ALGEBRAICALLY PARANORMAL OPERATORS ON BANACH SPACES

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Abstract. In this paper we show that a bounded linear operator on a Banach space $X$ is polaroid if and only if $p(T)$ is polaroid for some polynomial $p$. Consequently, algebraically paranormal operators defined on Banach spaces are hereditarily polaroid. Weyl type theorems are also established for perturbations $f(T + K)$, where $T$ is algebraically paranormal, $K$ is algebraic and commutes with $T$, and $f$ is an analytic function, defined on an open neighborhood of the spectrum of $T + K$, such that $f$ is nonconstant on each of the components of its domain. These results subsume recent results in this area.

1. Paranormal operators

There is a growing interest concerning paranormal operators, ([12, 14, 19, 7, 23]) and subclasses of paranormal operators ([17]), since the class of paranormal operators properly contains a relevant number of Hilbert space operators.

Paranormal operators are polaroid, where a bounded operator $T \in L(X)$ defined on a Banach space is said to be polaroid if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent. Polaroid operators have been studied in recent papers in relation with Weyl type theorems, see [16, 15, 3, 6]. In this note we show that algebraically paranormal operators on Banach spaces are hereditarily polaroid, extending previous results known for Hilbert space operators. This is a consequence of the following more general result: $T \in L(X)$ is polaroid if and only if $f(T)$ is polaroid for some analytic function $f$ (or equivalently, for some polynomial $p$), defined on an open neighborhood of $\sigma(T)$, such that $f$ is nonconstant on each of the components of its domain. These results are, in the final
part, applied for obtaining Weyl type theorems for operators $f(T + K)$, where $T$ is algebraically paranormal and $K$ is an algebraic operator which commutes with $T$.

We introduce the relevant terminology. A bounded linear operator $T \in L(X)$, $X$ an infinite dimensional complex Banach space, is said to be paranormal if

$$\|Tx\| \leq \|T^2x\| \|x\| \quad \text{for all } x \in X.$$ 

It is known that the property of being paranormal is not translation-invariant by scalars. The quasi-nilpotent part of an operator $T \in L(X)$ is the set

$$H_0(T) := \{x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$ 

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$.

An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$), if for every open disc $U$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$. Note that, that both $T$ and its dual $T^*$ (or in the case of Hilbert space operators, the adjoint $T'$) have SVEP at every isolated point of the spectrum $\sigma(T) = \sigma(T^*)$. Furthermore, the SVEP is inherited by the restrictions to closed invariant subspaces, i.e. if $T \in L(X)$ has the SVEP at $\lambda_0$ and $M$ is a closed $T$-invariant subspace then $T|_M$ has SVEP at $\lambda_0$.

The quasi-nilpotent part of an operator generally is not closed. We have

$$H_0(\lambda I - T) \text{ closed } \Rightarrow \ T \text{ has SVEP at } \lambda,$$  \hspace{1cm} (1.1)

see [5].

The following result is well-known, see [12, Corollary 2.10] and [7, p. 2445].

**Theorem 1.1.** Every paranormal operator on a separable Banach space has SVEP. Paranormal operators on Hilbert spaces have SVEP.

It is known that every paranormal operator $T \in L(X)$ is normaloid, i.e. $\|T\|$ is equal to the spectral radius of $T$. Consequently, if $T \in L(X)$

$$T \text{ quasi-nilpotent paranormal } \Rightarrow \ T = 0.$$  \hspace{1cm} (1.2)

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial $h$ such that $h(T)$ is paranormal is said to be algebraically paranormal. Note that algebraic paranormality is preserved under translation by scalars and under restriction to closed invariant subspaces.

