi(B,1)} \int_B |f(x)||g(x)|\varphi(x,1)\, dx \lesssim \|f\|_{\varphi,B}\|g\|_{\widetilde{\psi},B},$$

which completes the proof of Lemma 2.8. □

The following Lemmas 2.9 and 2.10 are, respectively, [30, Lemma 2.2] and [30, Theorem 2.7].

Lemma 2.9. (i) $\mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ *for* $1 \le p \le q < \infty$.

(ii) *If* $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ *with* $p \in (1,\infty)$*, then there exists* $q \in (1,p)$ *such that* $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$.

Lemma 2.10. *Let $\widetilde{p}_0, \widetilde{p}_1 \in (0,\infty)$, $\widetilde{p}_0 < \widetilde{p}_1$ and φ be a growth function with uniformly lower type p_0 and uniformly upper type p_1. If $0 < \widetilde{p}_0 < p_0 \le p_1 < \widetilde{p}_1 < \infty$ and T is a sublinear operator defined on $L^{\widetilde{p}_0}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{\widetilde{p}_1}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ satisfying that, for $i \in \{1,2\}$, all $\alpha \in (0,\infty)$ and $t \in (0,\infty)$,*

$$\varphi(\{x \in \mathbb{R}^n : \ |Tf(x)| > \alpha\}, t) \le C_i \alpha^{-\widetilde{p}_i} \int_{\mathbb{R}^n} |f(x)|^{\widetilde{p}_i} \varphi(x,t)\,dx,$$

where C_i is a positive constant independent of f, t and α. Then T is bounded on $L^\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that, for all $f \in L^\varphi(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi(x, |Tf(x)|)\,dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx.$$

By applying Lemmas 2.9 and 2.10, we have the following boundedness of S_α and $g^*_{\lambda,\alpha}$ on $L^\varphi(\mathbb{R}^n)$.

Proposition 2.11. *Let φ be a growth function with $1 < p_0 \le p_1 < \infty$, $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$, $\alpha \in (0,1]$ and $\lambda > \min\{\max\{2, p_1\}, 3 + 2\alpha/n\}$. Then there exists a positive constant C such that, for all $f \in L^\varphi(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi(x, S_\alpha(f)(x))\,dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx$$

and

$$\int_{\mathbb{R}^n} \varphi(x, g^*_{\lambda,\alpha}(f)(x))\,dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx.$$

Proof. For $\alpha \in (0,1]$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, it was proved in [49, Theorem 7.2] that

$$\|S_\alpha(f)\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{L^p_w(\mathbb{R}^n)}.$$

Since $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$ and $p_0 \in (1,\infty)$, by Lemma 2.9(ii), there exists some $\widetilde{p}_0 \in (1, p_0)$ such that $\varphi \in \mathbb{A}_{\widetilde{p}_0}(\mathbb{R}^n)$ and hence, for all $t \in (0,\infty)$, it holds true that

$$\int_{\mathbb{R}^n} [S_\alpha(f)(x)]^{\widetilde{p}_0} \varphi(x,t)\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^{\widetilde{p}_0} \varphi(x,t)\,dx. \tag{2.2}$$

On the other hand, by the fact that, for any $\widetilde{p}_1 \in (p_1,\infty)$, $\varphi(x,t) \in \mathbb{A}_{\widetilde{p}_0}(\mathbb{R}^n) \subset \mathbb{A}_{\widetilde{p}_1}(\mathbb{R}^n)$ (see Lemma 2.9(i)), we have

$$\int_{\mathbb{R}^n} [S_\alpha(f)(x)]^{\widetilde{p}_1} \varphi(x,t)\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^{\widetilde{p}_1} \varphi(x,t)\,dx. \tag{2.3}$$

From (2.2), (2.3) and Lemma 2.10, we deduce that

$$\int_{\mathbb{R}^n} \varphi(x, S_\alpha(f)(x))\,dx \lesssim \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx. \tag{2.4}$$

For $g^*_{\lambda,\alpha}$, by the definition, we know that, for all $x \in \mathbb{R}^n$,

$$[g^*_{\lambda,\alpha}(f)(x)]^2 = \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \,\frac{dy\,dt}{t^{n+1}}$$

$$+\sum_{j=1}^{\infty}\int_0^{\infty}\int_{2^{j-1}t\le|x-y|<2^jt}\cdots$$

$$\lesssim [S_\alpha(f)(x)]^2+\sum_{j=1}^{\infty}2^{-j\lambda n}[\mathcal{S}_{\alpha,2^j}(f)(x)]^2.$$

Thus, for all $x\in\mathbb{R}^n$, it holds true that

$$g^*_{\lambda,\alpha}(f)(x)\lesssim S_\alpha(f)(x)+\sum_{j=1}^{\infty}2^{-j\lambda n/2}\mathcal{S}_{\alpha,2^j}(f)(x). \tag{2.5}$$

In [49, Exericise 6.2], Wilson proved that, for all $x\in\mathbb{R}^n$,

$$\mathcal{S}_{\alpha,2^j}(f)(x)\lesssim 2^{j(\frac{3n}{2}+\alpha)}S_\alpha(f)(x),$$

where the implicit positive constant depends only on n and α. Hence, for all $x\in\mathbb{R}^n$, if $\lambda>3+2\alpha/n$, we have

$$g^*_{\lambda,\alpha}(f)(x)\lesssim\left[1+\sum_{j=1}^{\infty}2^{-\frac{jn}{2}(\lambda-3-\frac{2\alpha}{n})}\right]S_\alpha(f)(x)\lesssim S_\alpha(f)(x),$$

which, together with (2.4) and the nondecreasing property of $\varphi(x,\cdot)$ for all $x\in\mathbb{R}^n$, implies that

$$\int_{\mathbb{R}^n}\varphi(x,g^*_{\lambda,\alpha}(f)(x))\,dx\lesssim\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx.$$

On the other hand, by [47, Lemmas 4.1, 4.2 and 4.3], we know that, for all $p\in(1,\infty)$, $w\in A_p(\mathbb{R}^n)$ and $j\in\mathbb{N}$,

$$\|\mathcal{S}_{\alpha,2^j}(f)\|_{L^p_w(\mathbb{R}^n)}\lesssim(2^{jn}+2^{jnp/2})\|f\|_{L^p_w(\mathbb{R}^n)},$$

which, together with (2.5), implies that, if $\lambda>\max\{2,\,p\}$, then

$$\|g^*_{\lambda,\alpha}(f)\|_{L^p_w(\mathbb{R}^n)}\lesssim\|f\|_{L^p_w(\mathbb{R}^n)}.$$

By this and Lemma 2.10, we further see that, if $\lambda>\max\{2,p_1\}$, then

$$\int_{\mathbb{R}^n}\varphi(x,g^*_{\lambda,\alpha}(f)(x))\,dx\lesssim\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx,$$

which completes the proof of Proposition 2.11. □

One of the main results of this section is as follows.

Theorem 2.12. *Let $\alpha\in(0,1]$, φ be a growth function with $1<p_0\le p_1<\infty$, $\varphi\in\mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi:(0,\infty)\to(0,\infty)$ be nondecreasing. If there exists a positive constant C such that, for all $r\in(0,\infty)$,*

$$\int_r^{\infty}\frac{1}{\phi(t)t}\,dt\le C\frac{1}{\phi(r)},$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f\in\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\|S_\alpha(f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}\le\widetilde{C}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

Proof. Let $B := B(x_0, r_B)$ be any ball of $\mathbb{R}^n$, where $x_0 \in \mathbb{R}^n$ and $r_B \in (0,\infty)$. Decompose

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Since, for any $\alpha \in (0,1]$, S_α is sublinear, we see that, for all $x \in B$,

$$S_\alpha(f)(x) \le S_\alpha(f_1)(x) + S_\alpha(f_2)(x).$$

Let $\mu := \|f\|_{\varphi,2B} \neq 0$. By Proposition 2.11 and Lemma 2.7, we conclude that

$$\begin{aligned}\frac{1}{\varphi(B,1)}\int_B \varphi\left(x, \frac{S_\alpha(f_1)(x)}{\mu}\right)dx &\lesssim \frac{1}{\varphi(B,1)}\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f_1(x)|}{\mu}\right)dx \\ &\sim \frac{1}{\varphi(B,1)}\int_{2B} \varphi\left(x, \frac{|f(x)|}{\mu}\right)dx \lesssim 1.\end{aligned}$$

From this and Lemma 2.6(i), we deduce that $\|S_\alpha(f_1)\|_{\varphi,B} \lesssim \|f\|_{\varphi,2B}$. Therefore,

$$\begin{aligned}\phi(\varphi(B,1))\|S_\alpha(f_1)\|_{\varphi,B} &\lesssim \phi(\varphi(B,1))\|f\|_{\varphi,2B} \\ &\lesssim \frac{\phi(\varphi(B,1))}{\phi(\varphi(2B,1))}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.\end{aligned} \tag{2.6}$$

Next, we turn to estimate $S_\alpha(f_2)$. For any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$ and

$$(y,t) \in \Gamma(x) := \{(y,t) \in \mathbb{R}^n \times (0,\infty) :\ |y-x| < t\},$$

we have

$$\begin{aligned}\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} |f_2 * \theta_t(y)| &= \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{(2B)^\complement} \theta_t(y-z)f(z)\,dz\right| \\ &\le \sum_{k=1}^\infty \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{2^{k+1}B\setminus 2^kB} \theta_t(y-z)f(z)\,dz\right|.\end{aligned}$$

For any $k \in \mathbb{N}$, $x \in B$, $(y,t) \in \Gamma(x)$ and $z \in (2^{k+1}B\setminus 2^kB) \cap B(y,t)$, it holds true that

$$2t > |x-y| + |y-z| \ge |x-z| \ge |z-x_0| - |x-x_0| > 2^{k-1}r_B. \tag{2.7}$$

By this, the fact that $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded and the Minkowski inequality, we know that, for all $x \in B$,

$$\begin{aligned}&S_\alpha(f_2)(x) \\ &\quad\le \left\{\int_0^\infty \int_{|x-y|<t} \left[\sum_{k=1}^\infty \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{2^{k+1}B\setminus 2^kB} \theta_t(y-z)f(z)\,dz\right|\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\ &\quad\lesssim \sum_{k=1}^\infty \left\{\int_{2^{k-2}r_B}^\infty \int_{|x-y|<t} \left[t^{-n}\int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\ &\quad\lesssim \sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz \left(\int_{2^{k-2}r_B}^\infty \frac{dt}{t^{2n+1}}\right)^{1/2} \\ &\quad\lesssim \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|}\int_{2^{k+1}B\setminus 2^kB} |f(z)|\,dz.\end{aligned} \tag{2.8}$$

From this and Lemma 2.8, it follows that, for all $x \in B$,

$$S_\alpha(f_2)(x) \lesssim \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|} \|f\|_{\varphi,2^{k+1}B} \left\| \frac{1}{\varphi(\cdot,1)} \right\|_{\widetilde{\psi},2^{k+1}B}. \tag{2.9}$$

By $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n) \subset \mathbb{A}_{p_1}(\mathbb{R}^n)$ and Lemma 2.5, we conclude that

$$\begin{aligned}
&\frac{1}{\varphi(2^{k+1}B,1)} \int_{2^{k+1}B} \widetilde{\psi}\left(x, \frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|\varphi(x,1)}\right) \varphi(x,1)\,dx \\
&\quad \lesssim \frac{1}{\varphi(2^{k+1}B,1)} \int_{2^{k+1}B} \left\{ \left[\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|\varphi(x,1)}\right]^{p_1'} + \left[\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|\varphi(x,1)}\right]^{p_0'} \right\} \varphi(x,1)\,dx \\
&\quad \sim \left[\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \varphi(x,1)\,dx\right]^{p_0'-1} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} [\varphi(x,1)]^{1-p_0'}\,dx \\
&\qquad + \left[\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \varphi(x,1)\,dx\right]^{p_1'-1} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} [\varphi(x,1)]^{1-p_1'}\,dx \lesssim 1.
\end{aligned}$$

From this and Lemma 2.6(ii), we deduce that

$$\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|} \left\| \frac{1}{\varphi(\cdot,1)} \right\|_{\widetilde{\psi},2^{k+1}B} \lesssim 1, \tag{2.10}$$

which, together with (2.9), further implies that, for all $x \in B$,

$$\begin{aligned}
\phi(\varphi(B,1))S_\alpha(f_2)(x) &\lesssim \sum_{k=1}^{\infty} \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} \phi(\varphi(2^{k+1}B,1))\|f\|_{\varphi,2^{k+1}B} \\
&\lesssim \sum_{k=1}^{\infty} \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.
\end{aligned} \tag{2.11}$$

Recall that, for $r \in (1,\infty)$, a weight function w is said to satisfy the *reverse Hölder inequality,* denoted by $w \in \mathrm{RH}_r(\mathbb{R}^n)$, if there exists a positive constant C such that, for every ball $B \subset \mathbb{R}^n$,

$$\left\{\frac{1}{|B|} \int_B [w(x)]^r\,dx\right\}^{1/r} \le C \frac{1}{|B|} \int_B w(x)\,dx.$$

Since $\varphi(\cdot,1) \in A_{p_0}(\mathbb{R}^n)$, we know that there exists some $r \in (1,\infty)$ such that $\varphi(\cdot,1) \in \mathrm{RH}_r(\mathbb{R}^n)$, which, together with [18, p. 109], further implies that there exists a positive constant $\widetilde{C}$ such that, for any ball $B \subset \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$\frac{\varphi(2^kB,1)}{\varphi(2^{k+1}B,1)} \le \widetilde{C} \left(\frac{|2^kB|}{|2^{k+1}B|}\right)^{(r-1)/r}.$$

By choosing $j_0 \in (\frac{r}{n(r-1)} \log \widetilde{C}, \infty) \cap \mathbb{N}$, we see that, for all $k \in \mathbb{N}$,

$$\frac{\varphi(2^{(k+1)j_0}B,1)}{\varphi(2^{kj_0}B,1)} \ge 2^{nj_0(r-1)/r}/\widetilde{C} > 1,$$

which further implies that

$$\log\left(\frac{\varphi(2^{(k+1)j_0}B,1)}{\varphi(2^{kj_0}B,1)}\right)\gtrsim 1. \tag{2.12}$$

By (2.12) and the assumptions of ϕ, we know that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))}&\le\sum_{l=0}^{\infty}\sum_{i=lj_0+1}^{(l+1)j_0}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{i}B,1))}\\&\lesssim\sum_{i=1}^{j_0}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{i}B,1))}+j_0\sum_{l=1}^{\infty}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{lj_0}B,1))}\\&\lesssim 1+\sum_{l=1}^{\infty}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{lj_0}B,1))}\int_{\varphi(2^{(l-1)j_0}B,1)}^{\varphi(2^{lj_0}B,1)}\frac{dt}{t}\\&\lesssim 1+\phi(\varphi(B,1))\sum_{l=1}^{\infty}\int_{\varphi(2^{(l-1)j_0}B,1)}^{\varphi(2^{lj_0}B,1)}\frac{dt}{\phi(t)t}\\&\lesssim 1+\phi(\varphi(B,1))\int_{\varphi(B,1)}^{\infty}\frac{1}{\phi(t)t}\,dt\lesssim 1.\end{aligned} \tag{2.13}$$

From this and (2.11), we deduce that, for all $x\in B$,

$$\phi(\varphi(B,1))S_\alpha(f_2)(x)\lesssim\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \tag{2.14}$$

Therefore,

$$\frac{1}{\varphi(B,1)}\int_B\varphi\left(x,\frac{\phi(\varphi(B,1))S_\alpha(f_2)(x)}{\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}}\right)dx\lesssim 1,$$

which, together with Lemma 2.6(i), further implies that

$$\phi(\varphi(B,1))\|S_\alpha(f_2)\|_{\varphi,B}\lesssim\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

This, combined with (2.6) and Remark 2.4(i), finishes the proof of Theorem 2.12. □

For a growth function φ and a function $\phi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$, the *space* $\widetilde{M}^{\varphi,\phi}(\mathbb{R}^n)$ is defined by the same way as Definition 2.3, via using $\phi(c_B,\varphi(B,1))$ instead of $\phi(\varphi(B,1))$, where c_B is the center of the ball B. Then, by an argument similar to that used in the proof of Theorem 2.12, we have the following boundedness of S_α on $\widetilde{M}^{\varphi,\phi}(\mathbb{R}^n)$, the details being omitted.

Theorem 2.13. *Let $\alpha\in(0,1]$, φ be a growth function with $1<p_0\le p_1<\infty$, and $\varphi\in\mathbb{A}_{p_0}(\mathbb{R}^n)$. If there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $0<r\le s<\infty$,*

$$\int_r^\infty\frac{1}{\phi(x,t)t}\,dt\le C\frac{1}{\phi(x,r)}\quad\textit{and}\quad\phi(x,r)\le C\phi(x,s),$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f\in\widetilde{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\|S_\alpha(f)\|_{\widetilde{M}^{\varphi,\phi}(\mathbb{R}^n)}\le\widetilde{C}\|f\|_{\widetilde{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

For example, let $\phi(x,r) := r^{\lambda(x)}$ for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ and

$$\inf_{x\in\mathbb{R}^n} \lambda(x) > 0.$$

Then ϕ satisfies the assumptions of Theorem 2.13.

Observe that, for all $x \in \mathbb{R}^n$, $g_\alpha(f)(x)$ and $S_\alpha(f)(x)$ are pointwise comparable (see [48, p. 774]), which, together with Theorem 2.12, immediately implies the following conclusion, the details being omitted.

Corollary 2.14. *Let $\alpha \in (0,1]$, φ be a growth function with $1 < p_0 \le p_1 < \infty$, $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi : [0,\infty) \to [0,\infty)$ be nondecreasing. If there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_r^\infty \frac{1}{\phi(t)t}\,dt \le C\frac{1}{\phi(r)},$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f \in \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\|g_\alpha(f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \le \widetilde{C}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

Similarly, there exists a corollary similar to Corollary 2.14 of Theorem 2.13, the details being omitted.

Theorem 2.15. *Let $\alpha \in (0,1]$, φ be a growth function with $1 < p_0 \le p_1 < \infty$, $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi : [0,\infty) \to [0,\infty)$ be nondecreasing. If $\lambda > \min\{\max\{3, p_1\}, 3 + 2\alpha/n\}$ and there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_r^\infty \frac{1}{\phi(t)t}\,dt \le C\frac{1}{\phi(r)},$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f \in \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\|g^*_{\lambda,\alpha}(f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \le \widetilde{C}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

Proof. Fix any ball $B := B(x_0, r_B) \subset \mathbb{R}^n$, with $x_0 \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and decompose

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Then, for all $x \in B$,

$$g^*_{\lambda,\alpha}(f)(x) \le g^*_{\lambda,\alpha}(f_1)(x) + g^*_{\lambda,\alpha}(f_2)(x).$$

Similar to the estimate for f_1 in the proof of Theorem 2.12, by Proposition 2.11 and Lemmas 2.6(i) and 2.7, if $\lambda > \min\{\max\{2, p_1\}, 3 + 2\alpha/n\}$, we have

$$\phi(\varphi(B,1))\|g^*_{\lambda,\alpha}(f_1)\|_{\varphi,B} \lesssim \frac{\phi(\varphi(B,1))}{\phi(\varphi(2B,1))}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \quad (2.15)$$

Next, replacing f in (2.5) by f_2, we know that, for all $x \in B$,

$$g^*_{\lambda,\alpha}(f_2)(x) \lesssim S_\alpha(f_2)(x) + \sum_{j=1}^\infty 2^{-j\lambda n/2}\mathcal{S}_{\alpha,2^j}(f_2)(x). \quad (2.16)$$

Let k, $j \in \mathbb{N}$. For any $x \in B$,

$$(y,t) \in \Gamma_{2^j}(x) := \{(y,t) \in \mathbb{R}^n \times [0,\infty) : \ |y - x| < 2^j t\}$$

and $z \in (2^{k+1}B \backslash 2^k B) \cap B(y,t)$, we have

$$t + 2^j t > |x-y| + |y-z| \geq |x-z| \geq |z-x_0| - |x-x_0| > 2^{k-1} r_B.$$

From this, the Minkowski inequality and the fact that $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, it follows that, for all $x \in B$,

$$\begin{aligned}
&\mathcal{S}_{\alpha,2^j}(f_2)(x) \\
&\quad \leq \left\{ \int_0^\infty \int_{|x-y|<2^j t} \left[\sum_{k=1}^\infty \sup_{\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)} \left| \int_{2^{k+1}B \backslash 2^k B} \theta_t(y-z) f(z)\, dz \right| \right]^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2} \\
&\quad \lesssim \sum_{k=1}^\infty \left\{ \int_{2^{k-2-j} r_B}^\infty \int_{|x-y|<2^j t} \left| t^{-n} \int_{2^{k+1}B \backslash 2^k B} |f(z)|\, dz \right|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2} \\
&\quad \lesssim 2^{3jn/2} \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \backslash 2^k B} |f(z)|\, dz, \qquad (2.17)
\end{aligned}$$

which, together with Lemma 2.8, further implies that, for all $x \in B$,

$$\mathcal{S}_{\alpha,2^j}(f_2)(x) \lesssim 2^{3jn/2} \sum_{k=1}^\infty \frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|} \|f\|_{\varphi,2^{k+1}B} \left\| \frac{1}{\varphi(\cdot,1)} \right\|_{\widetilde{\psi},2^{k+1}B}.$$

By this, (2.10) and (2.13), we find that, for all $x \in B$,

$$\begin{aligned}
\phi(\varphi(B,1)) \mathcal{S}_{\alpha,2^j}(f_2)(x) &\lesssim 2^{3jn/2} \sum_{k=1}^\infty \phi(\varphi(B,1)) \|f\|_{\varphi,2^{k+1}B} \\
&\lesssim 2^{3jn/2} \sum_{k=1}^\infty \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \\
&\lesssim 2^{3jn/2} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)},
\end{aligned}$$

which further implies that

$$\frac{1}{\varphi(B,1)} \int_B \varphi\left(x, \frac{\phi(\varphi(B,1)) \mathcal{S}_{\alpha,2^j}(f_2)(x)}{2^{3jn/2} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}} \right) dx \lesssim 1.$$

From this and Lemma 2.6(i), we deduce that

$$\phi(\varphi(B,1)) \|\mathcal{S}_{\alpha,2^j}(f_2)\|_{\varphi,B} \lesssim 2^{3jn/2} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

Hence,

$$\|\mathcal{S}_{\alpha,2^j}(f_2)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \lesssim 2^{3jn/2} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)},$$

which, together with (2.16), Remark 2.4(i) and Theorem 2.12, further implies that there exists some $\gamma \in (0,1]$ such that, when $\lambda > 3$,

$$\begin{aligned}
\|g^*_{\lambda,\alpha}(f_2)\|^\gamma_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} &\lesssim \left\| S_\alpha(f_2) + \sum_{j=1}^\infty 2^{-j\lambda n/2} \mathcal{S}_{\alpha,2^j}(f_2) \right\|^\gamma \\
&\lesssim \|S_\alpha(f_2)\|^\gamma + \sum_{j=1}^\infty 2^{-j\gamma\lambda n/2} \left\| \mathcal{S}_{\alpha,2^j}(f_2) \right\|^\gamma
\end{aligned}$$

$$\sim \|S_\alpha(f_2)\|^\gamma_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} + \sum_{j=1}^\infty 2^{-j\gamma\lambda n/2} \left\|\mathcal{S}_{\alpha,2^j}(f_2)\right\|^\gamma_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}$$

$$\lesssim \|f\|^\gamma_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \left[1 + \sum_{j=1}^\infty 2^{-j\gamma(\lambda-3)n/2}\right] \lesssim \|f\|^\gamma_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

This, combined with (2.15) and Remark 2.4(i), finishes the proof of Theorem 2.15. □

The *space* $\mathrm{BMO}(\mathbb{R}^n)$, originally introduced by John and Nirenberg [22], is defined as the space of all locally integrable functions f such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and f_B as in (1.1). Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The commutators generated by b and intrinsic Littlewood–Paley functions are defined, respectively, by setting, for all $x \in \mathbb{R}^n$,

$$[b, S_\alpha](f)(x) := \left[\iint_{\Gamma(x)} \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(x) - b(z)]\theta_t(y-z)f(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

$$[b, g_\alpha](f)(x) := \left[\int_0^\infty \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(x) - b(z)]\theta_t(y-z)f(z)\,dz\right|^2 \frac{dt}{t}\right]^{1/2}$$

and

$$[b, g^*_{\lambda,\alpha}](f)(x) := \left[\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \right.$$
$$\left. \times \sup_{\phi\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(x) - b(z)]\phi_t(y-z)f(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2}.$$

Now we establish the boundedness of these commutators on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$. To this end, we first recall the following well-known property of $\mathrm{BMO}(\mathbb{R}^n)$ functions (see, for example, [12, Corollary 6.12]).

Proposition 2.16. *Assume that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, for any $p \in [1,\infty)$, there exists a positive constant C such that*

$$\sup_{B\subset\mathbb{R}^n} \left[\frac{1}{|B|}\int_B |b(x) - b_B|^p\,dx\right]^{1/p} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and b_B as in (1.1) with f replaced by b.

If $\alpha \in (0,1]$, $\lambda > \max\{2,p\}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, it was proved in [47, Theorem 3.1] that the commutators $[b, S_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ are bounded on $L^p_w(\mathbb{R}^n)$ for all $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$. By this and Lemma 2.10, we have the following boundedness of $[b, S_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ on $L^\varphi(\mathbb{R}^n)$, whose proof is similar to that of Proposition 2.11, the details being omitted.

Proposition 2.17. *Let φ be a growth function with $1 < p_0 \le p_1 < \infty$, $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\lambda > \min\{\max\{2, p_1\}, 3 + 2\alpha/n\}$. Then there exists a positive constant C such that, for all $f \in L^\varphi(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi(x, [b, S_\alpha](f)(x))\, dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx$$

and

$$\int_{\mathbb{R}^n} \varphi(x, [b, g^*_{\lambda,\alpha}](f)(x))\, dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx.$$

Theorem 2.18. *Let $\alpha \in (0, 1]$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, φ be a growth function with $1 < p_0 \le p_1 < \infty$, $\varphi \in \mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi : [0, \infty) \to [0, \infty)$ be nondecreasing. If there exists a positive constant C such that, for all $r \in (0, \infty)$,*

$$\int_r^\infty \frac{1}{\phi(t)t}\, dt \le C \frac{1}{\phi(r)},$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f \in \mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$\|[b, S_\alpha](f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \le \widetilde{C} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$; otherwise, we replace b by $b/\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Fix any ball $B := B(x_0, r_B) \subset \mathbb{R}^n$ for some $x_0 \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ and let

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Since, for all $\alpha \in (0, 1]$, $[b, S_\alpha]$ is sublinear, it follows that, for all $x \in B$,

$$[b, S_\alpha](f)(x) \le [b, S_\alpha](f_1)(x) + [b, S_\alpha](f_2)(x).$$

Taking $\mu := \|f\|_{\varphi,2B} \ne 0$, by Proposition 2.17 and Lemma 2.7, we obtain

$$\begin{aligned}\frac{1}{\varphi(B,1)} \int_B \varphi\left(x, \frac{[b, S_\alpha](f_1)(x)}{\mu}\right) dx &\lesssim \frac{1}{\varphi(B,1)} \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f_1(x)|}{\mu}\right) dx \\ &\sim \frac{1}{\varphi(B,1)} \int_{2B} \varphi\left(x, \frac{|f(x)|}{\mu}\right) dx \lesssim 1.\end{aligned}$$

From this and Lemma 2.6(i), we deduce that $\|[b, S_\alpha](f_1)\|_{\varphi,B} \lesssim \|f\|_{\varphi,2B}$. Therefore,

$$\begin{aligned}\phi(\varphi(B,1))\|[b, S_\alpha](f_1)\|_{\varphi,B} &\lesssim \phi(\varphi(B,1))\|f\|_{\varphi,2B} \\ &\lesssim \frac{\phi(\varphi(B,1))}{\phi(\varphi(2B,1))}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \qquad (2.18)\end{aligned}$$

Next, we turn to estimate $[b, S_\alpha](f_2)$. Since, for any $x \in B$ and $(y, t) \in \Gamma(x)$,

$$\begin{aligned}&\sup_{\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} [b(x) - b(z)]\theta_t(y - z) f_2(z)\, dz \right| \\ &\quad \le |b(x) - b_B| \sup_{\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} \theta_t(y - z) f_2(z)\, dz \right|\end{aligned}$$

$$+\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\theta_t(y-z)f_2(z)\,dz\right|, \tag{2.19}$$

where b_B is as in (1.1) with f replaced by b, it follows that, for all $x\in B$,

$$\begin{aligned}[b,S_\alpha](f_2)(x) &\le |b(x)-b_B|S_\alpha(f_2)(x)\\ &\quad+\left\{\iint_{\Gamma(x)}\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\theta_t(y-z)f_2(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\ &=: \mathrm{I}_1(x)+\mathrm{I}_2(x).\end{aligned}$$

For $\mathrm{I}_1(x)$, by (2.14), we see that, for all $x\in B$,

$$\phi(\varphi(B,1))\mathrm{I}_1(x)\lesssim|b(x)-b_B|\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.$$