Two classical quantities associated with a linear operator $T$ are the ascent $p := p(T)$, defined as the smallest non-negative integer $p$ (if it does exist) such that $\ker T^p = \ker T^{p+1}$, and the descent $q := q(T)$, defined as the smallest non-negative integer $q$ (if it does exists) such that $T^q(X) = T^{q+1}(X)$. It is well-known that if $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite then $p(\lambda I - T) = q(\lambda I - T)$ and $\lambda$ is a pole of the the function resolvent $\lambda \to (\lambda I - T)^{-1}$, in particular an isolated
Suppose that $T$ is nilpotent. Then from the spectral mapping theorem we have \[\tau(T) = \{0\}.\] 

**Lemma 1.2.** Suppose that $T \in L(X)$ is algebraically paranormal and quasi-nilpotent. Then $T$ is nilpotent.

**Proof.** Suppose that $h$ is a polynomial for which $h(T)$ is paranormal. From the spectral mapping theorem we have 
\[\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}.\]

We claim that $h(T) = h(0)I$. To see that let us consider the two possibilities: $h(0) = 0$ or $h(0) \neq 0$.

If $h(0) = 0$ then $h(T)$ is quasi-nilpotent, so from the implication (1.2), we deduce that $h(T) = 0$, hence the equality $h(T) = h(0)I$ trivially holds.

Suppose the other case $h(0) \neq 0$, and set $h_1(T) := \frac{1}{h(0)} h(T)$. Clearly, $h_1(T)$ has spectrum $\{1\}$ and $\|h_1(T)\| = 1$. Moreover, $h_1(T)$ is invertible and also its inverse $h_1(T)^{-1}$ has norm 1. The operator $h_1(T)$ is then doubly power-bounded and by a classical theorem due to Gelfand, see [20, Theorem 1.5.14] for a proof, it then follows that $h_1(T) = I$, and hence $h(T) = h(0)I$, as claimed.

Now, from the equality $h(0)I - h(T) = 0$, we see that there exist some natural $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$ for which 
\[0 = h(0)I - h(T) = \mu T^n \prod_{i=1}^{n} (\lambda_i I - T)\]
with $\lambda_i \neq 0$,
where all $\lambda_i I - T$ are invertible. This obviously implies that $T^n = 0$, so $T$ is nilpotent.

Recall first that if $T \in L(X)$, the analytic core $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

**Theorem 1.3.** If $T \in L(X)$ is algebraically paranormal then every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent; i.e. $T$ is polaroid.

**Proof.** We show that for every isolated point $\lambda$ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Let $\lambda$ be an isolated point of $\sigma(T)$, and denote by $P_\lambda$ denote the spectral projection associated with $\{\lambda\}$. Then $M := K(\lambda I - T) = \ker P_\lambda$ and $N := H_0(\lambda I - T) = P_\lambda(X)$, see [1, Theorem 3.74]. Therefore, $H = H_0(\lambda I - T) \oplus K(\lambda I - T)$. Furthermore, since $\sigma(T|N) = \{\lambda\}$, while $\sigma(T|M) = \sigma(T) \setminus \{\lambda\}$, so the restriction $\lambda I - T|N$ is quasi-nilpotent and $\lambda I - T|M$ is invertible. Since $\lambda I - T|N$ is algebraically paranormal then Lemma 1.2 implies that $\lambda I - T|N$ is nilpotent. In other worlds, $\lambda I - T$ is an operator of Kato Type, see [1, Chapter
1] for details.

Now, both $T$ and its dual $T^*$ have SVEP at $\lambda$, since $\lambda$ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.16 and Theorem 3.17 of [1], that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Therefore, $\lambda$ is a pole of the resolvent. \hfill \Box

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be Drazin invertible (with a finite index) if and only if $p(T) = q(T) < \infty$.

**Definition 1.4.** $T \in L(X)$ is said to be left Drazin invertible if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be right Drazin invertible if $q := q(T) < \infty$ and $T^q(X)$ is closed.

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if $T$ is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$, see [18, Prop. 50.2].

The concepts of left or right Drazin invertibility lead to the concepts of left or right pole. Let us denote by $\sigma_a(T)$ the classical approximate point spectrum and by $\sigma_s(T)$ the surjectivity spectrum. It is well known that $\sigma_a(T^*) = \sigma_s(T)$ and $\sigma_a(T^*) = \sigma_a(T)$.

**Definition 1.5.** Let $T \in L(X)$, $X$ a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_a(T)$ then $\lambda$ is said to be a left pole of the resolvent of $T$. A left pole $\lambda$ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_s(T)$ then $\lambda$ is said to be a right pole of the resolvent of $T$. A right pole $\lambda$ is said to have finite rank if $\beta(\lambda I - T) < \infty$.