By this and the fact that φ is of uniformly lower type p_0 and upper type p_1, we know that

$$\begin{aligned}&\frac{1}{\varphi(B,1)}\int_B\varphi\left(x,\frac{\phi(\varphi(B,1))\mathrm{I}_1(x)}{\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}}\right)dx\\ &\quad\lesssim\frac{1}{\varphi(B,1)}\int_B\left[|b(x)-b_B|^{p_1}+|b(x)-b_B|^{p_0}\right]\varphi(x,1)\,dx.\end{aligned}\tag{2.20}$$

Since $\varphi(\cdot,1)\in A_{p_0}(\mathbb{R}^n)\subset A_{p_1}(\mathbb{R}^n)$, there exists some $r\in(1,\infty)$ such that $\varphi(\cdot,1)\in \mathrm{RH}_r(\mathbb{R}^n)$. From this, the Hölder inequality and Proposition 2.16, we deduce that

$$\begin{aligned}&\left[\frac{1}{\varphi(B,1)}\int_B|b(x)-b_B|^{p_i}\varphi(x,1)\,dx\right]^{1/p_i}\\ &\quad\le\frac{1}{[\varphi(B,1)]^{1/p_i}}\left[\int_B|b(x)-b_B|^{r'p_i}\,dx\right]^{1/(r'p_i)}\left\{\int_B[\varphi(x,1)]^r\,dx\right\}^{1/(rp_i)}\\ &\quad\lesssim\left[\frac{1}{|B|}\int_B|b(x)-b_B|^{r'p_i}\,dx\right]^{1/(r'p_i)}\lesssim1,\end{aligned}\tag{2.21}$$

where $i\in\{0,1\}$. By this, (2.20) and Lemma 2.6(i), we have

$$\phi(\varphi(B,1))\|\mathrm{I}_1\|_{\varphi,B}\lesssim\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \tag{2.22}$$

On the other hand, from (2.7), the Minkowski inequality and the fact that $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, it follows that, for all $x\in B$,

$$\begin{aligned}&\mathrm{I}_2(x)\\ &\le\left\{\iint_{\Gamma(x)}\left[\sum_{k=1}^\infty\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\left|\int_{2^{k+1}B\setminus2^kB}[b(z)-b_B]\theta_t(y-z)f_2(z)\,dz\right|\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\ &\lesssim\sum_{k=1}^\infty\left[\int_{2^{k+1}B\setminus2^kB}|b(z)-b_B||f(z)|\,dz\right]\left\{\int_{2^{k-2}r}^\infty\int_{|x-y|<t}\frac{dy\,dt}{t^{3n+1}}\right\}^{1/2}\\ &\lesssim\sum_{k=1}^\infty\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|}|b_{2^{k+1}B}-b_B|\int_{2^{k+1}B}|f(z)|\,dz$$
$$=: \mathrm{J}_1(x)+\mathrm{J}_2(x).$$

By Lemma 2.8, we know that, for all $x\in B$,

$$\mathrm{J}_1(x)\lesssim\sum_{k=1}^{\infty}\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|}\|f\|_{\varphi,2^{k+1}B}\left\||b(\cdot)-b_{2^{k+1}B}|\frac{1}{\varphi(\cdot,1)}\right\|_{\widetilde{\psi},2^{k+1}B}. \tag{2.23}$$

From Lemma 2.5, it follows that

$$\begin{aligned}&\frac{1}{\varphi(2^{k+1}B,1)}\int_{2^{k+1}B}\widetilde{\psi}\left(x,\frac{\varphi(2^{k+1}B,1)|b(x)-b_{2^{k+1}B}|}{|2^{k+1}B|\varphi(x,1)}\right)\varphi(x,1)\,dx\\&\quad\lesssim\frac{1}{\varphi(2^{k+1}B,1)}\\&\qquad\times\int_{2^{k+1}B}\left\{\sum_{i=0}^{1}\left[\frac{|b(x)-b_{2^{k+1}B}|}{[\varphi(x,1)]}\right]^{p_i'}\left[\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|}\right]^{p_i'}\right\}\varphi(x,1)\,dx. \end{aligned}\tag{2.24}$$

Since $\varphi(\cdot,1)\in A_{p_0}(\mathbb{R}^n)\subset A_{p_1}(\mathbb{R}^n)$, we know that

$$w_i(\cdot):=[\varphi(\cdot,1)]^{-p_i'/p_i}\in A_{p_i'}(\mathbb{R}^n)$$

for $i\in\{0,1\}$ (see, for example, [12, p. 136]). By this, the Hölder inequality and (2.21) with p_i replaced by p_i', we conclude that, for $i\in\{0,1\}$,

$$\begin{aligned}&\frac{1}{\varphi(2^{k+1}B,1)}\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^{p_i'}\left[\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|}\right]^{p_i'}\frac{1}{[\varphi(x,1)]^{p_i'}}\varphi(x,1)\,dx\\&\sim\left[\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\varphi(x,1)\,dx\right]^{p_i'-1}\left\{\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}[\varphi(x,1)]^{1-p_i'}\,dx\right\}\\&\quad\times\frac{1}{w_i(2^{k+1}B)}\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^{p_i'}w_i(x)\,dx\lesssim 1,\end{aligned}$$

where

$$w_i(2^{k+1}B):=\int_{2^{k+1}B}w_i(x)\,dx.$$

From this and (2.24), it follows that

$$\frac{1}{\varphi(2^{k+1}B,1)}\int_{2^{k+1}B}\widetilde{\psi}\left(x,\frac{\varphi(2^{k+1}B,1)|b(x)-b_{2^{k+1}B}|}{|2^{k+1}B|\varphi(x,1)}\right)\varphi(x,1)\,dx\lesssim 1,$$

which, together with Lemma 2.6(ii), further implies that

$$\frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|}\left\||b(\cdot)-b_{2^{k+1}B}|\frac{1}{\varphi(\cdot,1)}\right\|_{\widetilde{\psi},2^{k+1}B}\lesssim 1. \tag{2.25}$$

By this, (2.23) and (2.13), we conclude that, for all $x\in B$,

$$\phi(\varphi(B,1))\mathrm{J}_1(x)\lesssim\sum_{k=1}^{\infty}\frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))}\phi(\varphi(2^{k+1}B,1))\|f\|_{\varphi,2^{k+1}B}$$

$$\lesssim \sum_{k=1}^{\infty} \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \tag{2.26}$$

For $\mathrm{J}_2(x)$, since $b \in \mathrm{BMO}(\mathbb{R}^n)$, we have

$$|b_{2^{k+1}B} - b_B| \lesssim (k+1)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

By this, Lemma 2.8 and (2.10), we know that, for all $x \in B$,

$$\begin{aligned}
\mathrm{J}_2(x) &\lesssim \sum_{k=1}^{\infty} (k+1) \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(z)|\,dz \\
&\lesssim \sum_{k=1}^{\infty} (k+1) \frac{\varphi(2^{k+1}B,1)}{|2^{k+1}B|} \|f\|_{\varphi,2^{k+1}B} \left\| \frac{1}{\varphi(\cdot,1)} \right\|_{\widetilde{\psi},2^{k+1}B} \\
&\lesssim \sum_{k=1}^{\infty} \frac{k+1}{\phi(\varphi(2^{k+1}B,1))} \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.
\end{aligned}$$

From (2.12), we deduce that there exists some $j_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, it holds true that $1 \lesssim \log(\frac{\varphi(2^{(k+1)j_0}B,1)}{\varphi(2^{kj_0}B,1)})$, which further implies that

$$k \lesssim \int_{\varphi(B,1)}^{\varphi(2^{kj_0}B,1)} \frac{1}{s}\,ds.$$

By this, (2.12) and the assumptions of ϕ, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} (k+1) \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} \\
&= \sum_{k=1}^{2j_0-1} (k+1) \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} + \sum_{k=1}^{\infty} \sum_{i=(k+1)j_0}^{(k+2)j_0-1} \frac{(i+1)\phi(\varphi(B,1))}{\phi(\varphi(2^{i}B,1))} \\
&\le \sum_{k=1}^{2j_0-1} (k+1) \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{k+1}B,1))} + j_0^2 \sum_{k=1}^{\infty} 2(k+1) \frac{\phi(\varphi(B,1))}{\phi(\varphi(2^{(k+1)j_0}B,1))} \\
&\lesssim 1 + \phi(\varphi(B,1)) \sum_{k=1}^{\infty} \frac{k+1}{\phi(\varphi(2^{(k+1)j_0}B,1))} \\
&\lesssim 1 + \phi(\varphi(B,1)) \sum_{k=1}^{\infty} \frac{k+1}{\phi(\varphi(2^{(k+1)j_0}B,1))} \int_{\varphi(2^{kj_0}B,1)}^{\varphi(2^{(k+1)j_0}B,1)} \frac{dt}{t} \\
&\lesssim 1 + \phi(\varphi(B,1)) \sum_{k=1}^{\infty} \int_{\varphi(2^{kj_0}B,1)}^{\varphi(2^{(k+1)j_0}B,1)} (k+1) \frac{1}{\phi(t)t}\,dt \\
&\lesssim 1 + \phi(\varphi(B,1)) \sum_{k=1}^{\infty} \int_{\varphi(2^{kj_0}B,1)}^{\varphi(2^{(k+1)j_0}B,1)} \frac{1}{\phi(t)t}\,dt \int_{\varphi(B,1)}^{\varphi(2^{kj_0}B,1)} \frac{1}{s}\,ds \\
&\lesssim 1 + \phi(\varphi(B,1)) \sum_{k=1}^{\infty} \int_{\varphi(2^{kj_0}B,1)}^{\varphi(2^{(k+1)j_0}B,1)} \frac{1}{\phi(t)t} \int_{\varphi(B,1)}^{t} \frac{1}{s}\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&\sim 1+\phi(\varphi(B,1))\int_{\varphi(B,1)}^{\infty}\frac{1}{\phi(t)t}\int_{\varphi(B,1)}^{t}\frac{1}{s}\,ds\,dt\\
&\sim 1+\phi(\varphi(B,1))\int_{\varphi(B,1)}^{\infty}\frac{1}{s}\int_{s}^{\infty}\frac{1}{\phi(t)t}\,dt\,ds\\
&\lesssim 1+\phi(\varphi(B,1))\int_{\varphi(B,1)}^{\infty}\frac{1}{\phi(s)s}\,ds\lesssim 1.
\end{aligned}
$$

Thus, for all $x\in B$,

$$
\phi(\varphi(B,1))\mathrm{J}_2(x)\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}. \tag{2.27}
$$

Combining (2.26) and (2.27), we see that, for all $x\in B$,

$$
\phi(\varphi(B,1))\mathrm{I}_2(x)\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)},
$$

which further implies that

$$
\phi(\varphi(B,1))\|\mathrm{I}_2\|_{\varphi,B}\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.
$$

From this and (2.22), we deduce that

$$
\phi(\varphi(B,1))\|[b,S_\alpha](f_2)\|_{\varphi,B}\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)},
$$

which, combined with (2.18), completes the proof of Theorem 2.18. □

By using an argument similar to that used in the proof of Theorem 2.18, we can prove that $[b,g^*_{\lambda,\alpha}]$ and $[b,g_\alpha]$ are, respectively, bounded on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ as following, the details being omitted.

Proposition 2.19. *Let $\alpha\in(0,1]$, $b\in\mathrm{BMO}(\mathbb{R}^n)$, φ be a growth function with $1<p_0\le p_1<\infty$, $\varphi\in\mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi:[0,\infty)\to[0,\infty)$ be nondecreasing. If there exists a positive constant C such that, for all $r\in(0,\infty)$,*

$$
\int_r^\infty \frac{1}{\phi(t)t}\,dt\le C\frac{1}{\phi(r)},
$$

then there exists a positive constant $\widetilde{C}$ such that, for all $f\in\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$
\|[b,g_\alpha](f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}\le \widetilde{C}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.
$$

Proposition 2.20. *Let $\alpha\in(0,1]$, $b\in\mathrm{BMO}(\mathbb{R}^n)$, φ be a growth function with $1<p_0\le p_1<\infty$, $\varphi\in\mathbb{A}_{p_0}(\mathbb{R}^n)$ and $\phi:[0,\infty)\to[0,\infty)$ be nondecreasing. If there exists a positive constant C such that, for all $r\in(0,\infty)$,*

$$
\int_r^\infty \frac{1}{\phi(t)t}\,dt\le C\frac{1}{\phi(r)}
$$

and $\lambda>\min\{\max\{3,\,p_1\},3+2\alpha/n\}$, then there exists a positive constant $\widetilde{C}$ such that, for all $f\in\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$,

$$
\|[b,g^*_{\lambda,\alpha}](f)\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}\le \widetilde{C}\|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)}.
$$

Remark 2.21. In [47], Wang established the boundedness of $g^*_{\lambda,\alpha}$ and $[b, g^*_{\lambda,\alpha}]$ on weighted Morrey space $\mathcal{M}^{p,\kappa}_w(\mathbb{R}^n)$ with $\lambda > \max\{3, p\}$. This corresponds to Theorem 2.15 and Proposition 2.20 in the case when φ is as in (1.2) with $w \in A_p(\mathbb{R}^n)$, $p \in (1, \infty)$ and ϕ as in Remark 2.4(ii). We point out that Theorem 2.15 and Proposition 2.20, even for this special case, also improve the range of $\lambda > p$ in [47] to a wider range $\lambda > 3 + 2\alpha/n$ when $p > 3 + 2\alpha/n$.

3. Boundedness of intrinsic Littlewood–Paley functions on weighted Orlicz–Morrey spaces

In this section, motivated by Nakai [37], we introduce the weighted Orlicz–Morrey space $M^{\Phi,\phi}_w(\mathbb{R}^n)$ and establish the boundedness on $M^{\Phi,\phi}_w(\mathbb{R}^n)$ of intrinsic Littlewood–Paley functions and their commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions.

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called a *Young function* (or *N-function*), if it is increasing and convex, and satisfies that

$$\Phi(0) = 0, \ \lim_{t\to 0} \Phi(t)/t = 0 \text{ and } \lim_{t\to\infty} \Phi(t)/t = \infty$$

(see, for example, [15, p. 436]). Obviously, any Young function is continuous and strictly increasing, and hence bijective. The *complementary function* of Φ is defined by, for all $r \in [0, \infty)$,

$$\widetilde{\Phi}(r) := \sup_{s\in[0,\infty)} \{rs - \Phi(s)\}.$$

It is well known that $\widetilde{\Phi}$ is also a Young function and, for all $r \in (0, \infty)$,

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \tag{3.1}$$

(see, for example, [41, pp. 13–14]), where Φ^{-1} denotes the *inverse function* of Φ. Moreover, by Lemma 2.5(ii) with $\varphi(x, t) := \Phi(t)$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, we know that, if Φ is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\widetilde{\Phi}$ is of lower type p_1' and upper type p_0'. In this case, $\Phi \in \Delta_2 \cap \nabla_2$ (see [41] for the definitions of the conditions Δ_2 and ∇_2). Conversely, if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is of lower type p_0 and upper type p_1 for some p_0 and p_1 with $1 < p_0 \le p_1 < \infty$ (see [23, Lemma 1.3.2]).

Definition 3.1. Let Φ be a Young function, $\phi : (0, \infty) \to (0, \infty)$ be nonincreasing and $w \in A_\infty(\mathbb{R}^n)$. The *weighted Orlicz–Morrey space* $M^{\Phi,\phi}_w(\mathbb{R}^n)$ is defined by

$$M^{\Phi,\phi}_w(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ \|f\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \|f\|_{\Phi,\phi,B} < \infty\},$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and

$$\|f\|_{\Phi,\phi,B} := \inf \left\{\mu \in (0, \infty) : \ \frac{1}{w(B)\phi(w(B))} \int_B \Phi\left(\frac{|f(x)|}{\mu}\right) w(x)\, dx \le 1\right\}.$$

Here and in what follows, for any ball B of $\mathbb{R}^n$ and $w \in A_\infty(\mathbb{R}^n)$,

$$w(B) := \int_B w(x)\, dx.$$

Remark 3.2. (i) Since Φ is convex, we know that $\|\cdot\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}$ is a norm.

(ii) If a function $\Phi : [0,\infty) \to [0,\infty)$ is quasi-convex, namely, there exist a convex function Φ_0 and a positive constant C such that

$$\Phi_0(C^{-1}r) \le \Phi(r) \le \Phi_0(Cr) \quad \text{for all } r \in [0,\infty),$$

then the corresponding functionals $\|\cdot\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}$ and $\|\cdot\|_{M_w^{\Phi_0,\phi}}$ are equivalent. Therefore, all the results in this section also hold true for any quasi-convex function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$.

(iii) If $w \equiv 1$, then $M_w^{\Phi,\phi}(\mathbb{R}^n)$ coincides with the Orlicz–Morrey space $L^{(\Phi,\phi)}(\mathbb{R}^n)$ in [37] with equivalent norms.

(iv) If $\phi(r) := 1/r$ for all $r \in (0,\infty)$, then $M_w^{\Phi,\phi}(\mathbb{R}^n)$ coincides with the weighted Orlicz space $L_w^{\Phi}(\mathbb{R}^n)$. In this case, ϕ satisfies the assumptions for all the theorems in this section. Therefore, all the results in this section hold true for any $L_w^{\Phi}(\mathbb{R}^n)$ with Young function Φ of lower type p_0 and upper type p_1 and $w \in A_{p_0}(\mathbb{R}^n)$, where $1 < p_0 \le p_1 < \infty$.

(v) Even if

$$\varphi(x,t) := w(x)\Phi(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in [0,\infty), \tag{3.2}$$

with Φ being a Young function and $w \in A_\infty(\mathbb{R}^n)$, $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ as in Section 2 may not coincide with $M_w^{\Phi,\phi}(\mathbb{R}^n)$.

Before proving the main results of this section, we first state the following technical lemmas whose proofs are, respectively, similar to those of Lemmas 2.6, 2.7 and 2.8, the details being omitted.

Lemma 3.3. *Let Φ be a Young function which is of lower type p_0 and upper type p_1 with $0 < p_0 \le p_1 < \infty$ and ϕ, w be as in Definition 3.1. Let $\widetilde{C}$ be a positive constant. Then there exists a positive constant C such that*

(i) *for any ball B of $\mathbb{R}^n$ and $\mu \in (0,\infty)$,*

$$\frac{1}{w(B)\phi(w(B))}\int_B \Phi\left(\frac{|f(x)|}{\mu}\right) w(x)\,dx \le \widetilde{C}$$

implies that $\|f\|_{\Phi,\phi,B} \le C\mu$;

(ii) *for any ball B of $\mathbb{R}^n$ and $\mu \in (0,\infty)$,*

$$\frac{1}{w(B)\phi(w(B))}\int_B \widetilde{\Phi}\left(\frac{|f(x)|}{\mu}\right) w(x)\,dx \le \widetilde{C}$$

implies that $\|f\|_{\widetilde{\Phi},\phi,B} \le C\mu$.

Lemma 3.4. *Let Φ be as in Lemma 3.3 and ϕ, w as in Definition 3.1. Then, for any ball B of $\mathbb{R}^n$ and $\|f\|_{\Phi,\phi,B} \ne 0$, it holds true that*

$$\frac{1}{w(B)\phi(w(B))}\int_B \Phi\left(\frac{|f(x)|}{\|f\|_{\Phi,\phi,B}}\right) w(x)\,dx = 1$$

and, for all $\|f\|_{\widetilde{\Phi},\phi,B} \ne 0$, it holds true that

$$\frac{1}{w(B)\phi(w(B))}\int_B \widetilde{\Phi}\left(\frac{|f(x)|}{\|f\|_{\widetilde{\Phi},\phi,B}}\right) w(x)\,dx = 1.$$

Lemma 3.5. *Let Φ be as in Lemma 3.3 and ϕ, w as in Definition 3.1. Then, for any ball B of $\mathbb{R}^n$ and $f, g \in L^1_{\rm loc}(\mathbb{R}^n)$,*

$$\frac{1}{w(B)\phi(w(B))}\int_B |f(x)||g(x)|w(x)\,dx \le 2\|f\|_{\Phi,\phi,B}\|g\|_{\widetilde{\Phi},\phi,B}.$$

One of the main results of this section is as follows.

Theorem 3.6. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}(\mathbb{R}^n)$ and ϕ be nonincreasing. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le \widetilde{C}\phi(r) \quad \text{and} \quad \phi(r)r \le \widetilde{C}\phi(s)s.$$

Then there exists a positive constant C such that, for all $f \in M_w^{\Phi,\phi}(\mathbb{R}^n)$,

$$\|S_\alpha(f)\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

Proof. Fix any ball $B := B(x_0, r_B)$, with $x_0 \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and decompose

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Since, for any $\alpha \in (0,1]$, S_α is sublinear, we see that, for all $x \in B$,

$$S_\alpha(f)(x) \le S_\alpha(f_1)(x) + S_\alpha(f_2)(x).$$

Let $\mu := \|f\|_{\Phi,\phi,2B}$. From Proposition 2.11 with φ being as in (3.2), it follows that

$$\int_{\mathbb{R}^n} \Phi(S_\alpha(f)(x))w(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x)\,dx,$$

which, together with Lemma 3.4 and the fact that ϕ is decreasing, further implies that

$$\begin{aligned}
&\frac{1}{w(B)\phi(w(B))}\int_B \Phi\left(\frac{S_\alpha(f_1)(x)}{\mu}\right)w(x)\,dx\\
&\quad\lesssim \frac{1}{w(B)\phi(w(B))}\int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\mu}\right)w(x)\,dx\\
&\quad\sim \frac{1}{w(B)\phi(w(B))}\int_{2B} \Phi\left(\frac{|f(x)|}{\mu}\right)w(x)\,dx \lesssim \frac{w(2B)\phi(w(2B))}{w(B)\phi(w(B))} \lesssim 1.
\end{aligned}$$

By this and Lemma 3.3(i), we have $\|S_\alpha(f_1)\|_{\Phi,\phi,B} \lesssim \|f\|_{\Phi,\phi,2B}$. Therefore,

$$\|S_\alpha(f_1)\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}. \tag{3.3}$$

From (2.8) and Lemma 3.5, it follows that, for all $x \in B$,

$$S_\alpha(f_2)(x) \lesssim \sum_{k=1}^\infty \frac{w(2^{k+1}B)\phi(w(2^{k+1}B))}{|2^{k+1}B|}\|f\|_{\Phi,\phi,2^{k+1}B}\|w^{-1}\|_{\widetilde{\Phi},\phi,2^{k+1}B}. \tag{3.4}$$

By the fact that $\widetilde{\Phi}$ is of uniformly lower type p_1' and upper type p_0' and the fact that $w \in A_{p_0}(\mathbb{R}^n) \subset A_{p_1}(\mathbb{R}^n)$, we know that

$$\frac{1}{w(2^{k+1}B)\phi(w(2^{k+1}B))}\int_{2^{k+1}B} \widetilde{\Phi}\left(\frac{w(2^{k+1}B)\widetilde{\Phi}^{-1}(\phi(w(2^{k+1}B)))}{|2^{k+1}B|w(x)}\right)w(x)\,dx$$

$$\lesssim \frac{1}{w(2^{k+1}B)} \int_{2^{k+1}B} \left\{ \left[\frac{w(2^{k+1}B)}{|2^{k+1}B|w(x)}\right]^{p_1'} + \left[\frac{w(2^{k+1}B)}{|2^{k+1}B|w(x)}\right]^{p_0'} \right\} w(x)\,dx$$
$$\sim \left[\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} w(x)\,dx\right]^{p_0'-1} \frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} [w(x)]^{1-p_0'}\,dx$$
$$+ \left[\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} w(x)\,dx\right]^{p_1'-1} \frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} [w(x)]^{1-p_1'}\,dx \lesssim 1.$$

From this and Lemma 3.3(ii), it follows that

$$\frac{w(2^{k+1}B)\widetilde{\Phi}^{-1}(\phi(w(2^{k+1}B)))}{|2^{k+1}B|}\|w^{-1}\|_{\widetilde{\Phi},\phi,2^{k+1}B} \lesssim 1. \tag{3.5}$$

By this, (3.4) and (3.1), we conclude that, for all $x \in B$,

$$S_\alpha(f_2)(x) \lesssim \sum_{k=1}^{\infty} \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \frac{\phi(w(2^{k+1}B))}{\widetilde{\Phi}^{-1}(\phi(w(2^{k+1}B)))}$$
$$\lesssim \sum_{k=1}^{\infty} \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \Phi^{-1}(\phi(w(2^{k+1}B))). \tag{3.6}$$

Recall that, by (2.12) with φ as in (3.2), there exists some $j_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$,

$$1 \lesssim \log\left(\frac{w(2^{(k+1)j_0}B)}{w(2^{kj_0}B)}\right).$$

Moreover, by this, the fact that $\Phi^{-1}(\phi(\cdot))$ is decreasing and the assumptions of ϕ, we have

$$\sum_{k=1}^{\infty} \Phi^{-1}(\phi(w(2^{k+1}B)))$$
$$= \sum_{l=0}^{\infty} \sum_{i=lj_0+1}^{(l+1)j_0} \Phi^{-1}(\phi(w(2^{i}B)))$$
$$\lesssim \sum_{i=1}^{j_0} \Phi^{-1}(\phi(w(2^{i}B))) + j_0 \sum_{l=1}^{\infty} \Phi^{-1}(\phi(w(2^{lj_0}B)))$$
$$\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{l=1}^{\infty} \Phi^{-1}(\phi(w(2^{lj_0}B))) \int_{w(2^{(l-1)j_0}B)}^{w(2^{lj_0}B)} \frac{dt}{t}$$
$$\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{l=1}^{\infty} \int_{w(2^{(l-1)j_0}B)}^{w(2^{lj_0}B)} \frac{\Phi^{-1}(\phi(t))}{t}\,dt$$
$$\lesssim \Phi^{-1}(\phi(w(B))) + \int_{w(B)}^{\infty} \frac{\Phi^{-1}(\phi(t))}{t}\,dt \lesssim \Phi^{-1}(\phi(w(B))), \tag{3.7}$$

where the last inequality is deduced from the fact that

$$\int_r^\infty \frac{\Phi^{-1}(\phi(t))}{t}\,dt \lesssim \Phi^{-1}(\phi(r)) \tag{3.8}$$

(see [37, Lemma 5.3] and the proof of [37, Corollary 3.2]). From (3.6) and (3.7), it follows that, for all $x \in B$,

$$S_\alpha(f_2)(x) \lesssim \Phi^{-1}(\phi(w(B)))\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)},$$

which further implies that

$$\Phi\left(\frac{S_\alpha(f_2)(x)}{\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}}\right) \lesssim \phi(w(B)).$$

Therefore,

$$\frac{1}{w(B)\phi(w(B))}\int_B \Phi\left(\frac{S_\alpha(f_2)(x)}{\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}}\right) w(x)\,dx \lesssim 1.$$

By this and Lemma 3.3(i), we have

$$\|S_\alpha(f_2)\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}, \tag{3.9}$$

which, together with (3.3), completes the proof of Theorem 3.6. □

For a Young function Φ, a function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a weight w on $\mathbb{R}^n$, the *space* $\widetilde{M}_w^{\Phi,\phi}(\mathbb{R}^n)$ is defined by a way same as Definition 3.1, via using $\phi(c_B, w(B))$ instead of $\phi(w(B))$, where c_B is the center of the ball B. Then, by an argument similar to that used in the proof of Theorem 3.6, we have the following boundedness of S_α on $\widetilde{M}_w^{\Phi,\phi}(\mathbb{R}^n)$, the details being omitted.

Theorem 3.7. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0 with $1 < p_0 \le p_1 < \infty$ and $w \in A_{p_0}(\mathbb{R}^n)$. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $x \in \mathbb{R}^n$ and $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le \widetilde{C}\phi(x,r),\ \ \phi(x,s) \le \widetilde{C}\phi(x,r) \ \ \text{and} \ \ \phi(x,r)r \le \widetilde{C}\phi(x,s)s.$$

Then there exists a positive constant C such that, for all $f \in \widetilde{M}_w^{\Phi,\phi}(\mathbb{R}^n)$,

$$\|S_\alpha(f)\|_{\widetilde{M}_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{\widetilde{M}_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

For example, let $\phi(x,r) := r^{\lambda(x)}$ for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ with $-1 \le \lambda(x) < 0$ and $\sup_{x\in\mathbb{R}^n} \lambda(x) < 0$. Then ϕ satisfies all the assumptions of Theorem 3.7.

Since $g_\alpha(f)$ is pointwise comparable to $S_\alpha(f)$, we have the following corollary of Theorem 3.6, the details being omitted.

Corollary 3.8. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}(\mathbb{R}^n)$ and ϕ be nonincreasing. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le \widetilde{C}\phi(r) \ \ \text{and} \ \ \phi(r)r \le \widetilde{C}\phi(s)s.$$

Then there exists a positive constant C such that, for all $f \in M_w^{\Phi,\phi}(\mathbb{R}^n)$,

$$\|g_\alpha(f)\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

Similarly, there exists a corollary similar to Corollary 3.8 of Theorem 3.7, the details being omitted.

Theorem 3.9. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}(\mathbb{R}^n)$ and ϕ be nonincreasing. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le \widetilde{C}\phi(r) \quad \text{and} \quad \phi(r)r \le \widetilde{C}\phi(s)s.$$

If $\lambda > \min\{\max\{3, p_1\}, 3 + 2\alpha/n\}$, then there exists a positive constant C such that, for all $f \in M_w^{\Phi,\phi}(\mathbb{R}^n)$,

$$\|g^*_{\lambda,\alpha}(f)\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

Proof. For any ball $B := B(x_0, r_B) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, let

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Since, for any $\alpha \in (0,1]$, $g^*_{\lambda,\alpha}$ is sublinear, we know that, for all $x \in B$,

$$g^*_{\lambda,\alpha}(f)(x) \le g^*_{\lambda,\alpha}(f_1)(x) + g^*_{\lambda,\alpha}(f_2)(x).$$

Similar to the estimate for f_1 in the proof of Theorem 3.6, by Proposition 2.11 with φ as in (3.2), Lemma 3.4, the fact that ϕ is nonincreasing and Lemma 3.3(i), if $\lambda > \min\{\max\{\{2, p_1\}, 3 + 2\alpha/n\}$, we conclude that

$$\|g^*_{\lambda,\alpha}(f_1)\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}. \tag{3.10}$$

To estimate f_2, from (2.17), Lemma 3.5, (3.1) and (3.7), we deduce that, for all $j \in \mathbb{Z}_+$ and $x \in B$,

$$\begin{aligned}\mathcal{S}_{\alpha,2^j}(f_2)(x) &\lesssim 2^{3jn/2}\sum_{k=1}^\infty \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \frac{\phi(w(2^{k+1}B))}{\widetilde{\Phi}^{-1}(\phi(w(2^{k+1}B)))}\\ &\lesssim 2^{3jn/2}\sum_{k=1}^\infty \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \Phi^{-1}(\phi(w(2^{k+1}B)))\\ &\lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\Phi^{-1}(\phi(w(B))).\end{aligned} \tag{3.11}$$

By this, we further see that

$$\Phi\left(\frac{\mathcal{S}_{\alpha,2^j}(f_2)(x)}{2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}}\right) \lesssim \phi(w(B)),$$

which further implies that

$$\frac{1}{w(B)\phi(w(B))}\int_B \Phi\left(\frac{\mathcal{S}_{\alpha,2^j}(f_2)(x)}{2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}}\right) w(x)\,dx \lesssim 1.$$

From this and Lemma 3.3(i), we deduce that

$$\|\mathcal{S}_{\alpha,2^j}(f_2)\|_{\Phi,\phi,B} \lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

By this and (3.9), we know that, if $\lambda > 3$,

$$\|g^*_{\lambda,\alpha}(f_2)\|_{\Phi,\phi,B} \lesssim \left[1+\sum_{j=1}^{\infty} 2^{-j(\lambda-3)n/2}\right] \|f\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)} \lesssim \|f\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)},$$

which, combined with (3.10), completes the proof of Theorem 3.9. □

Theorem 3.10. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0, $1 < p_0 \le p_1 < \infty$ with $w \in A_{p_0}(\mathbb{R}^n)$ and ϕ be nonincreasing. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le \widetilde{C}\phi(r) \quad \text{and} \quad \phi(r)r \le \widetilde{C}\phi(s)s.$$

If $\lambda > \min\{\max\{3, p_1\}, 3+2\alpha/n\}$, then there exists a positive constant C such that, for all $f \in M^{\Phi,\phi}_w(\mathbb{R}^n)$,

$$\|[b, g^*_{\lambda,\alpha}](f)\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)} \le C\|f\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)}.$$

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$; otherwise, we replace b by $b/\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Fix any ball $B := B(x_0, r_B) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r_B \in (0,\infty)$. Let

$$f = f\chi_{2B} + f\chi_{(2B)^\complement} =: f_1 + f_2.$$

Since, for any $\alpha \in (0,1]$, $[b, S_\alpha]$ is sublinear, we know that, for all $x \in B$,

$$[b, g^*_{\lambda,\alpha}](f)(x) \le [b, g^*_{\lambda,\alpha}](f_1)(x) + [b, g^*_{\lambda,\alpha}](f_2)(x).$$

Let $\mu := \|f\|_{\Phi,\phi,2B}$. From Proposition 2.17 with φ as in (3.2), it follows that

$$\int_{\mathbb{R}^n} \Phi([b, g^*_{\lambda,\alpha}](f)(x))w(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x)\,dx,$$

which, combined with Lemma 3.4, further implies that

$$\begin{aligned}
&\frac{1}{w(B))\phi(w(B))}\int_B \Phi\left(\frac{[b, g^*_{\lambda,\alpha}](f_1)(x)}{\mu}\right)w(x)\,dx\\
&\quad\lesssim \frac{1}{w(B))\phi(w(B))}\int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\mu}\right)w(x)\,dx\\
&\quad\sim \frac{1}{w(B))\phi(w(B))}\int_{2B} \Phi\left(\frac{|f(x)|}{\mu}\right)w(x)\,dx \sim 1.
\end{aligned}$$

From this and Lemma 3.3(i), we further deduce that

$$\|[b, g^*_{\lambda,\alpha}](f_1)\|_{\Phi,\phi,B} \lesssim \|f\|_{\Phi,\phi,2B} \lesssim \|f\|_{M^{\Phi,\phi}_w(\mathbb{R}^n)}. \tag{3.12}$$

Next, we turn to estimate $[b, g^*_{\lambda,\alpha}](f_2)$. By (2.19), we know that, for all $x \in B$,

$$\begin{aligned}
[b, g^*_{\lambda,\alpha}](f_2)(x) &\le |b(x) - b_B|g^*_{\lambda,\alpha}(f_2)(x) + \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\
&\quad\left.\times \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(z)-b_B]\theta_t(y-z)f_2(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}
\end{aligned}$$

$$=: \mathrm{I}(x) + \mathrm{II}(x).$$

For any $x \in B$, by (2.16), (3.11) and $\lambda > 3$, we conclude that

$$g^*_{\lambda,\alpha}(f_2)(x) \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \Phi^{-1}(\phi(w(B))),$$

which further implies that, for all $x \in B$,

$$\mathrm{I}(x) \lesssim |b(x) - b_B| \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \Phi^{-1}(\phi(w(B))).$$

From this, the fact that Φ is lower type p_0 and upper type p_1 and (2.21) with $\varphi(x, 1)$ replaced by $w(x)$, it follows that

$$\begin{aligned}
&\frac{1}{\phi(w(B))w(B)} \int_B \Phi\left(\frac{\mathrm{I}(x)}{\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}}\right) w(x)\,dx \\
&\quad\lesssim \frac{1}{\phi(w(B))w(B)} \int_B \Phi\left(|b(x) - b_B|\Phi^{-1}(\phi(w(B)))\right) w(x)\,dx \\
&\quad\lesssim \frac{1}{w(B)} \int_B [|b(x) - b_B|^{p_0} + |b(x) - b_B|^{p_1}] w(x)\,dx \lesssim 1.
\end{aligned}$$

By this and Lemma 3.3(i), we know that

$$\|\mathrm{I}\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}. \tag{3.13}$$

For $\mathrm{II}(x)$, we find that, for all $x \in B$,

$$\begin{aligned}
\mathrm{II}(x) &\le \left\{\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t + |x-y|}\right)^{\lambda n}\right. \\
&\qquad \left.\times \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(z) - b_B]\theta_t(y-z) f_2(z)\,dz\right|\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&\qquad + \sum_{j=1}^\infty \left\{\int_0^\infty \int_{2^{j-1}t\le|x-y|<2^j t} \left(\frac{t}{t + |x-y|}\right)^{\lambda n}\right. \\
&\qquad \left.\times \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(z) - b_B]\theta_t(y-z) f_2(z)\,dz\right|\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&\le \sum_{j=0}^\infty 2^{-j\lambda n/2} \left\{\int_0^\infty \int_{|x-y|<2^j t}\right. \\
&\qquad \left.\times \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\int_{\mathbb{R}^n} [b(z) - b_B]\theta_t(y-z) f_2(z)\,dz\right|\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&=: \sum_{j=0}^\infty 2^{-j\lambda n/2} \mathrm{I}_j(x).
\end{aligned} \tag{3.14}$$

For $j\in\mathbb{Z}_+$, by the fact that $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, we know that, for all $x\in B$,

$$\begin{aligned}\mathrm{I}_j(x)&\lesssim\sum_{k=1}^\infty\left[\int_{2^{k+1}B\setminus 2^kB}|b(z)-b_B||f(z)|\,dz\right]\left\{\int_{2^{k-j-2}r}^\infty\int_{|x-y|<2^jt}\frac{dy\,dt}{t^{3n+1}}\right\}^{1/2}\\&\lesssim 2^{3jn/2}\sum_{k=1}^\infty\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\\&\quad+2^{3jn/2}\sum_{k=1}^\infty\frac{1}{|2^{k+1}B|}|b_{2^{k+1}B}-b_B|\int_{2^{k+1}B}|f(z)|\,dz\\&=:\mathrm{H}_j(x)+\mathrm{G}_j(x).\end{aligned}$$

For $\mathrm{H}_j(x)$, by Lemma 3.5, we know that

$$\begin{aligned}\mathrm{H}_j(x)\lesssim 2^{3jn/2}&\sum_{k=1}^\infty\frac{w(2^{k+1}B)\phi(w(2^{k+1}B))}{|2^{k+1}B|}\\&\times\left\||b(\cdot)-b_{2^{k+1}B}|\frac{1}{w(\cdot)}\right\|_{\widetilde{\Phi},\phi,2^{k+1}B}\|f\|_{\Phi,\phi,2^{k+1}B}.\end{aligned}$$

By an argument similar to that used in the estimate for (2.25), we have

$$\frac{w(2^{k+1}B)\phi(w(2^{k+1}B))}{|2^{k+1}B|}\left\||b(\cdot)-b_{2^{k+1}B}|\frac{1}{w(\cdot)}\right\|_{\widetilde{\Phi},\phi,2^{k+1}B}\lesssim\frac{\phi(w(2^{k+1}B))}{\widetilde{\Phi}^{-1}(\phi(w(2^{k+1}B)))}.$$

From this, (3.1) and (3.7), it follows that, for all $x\in B$,

$$\begin{aligned}\mathrm{H}_j(x)&\lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\sum_{k=1}^\infty\Phi^{-1}(\phi(w(2^{k+1}B)))\\&\lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\Phi^{-1}(\phi(w(B))).\end{aligned}\tag{3.15}$$

For $\mathrm{G}_j(x)$, by the fact that

$$|b_{2^{k+1}B}-b_B|\lesssim(k+1)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

Lemma 3.5 and (3.5), we conclude that, for all $x\in B$,

$$\begin{aligned}\mathrm{G}_j(x)&\le 2^{3jn/2}\sum_{k=1}^\infty(k+1)\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B\setminus 2^kB}|f(z)|\,dz\\&\lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\sum_{k=1}^\infty(k+1)\Phi^{-1}(\phi(w(2^{k+1}B))).\end{aligned}$$

By (2.12), we know that there exists some $j_0\in\mathbb{N}$ such that, for all $k\in\mathbb{N}$,

$$1\lesssim\log\left(\frac{w(2^{(k+1)j_0}B)}{w(2^{kj_0}B)}\right).$$

From this, (3.8) and the fact that $\Phi^{-1}(\phi(\cdot))$ is decreasing, it follows that

$$\sum_{k=1}^\infty(k+1)\Phi^{-1}(\phi(w(2^{k+1}B)))$$

$$
\begin{aligned}
&= \sum_{k=1}^{2j_0-1}(k+1)\Phi^{-1}(\phi(w(2^{k+1}B))) + \sum_{k=1}^{\infty}\sum_{i=(k+1)j_0}^{(k+2)j_0-1}(i+1)\Phi^{-1}(\phi(w(2^{i+1}B)))\\
&\le \sum_{k=1}^{2j_0-1}(k+1)\Phi^{-1}(\phi(w(2^{k+1}B))) + 2j_0^2\sum_{k=1}^{\infty}(k+1)\Phi^{-1}(\phi(w(2^{(k+1)j_0}B)))\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{k=1}^{\infty}(k+1)\Phi^{-1}(\phi(w(2^{(k+1)j_0}B)))\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{k=1}^{\infty}(k+1)\Phi^{-1}(\phi(w(2^{(k+1)j_0}B)))\int_{w(2^{kj_0}B)}^{w(2^{(k+1)j_0}B)}\frac{dt}{t}\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{k=1}^{\infty}(k+1)\int_{w(2^{kj_0}B)}^{w(2^{(k+1)j_0}B)}\frac{\Phi^{-1}(\phi(t))}{t}\,dt\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{k=1}^{\infty}\int_{w(2^{kj_0}B)}^{w(2^{(k+1)j_0}B)}\frac{\Phi^{-1}(\phi(t))}{t}\,dt\int_{w(B)}^{w(2^{kj_0}B)}\frac{1}{s}\,ds\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \sum_{k=1}^{\infty}\int_{w(2^{kj_0}B)}^{w(2^{(k+1)j_0}B)}\frac{\Phi^{-1}(\phi(t))}{t}\int_{w(B)}^{t}\frac{1}{s}\,ds\,dt\\
&\sim \Phi^{-1}(\phi(w(B))) + \int_{w(B)}^{\infty}\frac{1}{s}\int_{s}^{\infty}\frac{\Phi^{-1}(\phi(t))}{t}\,dt\\
&\lesssim \Phi^{-1}(\phi(w(B))) + \int_{w(B)}^{\infty}\frac{\Phi^{-1}(\phi(s))}{s}\,ds \lesssim \Phi^{-1}(\phi(w(B))).
\end{aligned}
$$

Thus, we find that, for all $x \in B$,

$$
\mathrm{G}_j(x) \lesssim 2^{3jn/2}\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\Phi^{-1}(\phi(w(B))).
$$

By this, (3.15) and (3.14), together with $\lambda > 3$, we see that, for all $x \in B$,

$$
\mathrm{II}(x) \lesssim \left[1 + \sum_{j=1}^{\infty}2^{-j(\lambda-3)n/2}\right]\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}\Phi^{-1}(\phi(w(B))),
$$

which, combined with Lemma 3.3(i), implies that

$$
\|\mathrm{II}\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.
$$

From this and (3.13), we deduce that

$$
\|[b, g^*_{\lambda,\alpha}](f_2)\|_{\Phi,\phi,B} \lesssim \|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)},
$$

which, combined with (3.12), completes the proof of Theorem 3.10. □

By using an argument similar to that used in the proof of Theorem 3.10, we can prove that $[b, S_\alpha]$ and $[b, g_\alpha]$ are bounded, respectively, on $M_w^{\Phi,\phi}(\mathbb{R}^n)$ as follows, the details being omitted.

Proposition 3.11. *Let $\alpha \in (0,1]$, Φ be a Young function which is of upper type p_1 and lower type p_0, $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}(\mathbb{R}^n)$ and ϕ be nonincreasing. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $0 < r \le s < \infty$,*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le \widetilde{C}\phi(r) \quad \textit{and} \quad \phi(r)r \le \widetilde{C}\phi(s)s.$$

Then there exists a positive constant C such that, for all $f \in M_w^{\Phi,\phi}(\mathbb{R}^n)$,

$$\|[b,S_\alpha](f)\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}$$

and

$$\|[b,g_\alpha](f)\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)} \le C\|f\|_{M_w^{\Phi,\phi}(\mathbb{R}^n)}.$$

4. Boundedness of intrinsic Littlewood–Paley functions on Musielak–Orlicz Campanato spaces

In this section, we establish the boundedness of intrinsic Littlewood–Paley functions on the Musielak–Orlicz Campanato space which was introduced in [31]. We begin with recalling the notion of Musielak–Orlicz Campanato spaces.

Definition 4.1. Let φ be a growth function satisfying $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$, $p \in [1,\infty)$ and $q \in [1,\infty)$. A locally integrable function f on $\mathbb{R}^n$ is said to belong to the *Musielak–Orlicz Campanato space* $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, if

$$\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{\int_B \left[\frac{|f(x)-f_B|}{\varphi(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\right]^q \varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\,dx\right\}^{1/q}$$

is finite, where the supremum is taken over all balls B of $\mathbb{R}^n$ and f_B as in (1.1).

Motivated by [20], we also introduce a subspace $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)$ of $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$.

Definition 4.2. Let φ be a growth function satisfying $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$, $p \in [1,\infty)$ and $q \in [1,\infty)$. A locally integrable function f on $\mathbb{R}^n$ is said to belong to $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)$, if

$$\|f\|_{\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{\int_B \left[\frac{f(x)-\operatorname*{ess\,inf}_{y\in B} f(y)}{\varphi(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\right]^q \varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\,dx\right\}^{1/q}$$

is finite, where the supremum is taken over all balls B of $\mathbb{R}^n$.

Remark 4.3. (i) Since the growth function here is slightly different from [31] (see Remark 2.2(i)), the Musielak–Orlicz Campanato space here is also slightly different from [31].

(ii) $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n) \subset \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$.

Before proving the main results of this section, we need the following technical lemma.

Lemma 4.4. *Let φ be a growth function satisfying $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$, $p \in [1, \infty)$ and $q \in [1, \infty)$. Then, for any ball $B := B(x_0, r) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ and $\beta \in (\max\{n(\frac{p}{p_0} - 1), 0\}, \infty)$, there exists a positive constant C, independent of f and B, such that*

$$\frac{r^\beta |B|}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{|f(y) - f_B|}{r^{n+\beta} + |y - x_0|^{n+\beta}} \, dy \leq C \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

Proof. Let $B := B(x_0, r) \subset \mathbb{R}^n$, with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, and $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$. Write

$$\begin{aligned} & r^\beta \int_{\mathbb{R}^n} \frac{|f(y) - f_B|}{r^{n+\beta} + |y - x_0|^{n+\beta}} \, dy \\ & \quad \leq r^\beta \int_B \frac{|f(y) - f_B|}{r^{n+\beta} + |y - x_0|^{n+\beta}} \, dy + \sum_{k=1}^{\infty} r^\beta \int_{2^k B \setminus 2^{k-1} B} \cdots \\ & \quad =: \mathrm{I}_0 + \sum_{k=1}^{\infty} \mathrm{I}_k. \end{aligned} \tag{4.1}$$

For I_0, by the Hölder inequality, we know that

$$\mathrm{I}_0 \lesssim \frac{1}{|B|} \int_B |f(y) - f_B| \, dy \lesssim \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \tag{4.2}$$

For any $k \in \mathbb{N}$, by the Hölder inequality again, we have

$$\begin{aligned} |f_{2^k B} - f_B| & \leq \sum_{j=1}^{k} |f_{2^j B} - f_{2^{j-1} B}| \leq \sum_{j=1}^{k} \frac{1}{|2^{j-1} B|} \int_{2^{j-1} B} |f(y) - f_{2^j B}| \, dy \\ & \lesssim \sum_{j=1}^{k} \frac{\|\chi_{2^j B}\|_{L^\varphi(\mathbb{R}^n)}}{|2^j B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \end{aligned} \tag{4.3}$$

Since $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ and φ is of uniformly lower type p_0, we see that, for all $j \in \mathbb{Z}_+$,

$$\varphi\left(2^j B, 2^{-jnp/p_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \lesssim 2^{-jnp} \varphi\left(2^j B, \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \lesssim 1,$$

which further implies that, for all $j \in \mathbb{Z}_+$,

$$\|\chi_{2^j B}\|_{L^\varphi(\mathbb{R}^n)} \lesssim 2^{jnp/p_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}.$$

By this, the Hölder inequality and (4.3), we conclude that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \int_{2^k B} |f(y) - f_B| \, dy & \leq \int_{2^k B} |f(y) - f_{2^k B}| \, dy + |2^k B| |f_{2^k B} - f_B| \\ & \lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)} \left[2^{kn\frac{p}{p_0}} + 2^{kn} \sum_{j=1}^{k} 2^{jn(\frac{p}{p_0} - 1)}\right] \\ & \lesssim 2^{kns} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}, \end{aligned} \tag{4.4}$$

where $s := \max\{1, p/p_0\}$. By (4.4) and $\beta \in (\max\{n(\frac{p}{p_0}-1), 0\}, \infty)$, we see that

$$\sum_{k=1}^{\infty} \mathrm{I}_k \le \sum_{k=1}^{\infty} \frac{r^\beta}{2^{k(n+\beta)} r^{n+\beta}} \int_{2^k B} |f(y) - f_B|\, dy$$
$$\lesssim \sum_{k=1}^{\infty} 2^{kn(s-1-\beta/n)} \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)} \lesssim \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},$$

which, together with (4.1) and (4.2), completes the proof of Lemma 4.4. □

One of the main results of this section is as follows.

Theorem 4.5. *Let $\alpha \in (0,1]$, $q \in (1,\infty)$, φ be a growth function as in Definition 2.1 and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$. If $n(\frac{p}{p_0}-1) < \alpha$ and $p \le q'$, then, for any $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, $g_\alpha(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|g_\alpha(f)\|_{\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)} \le C \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

Proof. We only need to show that, for all $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, if there exists some $u \in \mathbb{R}^n$ such that $g_\alpha(f)(u) < \infty$, then, for any ball $B := B(x_0, r) \subset \mathbb{R}^n$, with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$, and $B \ni u$,

$$\left\{\int_B \left[g_\alpha(f)(x) - \inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x})\right]^q \left[\varphi\left(x, \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\right]^{1-q} dx\right\}^{1/q}$$
$$\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

To this end, for any $x \in B$, since, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \theta(x)\, dx = 0$ and

$$\inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x}) \le g_\alpha(f)(u) < \infty,$$

we write

$$\begin{aligned}
&g_\alpha(f)(x) - \inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x}) \\
&\quad\le \left\{\int_0^r [A_\alpha([f-f_B]\chi_{2B})(x,t)]^2 \frac{dt}{t}\right\}^{1/2} \\
&\qquad + \left\{\int_0^r [A_\alpha([f-f_B]\chi_{(2B)^\complement})(x,t)]^2 \frac{dt}{t}\right\}^{1/2} \\
&\qquad + \sup_{\widetilde{x}\in B} \left| \left\{\int_r^\infty [A_\alpha(f)(x,t)]^2 \frac{dt}{t}\right\}^{1/2} - \left\{\int_r^\infty [A_\alpha(f)(\widetilde{x},t)]^2 \frac{dt}{t}\right\}^{1/2} \right| \\
&\quad =: \mathrm{I}_1(x) + \mathrm{I}_2(x) + \mathrm{I}_3(x).
\end{aligned} \tag{4.5}$$

Since $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ and $1 \le p \le q'$, we have $\varphi \in \mathbb{A}_{q'}(\mathbb{R}^n)$ and

$$[\varphi(\cdot, \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})]^{1-q} \in A_q(\mathbb{R}^n).$$

From (4.3), the fact that S_α is bounded on $L^q_w(\mathbb{R}^n)$ with $q \in (1,\infty)$ and $w \in A_q(\mathbb{R}^n)$ (see [49, Theorem 7.2]) and $g_\alpha(f)(x)$ and $S_\alpha(f)(x)$ are pointwise comparable for all $x \in \mathbb{R}^n$, it follows that

$$\begin{aligned}
&\left\{\int_B [\mathrm{I}_1(x)]^q \left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\lesssim \left\{\int_{\mathbb{R}^n} [g_\alpha([f-f_B]\chi_{2B})(x)]^q \left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\lesssim \left\{\int_{2B} |f(x)-f_B|^q \left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\lesssim \left\{\int_{2B} |f(x)-f_{2B}|^q \left[\varphi\left(x, \|\chi_{2B}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\quad + \left\{\int_{2B} |f_{2B}-f_B|^q \left[\varphi\left(x, \|\chi_{2B}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)} + |f_{2B}-f_B| \left\{\int_{2B} \left[\varphi\left(x, \|\chi_{2B}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q} \\
&\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)} \left(1 + \frac{1}{|2B|}\left\{\int_{2B} \left[\varphi\left(x, \|\chi_{2B}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q}\right) \\
&\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \qquad (4.6)
\end{aligned}$$

To estimate $\mathrm{I}_2(x)$, since, for any $z \in (2B)^\complement$, $x \in B$ and $t \in (0,r)$, we have $|x-z| \ge |x_0 - z| - |x - x_0| > 2r - r > t$, by the fact that, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, supp $\theta \subset B(0,1)$, we conclude that

$$A_\alpha([f-f_B]\chi_{(2B)^\complement})(x,t) = \sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \left|\frac{1}{t^n}\int_{(2B)^\complement} \theta\left(\frac{x-z}{t}\right)[f(z)-f_B]\,dz\right| = 0.$$

Thus, for all $x \in B$, $\mathrm{I}_2(x) \equiv 0$

For any $x, \widetilde{x} \in B$, from the Minkowski inequality and the fact that, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\theta(x)\,dx = 0$, we deduce that

$$\begin{aligned}
&\left|\left\{\int_r^\infty [A_\alpha(f)(x,t)]^2 \frac{dt}{t}\right\}^{1/2} - \left\{\int_r^\infty [A_\alpha(f)(\widetilde{x},t)]^2 \frac{dt}{t}\right\}^{1/2}\right| \\
&\quad \le \left\{\int_r^\infty [A_\alpha(f)(x,t) - A_\alpha(f)(\widetilde{x},t)]^2 \frac{dt}{t}\right\}^{1/2} \\
&\quad \le \left\{\int_r^\infty \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\theta_t(x-z) - \theta_t(\widetilde{x}-z)||f(z)-f_B|\,dz\right]^2 \frac{dt}{t}\right\}^{1/2} \\
&\quad \le \left\{\int_r^\infty \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \int_B |\theta_t(x-z) - \theta_t(\widetilde{x}-z)||f(z)-f_B|\,dz\right]^2 \frac{dt}{t}\right\}^{1/2}
\end{aligned}$$

$$+\left\{\int_r^\infty\left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\int_{B^\complement}\cdots\,dz\right]^2\frac{dt}{t}\right\}^{1/2}=:\mathrm{J}_1+\mathrm{J}_2.$$

For J_1, since $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, we have

$$\begin{aligned}\mathrm{J}_1&\lesssim\left\{\int_r^\infty\left[\int_B\frac{1}{t^n}|f(z)-f_B|\,dz\right]^2\frac{dt}{t}\right\}^{1/2}\\&\lesssim\left\{\int_r^\infty\frac{dt}{t^{2n+1}}\right\}^{1/2}\int_B|f(z)-f_B|\,dz\\&\lesssim\frac{1}{|B|}\int_B|f(z)-f_B|\,dz\lesssim\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.\end{aligned}\tag{4.7}$$

For J_2, since, for all $t\in(r,\infty)$, x, $\widetilde{x}\in B$ and $z\in B^\complement$, we have $t+|x-z|>|x_0-z|$ and $t+|\widetilde{x}-z|>|x_0-z|$, from this, the Minkowski inequality, the fact that, for $\alpha\in(0,1]$, $\epsilon\in(\max\{n(p/p_0-1),0\},\alpha)$, there exists a positive constant C such that, for any $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$, and x_1, $x_2\in\mathbb{R}^n$,

$$|\theta(x_1)-\theta(x_2)|\le C|x_1-x_2|^\alpha[(1+|x_1|)^{-n-\epsilon}+(1+|x_2|)^{-n-\epsilon}]\tag{4.8}$$

(see [48, p. 775]) and Lemma 4.4, it follows that, for any x, $\widetilde{x}\in B$,

$$\begin{aligned}\mathrm{J}_2&\lesssim\left(\int_r^\infty\left\{\int_{B^\complement}\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left[\left(\frac{t}{t+|x-z|}\right)^{n+\epsilon}\right.\right.\right.\\&\qquad\left.\left.\left.+\left(\frac{t}{t+|\widetilde{x}-z|}\right)^{n+\epsilon}\right]|f(z)-f_B|\,dz\right\}^2\frac{dt}{t}\right)^{1/2}\\&\lesssim\left\{\int_r^\infty\left[\int_{B^\complement}\frac{1}{t^n}\left(\frac{r}{t}\right)^\alpha\left(\frac{t}{|x_0-z|}\right)^{n+\epsilon}|f(z)-f_B|\,dz\right]^2\frac{dt}{t}\right\}^{1/2}\\&\lesssim\int_{B^\complement}|f(z)-f_B|\left\{\int_r^\infty\frac{1}{t^{2n}}\left(\frac{r}{t}\right)^{2\alpha}\left(\frac{t}{|x_0-z|}\right)^{2(n+\epsilon)}\frac{dt}{t}\right\}^{1/2}dz\\&\lesssim\int_{B^\complement}\frac{r^\epsilon|f(z)-f_B|}{|x_0-z|^{n+\epsilon}}\,dz\lesssim\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},\end{aligned}\tag{4.9}$$

which, together with (4.7), $\varphi\in\mathbb{A}_p(\mathbb{R}^n)\subset\mathbb{A}_{q'}(\mathbb{R}^n)$ and $\varphi(B,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})=1$, further implies that

$$\begin{aligned}&\left\{\int_B[\mathrm{I}_3(x)]^q\left[\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}dx\right\}^{1/q}\\&\quad\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}\frac{1}{|B|}\left\{\int_B\left[\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}dx\right\}^{1/q}\\&\quad\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.\end{aligned}\tag{4.10}$$

Combining (4.5), (4.6) and (4.10), we know that

$$\left\{\int_B \left[g_\alpha(f)(x)-\inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x})\right]^q \left[\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}\,dx\right\}^{1/q}$$
$$\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},$$

which completes the proof of Theorem 4.5. □

Corollary 4.6. *Let $\alpha\in(0,1]$, φ be a growth function satisfying $0<p_0\le p_1\le 1$ and $\varphi\in\mathbb{A}_p(\mathbb{R}^n)$ with $p\in[1,\infty)$. If $n(\frac{p}{p_0}-1)<\alpha$, then, for any $f\in\mathcal{L}^{\varphi,1}(\mathbb{R}^n)$, $g_\alpha(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|g_\alpha(f)\|_{\mathcal{L}^{\varphi,1}_*(\mathbb{R}^n)}\le C\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}.$$

Proof. We only need to show that, for all $f\in\mathcal{L}^{\varphi,1}(\mathbb{R}^n)$, if there exists some $u\in\mathbb{R}^n$ such that $g_\alpha(f)(u)<\infty$, then, for any ball $B:=B(x_0,r)\subset\mathbb{R}^n$, with $x_0\in\mathbb{R}^n$ and $r\in(0,\infty)$, and $B\ni u$,

$$\left\{\int_B\left[g_\alpha(f)(x)-\inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x})\right]dx\right\}\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}.$$

Since $0<p_0\le p_1\le 1$, by [31, Theorem 2.7], we find that, for any $q\in(1,p')$, $\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}\sim\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}$. By this, the Hölder inequality, $\varphi(B,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})=1$ and Theorem 4.5, we have

$$\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\int_B\left[g_\alpha(f)(x)-\inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x})\right]dx$$
$$\le\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\left\{\int_B\left[g_\alpha(f)(x)-\inf_{\widetilde{x}\in B} g_\alpha(f)(\widetilde{x})\right]^q\left[\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}dx\right\}^{1/q}$$
$$\lesssim\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}\sim\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}.$$

This finishes the proof of Corollary 4.6. □

On S_α, we have the following boundedness from $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ to $\mathcal{L}^{\varphi,q}_*(\mathbb{R}^n)$.

Theorem 4.7. *Let $\alpha\in(0,1]$, $q\in(1,\infty)$, φ be a growth function as in Definition 2.1 and $\varphi\in\mathbb{A}_p(\mathbb{R}^n)$ with $p\in[1,\infty)$. If $n(\frac{p}{p_0}-1)<\alpha$ and $p\le q'$, then, for any $f\in\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, $S_\alpha(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|S_\alpha(f)\|_{\mathcal{L}^{\varphi,q}_*(\mathbb{R}^n)}\le C\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

Proof. We only need to show that, for all $f\in\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, if there exists some $u\in\mathbb{R}^n$ such that $S_\alpha(f)(u)<\infty$, then, for all balls $B:=B(x_0,r)\subset\mathbb{R}^n$, with $x_0\in\mathbb{R}^n$ and $r\in(0,\infty)$, and $B\ni u$,

$$\left\{\int_B\left[S_\alpha(f)(x)-\inf_{\widetilde{x}\in B} S_\alpha(f)(\widetilde{x})\right]^q\left[\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}dx\right\}^{1/q}$$
$$\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

To this end, for any $x \in B$, since, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \theta(x)\,dx = 0$ and

$$\inf_{\widetilde{x}\in B} S_\alpha(f)(\widetilde{x}) \le S_\alpha(f)(u) < \infty,$$

we write

$$\begin{aligned}
&S_\alpha(f)(x) - \inf_{\widetilde{x}\in B} S_\alpha(f)(\widetilde{x})\\
&\quad\le \left\{\int_0^{r/2}\int_{|x-y|<t} [A_\alpha([f-f_B]\chi_{2B})(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}\\
&\qquad+\left\{\int_0^{r/2}\int_{|x-y|<t} [A_\alpha([f-f_B]\chi_{(2B)^\complement})(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}\\
&\qquad+\sup_{\widetilde{x}\in B}\left|\left\{\int_{r/2}^{\infty}\int_{|x-y|<t} [A_\alpha(f)(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}\right.\\
&\qquad\left.-\left\{\int_{r/2}^{\infty}\int_{|\widetilde{x}-y|<t} [A_\alpha(f)(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}\right|\\
&\quad=: \mathrm{I}_1(x)+\mathrm{I}_2(x)+\mathrm{I}_3(x). \qquad (4.11)
\end{aligned}$$

For $\mathrm{I}_1(x)$, by using an argument similar to that used in the estimate for (4.6), we have

$$\begin{aligned}
&\left\{\int_B [\mathrm{I}_1(x)]^q\left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q}\,dx\right\}^{1/q}\\
&\qquad\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \qquad (4.12)
\end{aligned}$$

For $\mathrm{I}_2(x)$, $x\in B$, noticing that, for any $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$, $\operatorname{supp}\theta\subset B(0,1)$, $|x-y|<t$ and $t\in(0,r/2)$, we have $|y-x_0|<3r/2$, by this, together with $z\in(2B)^\complement$, we further see that $|y-z|\ge|z-x_0|-|x_0-y|>2r-\frac{3r}{2}>t$ and hence

$$A_\alpha([f-f_B]\chi_{(2B)^\complement})(y,t)=\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\frac{1}{t^n}\left|\int_{(2B)^\complement}\theta\left(\frac{y-z}{t}\right)[f(z)-f_B]\,dz\right|=0.$$

Thus, for all $x\in B$, $\mathrm{I}_2(x)\equiv 0$.

For any $x,\widetilde{x}\in B$, from the Minkowski inequality and the fact that, for any $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\theta(x)\,dx=0$, we deduce that

$$\begin{aligned}
&\left|\left\{\int_{r/2}^{\infty}\int_{|x-y|<t}[A_\alpha(f)(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}-\left\{\int_{r/2}^{\infty}\int_{|\widetilde{x}-y|<t}[A_\alpha(f)(y,t)]^2\,\frac{dt}{t^{n+1}}\right\}^{1/2}\right|\\
&\quad=\left|\left\{\int_{r/2}^{\infty}\int_{B(x_0,t)}[A_\alpha(f)(y-x_0+x,\,t)]^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right.\\
&\qquad\left.-\left\{\int_{r/2}^{\infty}\int_{B(x_0,t)}[A_\alpha(f)(y-x_0+\widetilde{x},\,t)]^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right|\\
&\quad\le\left\{\int_{r/2}^{\infty}\int_{B(x_0,t)}\left|A_\alpha(y-x_0+x,\,t)-A_\alpha(y-x_0+\widetilde{x},\,t)\right|^2\,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}
\end{aligned}$$

$$\lesssim \left\{\int_{r/2}^{\infty}\int_{B(x_0,t)}\left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\int_B|\theta_t(y-x_0+x-z)\right.\right.$$
$$\left.\left.-\theta_t(y-x_0+\widetilde{x}-z)||f(z)-f_B|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$+\left\{\int_{r/2}^{\infty}\int_{B(x_0,t)}\left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\int_{B^\complement}\cdots\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$=: \mathrm{J}_1+\mathrm{J}_2.$$

For J_1, since $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, by using an argument similar to that used in the estimate for (4.7), we have

$$\mathrm{J}_1\lesssim\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \tag{4.13}$$

For J_2, from (4.8), we deduce that, for any x, $\widetilde{x}\in B$, $y\in B(x_0,t)$, $t\in(r/2,\infty)$, $z\in B^\complement$ and $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$,

$$|x_0-z|<3t+|y-x_0+x-z|,\quad |x_0-z|<3t+|y-x_0+\widetilde{x}-z|$$

and hence

$$|\theta_t(y-x_0+x-z)-\theta_t(y-x_0+\widetilde{x}-z)|$$
$$\lesssim\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left[\left(\frac{t}{t+|y-x_0+x-z|}\right)^{n+\epsilon}+\left(\frac{t}{t+|y-x_0+\widetilde{x}-z|}\right)^{n+\epsilon}\right]$$
$$\lesssim\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left(\frac{t}{|x_0-z|}\right)^{n+\epsilon},$$

which, together with Lemma 4.4 and an argument similar to that used in the estimate for (4.9), further implies that

$$\mathrm{J}_2\lesssim\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \tag{4.14}$$

Combining (4.13) with (4.14), by an argument similar to that used in the estimate for (4.10), we obtain

$$\left\{\int_B[\mathrm{I}_3(x)]^q\left[\varphi\left(x,\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\right]^{1-q}dx\right\}^{1/q}$$
$$\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}\frac{1}{|B|}\left\{\int_B\left[\varphi\left(x,\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\right]^{1-q}dx\right\}^{1/q}$$
$$\lesssim\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},$$

which, together with (4.11) and (4.12), completes the proof of Theorem 4.7. □

By Theorem 4.7 and an argument similar to that used in the proof of Corollary 4.6, we can prove that S_α is bounded from $\mathcal{L}^{\varphi,1}(\mathbb{R}^n)$ to $\mathcal{L}_*^{\varphi,1}(\mathbb{R}^n)$ as follows, the details being omitted.

Corollary 4.8. *Let $\alpha \in (0,1]$ and φ be a growth function satisfying $0 < p_0 \le p_1 \le 1$ and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$. If $n(\frac{p}{p_0}-1) < \alpha$, then, for any $f \in \mathcal{L}^{\varphi,1}(\mathbb{R}^n)$, $S_\alpha(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|S_\alpha(f)\|_{\mathcal{L}_*^{\varphi,1}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}.$$

Finally, we have the following boundedness of $g^*_{\lambda,\alpha}$ from $\mathcal{L}^{\varphi,q}(\mathbb{R}^n)$ to $\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)$.

Theorem 4.9. *Let $\alpha \in (0,1]$, $q \in (1,\infty)$ and φ be a growth function as in Definition 2.1 and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$. If $n(\frac{p}{p_0}-1) < \alpha$, $p \le q'$ and $\lambda \in (3+\frac{2\alpha}{n},\infty)$, then, for any $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, $g^*_{\lambda,\alpha}(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|g^*_{\lambda,\alpha}(f)\|_{\mathcal{L}_*^{\varphi,q}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

Proof. We only need to show that, for all $f \in \mathcal{L}^{\varphi,q}(\mathbb{R}^n)$, if there exists some $u \in \mathbb{R}^n$ such that $g^*_{\lambda,\alpha}(f)(u) < \infty$, then, for any ball $B := B(x_0,r) \subset \mathbb{R}^n$, with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$, and $B \ni u$,

$$\left\{\int_B \left[g^*_{\lambda,\alpha}(f)(x) - \inf_{\widetilde{x}\in B} g^*_{\lambda,\alpha}(f)(\widetilde{x})\right]^q \left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q}$$
$$\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

To this end, for any $x \in B$, since, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\theta(x)\,dx = 0$ and

$$\inf_{\widetilde{x}\in B} g^*_{\lambda,\alpha}(f)(\widetilde{x}) \le g^*_{\lambda,\alpha}(f)(u) < \infty,$$

we write

$$\begin{aligned}
&g^*_{\lambda,\alpha}(f)(x) - \inf_{\widetilde{x}\in B} g^*_{\lambda,\alpha}(f)(\widetilde{x})\\
&\le \left\{\int_0^r \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\quad + \sup_{\widetilde{x}\in B}\left|\left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right.\\
&\quad \left. - \left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|\widetilde{x}-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right| =: \mathrm{I}(x) + \mathrm{II}(x).
\end{aligned}$$

For any $x \in B$, by the fact that, for all $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\theta(x)\,dx = 0$, we know that

$$\begin{aligned}
\mathrm{I}(x) &= \left\{\int_0^r \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_\alpha(f-f_B)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&= \left\{\int_0^r \int_{\mathbb{R}^n} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha(f-f_B)(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}
\end{aligned}$$

$$\leq \left\{\int_0^r \int_{2B} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha(f-f_B)(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$+\sum_{k=1}^\infty \left\{\int_0^r \int_{2^{k+1}B\setminus 2^kB} \cdots \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} =: \mathrm{I}_0(x)+\sum_{k=1}^\infty \mathrm{I}_k(x).$$

For $\mathrm{I}_0(x)$, we further have

$$\mathrm{I}_0(x)$$
$$\leq \left\{\int_0^r \int_{2B} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha([f-f_B]\chi_{4B})(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$+\left\{\int_0^r \int_{2B} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha([f-f_B]\chi_{(4B)^\complement})(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$=: \mathrm{J}_1(x)+\mathrm{J}_2(x).$$

For any $t\in(0,r)$, $x\in B$, $y\in 2B$ and $z\in(4B)^\complement$, it holds true that

$$|y-x_0+x-z| \geq |x-z|-|x_0-y| > |x_0-z|-|x-x_0|-2r > 4r-r-2r > t.$$

From this and the fact that, for any $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$, $\operatorname{supp}\theta\in B(0,1)$, we deduce that $\theta(\frac{y-x_0+x-z}{t})=0$, which further implies that $\mathrm{J}_2(x)\equiv 0$. By this,

$$\mathrm{J}_1(x) \leq g^*_{\lambda,\alpha}([f-f_B]\chi_{4B})(x) \text{ for all } x\in B,$$

the fact that, when $\lambda\in(3+\frac{2\alpha}{n},\infty)$, $g^*_{\lambda,\alpha}$ is bounded on $L^q_w(\mathbb{R}^n)$ with $q\in(1,\infty)$ and $w\in A_q(\mathbb{R}^n)$, and an argument similar to that used in the estimate for (4.6), we know that

$$\left\{\int_B [\mathrm{I}_0(x)]^q \left[\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{1-q} dx\right\}^{1/q}$$
$$\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \tag{4.15}$$

As for $\mathrm{I}_k(x)$, we have

$$\mathrm{I}_k(x) \leq \left\{\int_0^r \int_{2^{k+1}B\setminus 2^kB} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\right.$$
$$\left.\times [A_\alpha([f-f_B]\chi_{2^{k+2}B})(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$+\left\{\int_0^r \int_{2^{k+1}B\setminus 2^kB} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\right.$$
$$\left.\times [A_\alpha([f-f_B]\chi_{(2^{k+2}B)^\complement})(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$
$$=: \mathrm{H}_k(x)+\mathrm{G}_k(x).$$

By using an argument similar to that used in the estimate for $\mathrm{J}_2(x)$, we have $\mathrm{G}_k(x) \equiv 0$ for all $x \in B$. Thus, if $\lambda > 3$, by the fact that $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, we then see that

$$\begin{aligned}
\mathrm{I}_k(x) &= \mathrm{H}_k(x)\\
&\le \left\{\int_0^r \int_{2^{k+1}B\setminus 2^k B} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} \left[\int_{2^{k+2}B} \frac{1}{t^n}|f(z)-f_B|\,dz\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\
&\lesssim \left\{\int_0^r \left(\frac{t}{2^k r}\right)^{\lambda n} \frac{(2^k r)^{3n}}{t^{3n+1}}\,dt\right\}^{1/2} \left[\frac{1}{|2^{k+2}B|}\int_{2^{k+2}B} |f(z)-f_B|\,dz\right]\\
&\sim 2^{-\frac{kn(\lambda-3)}{2}} \frac{1}{|2^{k+2}B|}\int_{2^{k+2}B} |f(z)-f_B|\,dz,
\end{aligned}$$

which, together with (4.4), $\lambda > 3+\frac{2\alpha}{n}$ and $\alpha > n(\frac{p}{p_0}-1)$, further implies that

$$\begin{aligned}
\sum_{k=1}^\infty \mathrm{I}_k &\lesssim \sum_{k=1}^\infty 2^{kn(s-\frac{\lambda-1}{2})} \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}\\
&\lesssim \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}, \qquad (4.16)
\end{aligned}$$

where $s := \max\{p/p_0, 1\}$.

Combining (4.15) and (4.16), we know that

$$\begin{aligned}
&\left\{\int_B [\mathrm{I}(x)]^q \left[\varphi\left(x, \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}{}^{-1}\right)\right]^{1-q} dx\right\}^{1/q}\\
&\quad\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}. \qquad (4.17)
\end{aligned}$$

To estimate $\mathrm{II}(x)$, for any $x, \widetilde{x} \in B$, from the Minkowski inequality and the fact that, for any $\theta \in \mathcal{C}_\alpha(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \theta(x)\,dx = 0$, we deduce that

$$\begin{aligned}
&\left|\left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right.\\
&\qquad \left. - \left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|\widetilde{x}-y|}\right)^{\lambda n} [A_\alpha(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right|\\
&\quad= \left|\left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha(f-f_B)(y-x_0+x,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right.\\
&\qquad \left. - \left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} [A_\alpha(f-f_B)(y-x_0+\widetilde{x},t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right|\\
&\quad\le \left\{\int_r^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x_0-y|}\right)^{\lambda n} \left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)} \int_B |\theta_t(y-x_0+x-z)\right.\right.
\end{aligned}$$

$$-\theta_t(y-x_0+\widetilde{x}-z)||f(z)-f_B|\,dz\Bigg]^2\frac{dy\,dt}{t^{n+1}}\Bigg\}^{1/2}$$

$$+\left\{\int_r^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\left[\sup_{\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)}\int_{B^\complement}\cdots\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}=:\mathrm{R}_1+\mathrm{R}_2.$$

For R_1, since $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$ is uniformly bounded, by an argument similar to that used in the estimate for (4.7), we have

$$\begin{aligned}\mathrm{R}_1&\le\left\{\int_r^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\left[\int_B\frac{1}{t^n}|f(z)-f_B|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\\&\le\left\{\int_r^\infty\left(\int_{0<|x_0-y|<t}+\sum_{j=1}^\infty\int_{2^{j-1}t\le|x_0-y|<2^jt}\right)\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\frac{dy\,dt}{t^{3n+1}}\right\}^{1/2}\\&\quad\times\left[\int_B|f(z)-f_B|\,dz\right]\\&\lesssim\frac{1}{|B|}\int_B|f(z)-f_B|\,dz\lesssim\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|}\|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.\end{aligned}$$

For R_2, from (4.8), we deduce that, for any $x,\widetilde{x}\in B$, $j\in\mathbb{Z}_+$, $y\in B(x_0,2^jt)$, $t\in(r,\infty)$, $z\in B^\complement$ and $\theta\in\mathcal{C}_\alpha(\mathbb{R}^n)$, it holds true that $|x_0-z|<2^jt+|y-x_0+x-z|+r$, $|x_0-z|<2^jt+|y-x_0+\widetilde{x}-z|+r$ and hence

$$\begin{aligned}&|\theta_t(y-x_0+x-z)-\theta_t(y-x_0+\widetilde{x}-z)|\\&\quad\lesssim\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left[\left(\frac{t}{t+|y-x_0+x-z|}\right)^{n+\epsilon}+\left(\frac{t}{t+|y-x_0+\widetilde{x}-z|}\right)^{n+\epsilon}\right]\\&\quad\lesssim\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left(\frac{2^jt}{|x_0-z|}\right)^{n+\epsilon},\end{aligned}$$

which, together with Lemma 4.4 and an argument similar to that used in the estimate for (4.9), further implies that

$$\begin{aligned}\mathrm{R}_2&\lesssim\Bigg[\int_r^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\\&\quad\times\left\{\int_{B^\complement}\frac{1}{t^n}\left(\frac{|x-\widetilde{x}|}{t}\right)^\alpha\left[\left(\frac{t}{t+|y-x_0+x-z|}\right)^{n+\epsilon}\right.\right.\\&\quad\left.\left.+\left(\frac{t}{t+|y-x_0+\widetilde{x}-z|}\right)^{n+\epsilon}\right]|f(z)-f_B|\,dz\right\}^2\frac{dy\,dt}{t^{n+1}}\Bigg]^{1/2}\\&\lesssim\int_{B^\complement}|f(z)-f_B|\left\{\int_r^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\frac{1}{t^{2n}}\left(\frac{r}{t}\right)^{2\alpha}\right.\end{aligned}$$

$$\times \left[\left(\frac{t}{t+|y-x_0+x-z|} \right)^{n+\epsilon} + \left(\frac{t}{t+|y-x_0+\widetilde{x}-z|} \right)^{n+\epsilon} \right]^2 \frac{dy\,dt}{t^{n+1}} \Bigg\}^{1/2} dz$$

$$\lesssim \int_{B^\complement} |f(z)-f_B| \sum_{j=1}^{\infty} \left\{ \int_r^\infty \int_{2^{j-1}t\le|x_0-y|<2^jt} \left(\frac{1}{2^j}\right)^{\lambda n} \frac{1}{t^{2n}} \left(\frac{r}{t}\right)^{2\alpha} \right.$$

$$\times \left(\frac{2^j t}{|x_0-z|} \right)^{2(n+\epsilon)} \frac{dy\,dt}{t^{n+1}} \Bigg\}^{1/2} dz$$

$$+ \int_{B^\complement} |f(z)-f_B| \left\{ \int_r^\infty \int_{|x_0-y|<t} \frac{1}{t^{2n}} \left(\frac{r}{t}\right)^{2\alpha} \left(\frac{t}{|x_0-z|}\right)^{2(n+\epsilon)} \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} dz$$

$$\lesssim \left[\sum_{j=1}^\infty \frac{1}{2^{jn(\lambda-3-2\epsilon/n)/2}} + 1 \right] \left\{ \int_r^\infty \frac{r^{2\alpha}}{t^{1+2\alpha-2\epsilon}}\,dt \right\}^{1/2} \int_{B^\complement} \frac{|f(z)-f_B|}{|x_0-z|^{n+\epsilon}}\,dz$$

$$\lesssim \int_{B^\complement} \frac{r^\epsilon |f(z)-f_B|}{|x_0-z|^{n+\epsilon}}\,dz \lesssim \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},$$

which, together with the estimate of R_1, further implies that, for all $x \in B$,

$$\mathrm{II}(x) \lesssim \frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{|B|} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)}.$$

Thus, we have

$$\left\{ \int_B [\mathrm{II}(x)]^q \left[\varphi\left(x, \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) \right]^{1-q} dx \right\}^{1/q} \lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} \|f\|_{\mathcal{L}^{\varphi,q}(\mathbb{R}^n)},$$

which, combined with (4.17), completes the proof of Theorem 4.9. □

By Theorem 4.9 and an argument similar to that used in the proof of Corollary 4.6, we can prove that $g^*_{\lambda,\alpha}$ is bounded from $\mathcal{L}^{\varphi,1}(\mathbb{R}^n)$ to $\mathcal{L}^{\varphi,1}_*(\mathbb{R}^n)$ as follows, the details being omitted.

Corollary 4.10. *Let $\alpha \in (0,1]$ and φ be a growth function satisfying $0 < p_0 \le p_1 \le 1$ and $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$. If $n(\frac{p}{p_0}-1) < \alpha$ and $\lambda \in (3+\frac{2\alpha}{n},\infty)$, then, for any $f \in \mathcal{L}^{\varphi,1}(\mathbb{R}^n)$, $g^*_{\lambda,\alpha}(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, there exists a positive constant C, independent of f, such that*

$$\|g^*_{\lambda,\alpha}(f)\|_{\mathcal{L}^{\varphi,1}_*(\mathbb{R}^n)} \le C\|f\|_{\mathcal{L}^{\varphi,1}(\mathbb{R}^n)}.$$

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References

1. D.R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.
2. D.R. Adams and J. Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), 201–230.
3. T. Aoki, *Locally bounded linear topological space*, Proc. Imp. Acad. Tokyo **18** (1942), 588–594.
4. Z. Birnbaum and W. Orlicz, *Über die verallgemeinerung des begriffes der zueinander konjugierten potenzen*, Studia Math. **3** (1931), 1–67.
5. A. Bonami, J. Feuto and S. Grellier, *Endpoint for the DIV-CURL lemma in Hardy spaces*, Publ. Mat. **54** (2010), 341–358.
6. A. Bonami and S. Grellier, *Hankel operators and weak factorization for Hardy-Orlicz spaces*, Colloq. Math. **118** (2010), 107–132.
7. A. Bonami, S. Grellier and L.D. Ky, *Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets*, J. Math. Pures Appl. **97** (2012), 230–241.
8. A. Bonami, T. Iwaniec, P. Jones and M. Zinsmeister, *On the product of functions in BMO and H^1*, Ann. Inst. Fourier (Grenoble) **57** (2007), 1405–1439.
9. S. Campanato, *Proprietá di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 137–160.
10. F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. (7) **7** (1987), 273–279 (1988).
11. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics, **242**, Springer-Verlag, Berlin-New York, 1971.
12. J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, Providence, Rhode Island, 2000.
13. C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.
14. J. Feuto, *Intrinsic square functions on functions spaces including weighted Morrey spaces*, Bull. Korean Math. Soc. (to appear) or arXiv: 1205.0129.
15. A. Fiorenza and M. Krbec, *Indices of Orlicz spaces and some applications*, Comment. Math. Univ. Carolin. **38** (1997), 433–451.
16. J. García–Cuerva, *Weighted H^p spaces*, Dissertationes Math. (Rozprawy Mat.) **162** (1979), 63 pp.
17. J. García–Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Publishing Co., Amsterdam, 1985.
18. R.F. Gundy and R.L. Wheeden, *Weighted integral inequalities for nontangential maximal function, Lusin area integral, and Walsh-Paley series*, Studia Math. **49** (1973/74), 107–124.
19. S. Hou, D. Yang and S. Yang, *Lusin area function and molecular characterizations of Musielak–Orlicz Hardy spaces and their applications*, Commun. Contemp. Math. (2013), DOI: 10.1142/S0219199713500296.
20. G. Hu, Y. Meng and D. Yang, *Estimates for Marcinkiewicz integrals in BMO and Campanto spaces*, Glasg. Math. J. **49** (2007), 167–187.
21. S. Janson, *Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation*, Duke Math. J. **47** (1980), 959–982.
22. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.
23. V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.
24. Y. Komori and S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2009), 219–231.
25. A. Kufner, *Weighted inequalities and spectral problems*, Banach J. Math. Anal. **4** (2010), 116–121.

26. L.D. Ky, *New Hardy spaces of Musielak–Orlicz type and boundedness of sublinear operators*, arXiv: 1103.3757.
27. L.D. Ky, *Bilinear decompositions and commutators of singular integral operators*, Trans. Amer. Math. Soc. **365** (2013), 2931–2958.
28. A.K. Lerner, *Sharp weighted norm inequalities for Littlewood–Paley operators and singular integrals*, Adv. Math. **226** (2011), 3912-3926.
29. A.K. Lerner, *On sharp aperture-weighted estimates for square functions*, arXiv: 1301.1051.
30. Y. Liang, J. Huang and D. Yang, *New real-variable characterizations of Musielak–Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), 413–428.
31. Y. Liang and D. Yang, *Musielak–Orlicz Campanato spaces and applications*, J. Math. Anal. Appl. **406** (2013), 307–322.
32. Y. Liang and D. Yang, *Intrinsic square function characterizations of Musielak–Orlicz Hardy spaces*, Trans. Amer. Math. Soc. (to appear).
33. H. Lin and D. Yang, *An interpolation theorem for sublinear operators on non-homogeneous metric measure spaces*, Banach J. Math. Anal. **6** (2012), 168–179.
34. Y. Meng, E. Nakai and D. Yang, *Estimates for Lusin-area and Littlewood–Paley g_λ^* functions over spaces of homogeneous type*, Nonlinear Anal. **72** (2010), 2721–2736.
35. C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
36. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, **1034**, Springer-Verlag, Berlin, 1983.
37. E. Nakai, *Calderón-Zygmund operators on Orlicz–Morrey spaces and modular inequalities*, Banach and function spaces II, 393-410, Yokohama Publ., Yokohama, 2008.
38. E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), 3665–3748.
39. W. Orlicz, *Über eine gewisse Klasse von Raumen vom Typus B*, Bull. Int. Acad. Pol. Ser. A **8** (1932), 207–220.
40. J. Prochno, *A combinatorial approach to Musielak–Orlicz spaces*, Banach J. Math. Anal. **7** (2013), 132–141.
41. M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, Basel and Hong Kong, 1991.
42. S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III **5** (1957), 471–473.
43. Y. Sawano, S. Sugano and H. Tanaka, *Orlicz–Morrey spaces and fractional operators*, Potential Anal. **36** (2012), 517–556.
44. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, **1381**, Springer-Verlag, Berlin, 1989.
45. Y. Sun, *On the existence and boundedness of square function operators on Campanato spaces*, Nagoya Math. J. **173** (2004), 139–151.
46. B.E. Viviani, *An atomic decomposition of the predual of $BMO(\rho)$*, Rev. Mat. Iberoamericana **3** (1987), 401–425.
47. H. Wang, *Intrinsic square functions on the weighted Morrey spaces*, J. Math. Anal. Appl. **396** (2012), 302–314.
48. M. Wilson, *The intrinsic square function*, Rev. Mat. Iberoam. **23** (2007), 771–791.
49. M. Wilson, *Weighted Littlewood–Paley Theory and Exponential-Square Integrability*, Lecture Notes in Mathematics, **1924**, Springer, Berlin, 2008.
50. M. Wilson, *How fast and in what sense(s) does the Calderón reproducing formula converge?* J. Fourier Anal. Appl. **16** (2010), 768–785.
51. M. Wilson, *Convergence and stability of the Calderón reproducing formula in H^1 and BMO*, J. Fourier Anal. Appl. **17** (2011), 801–820.
52. K. Yabuta, *Boundedness of Littlewood–Paley operators*, Math. Japon. **43** (1996), 143–150.

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