Evidently, $\lambda$ is a pole of $T$ if and only if $\lambda$ is both a left and a right pole of $T$. Moreover, $\lambda$ is a pole of $T$ if and only if $\lambda$ is a pole of $T'$. In the case of Hilbert space operators, $\lambda$ is a pole of $T'$ if and only if $\bar{\lambda}$ is a pole of $T^*$.

**Definition 1.6.** Let $T \in L(X)$. Then

(i) $T$ is said to be left polaroid if every isolated point of $\sigma_a(T)$ is a left pole of the resolvent of $T$.

(ii) $T \in L(X)$ is said to be right polaroid if every isolated point of $\sigma_s(T)$ is a right pole of the resolvent of $T$.

(iii) $T \in L(X)$ is said to be $a$-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of $T$.

Let $\text{iso} \sigma(T)$ denote the set of all isolated points of $\sigma(T)$. The condition of being polaroid may be characterized as follows:

**Theorem 1.7.** [6, Theorem 2.2] Suppose that $T \in L(X)$. Then we have:

(i) $T$ is polaroid if and only if for every $\lambda \in \text{iso} \sigma(T)$, there exists $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker (\lambda I - T)\nu$.

(ii) Suppose that $T$ is left polaroid. Then, for every $\lambda \in \text{iso} \sigma_a(T)$, there exists $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker (\lambda I - T)\nu$. 
Note that the concepts of left and right polaroid are dual each other, see [3]. If \( T \in L(X) \) then the following implications hold:

\[
T \text{ a-polaroid } \Rightarrow T \text{ left polaroid } \Rightarrow T \text{ polaroid.}
\]

Furthermore, if \( T \) is right polaroid then \( T \) is polaroid. The first implication is clear, since a pole is always a left pole. Assume that \( T \) is left polaroid and let \( \lambda \in \sigma(T) \). It is known that the boundary of the spectrum is contained in \( \sigma_a(T) \), in particular every isolated point of \( \sigma(T) \), thus \( \lambda \in \sigma_a(T) \) and hence \( \lambda \) is a left pole of the resolvent of \( T \). By part (ii) of Theorem 1.7, then there exists a natural \( \nu := \nu(\lambda I - T) \in \mathbb{N} \) such that \( H_0(\lambda I - T) = \ker (\lambda I - T)^\nu \). But \( \lambda \) is isolated in \( \sigma(T) \), so \( T \) is polaroid, by part (i) of Theorem 1.7.

To show the last assertion suppose that \( T \) is right polaroid. Then \( T^* \) is left polaroid and hence, by the first part, \( T^* \) is polaroid, or equivalently \( T \) is polaroid.

2. Weyl type theorems for perturbations of paranormal operators

Recall that an operator \( T \in L(X) \) is said to be Weyl (\( T \in W(X) \)), if \( T \) is Fredholm (i.e. \( \alpha(T) := \dim \ker T \) and \( \beta(T) := \text{codim} T(X) \) are both finite) and the index \( \text{ind} T := \alpha(T) - \beta(T) = 0 \). The Weyl spectrum of \( T \in L(X) \) is defined by

\[
\sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \notin W(X) \}.
\]

Following Coburn [13], we say that Weyl’s theorem holds for \( T \in L(X) \) if

\[
\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \tag{2.1}
\]

where

\[
\pi_{00}(T) := \{ \lambda \in \sigma(T) : 0 < \lambda \alpha(I - T) < \infty \}.
\]

The concept of Fredholm operators has been generalized by Berkani ([10]) in the following way: for every \( T \in L(X) \) and a nonnegative integer \( n \) let us denote by \( T_{[n]} \) the restriction of \( T \) to \( T_n(X) \) viewed as a map from the space \( T_n(X) \) into itself (we set \( T_{[0]} = T \)). \( T \in L(X) \) is said to be B-Fredholm if for some integer \( n \geq 0 \) the range \( T_n(X) \) is closed and \( T_{[n]} \) is a Fredholm operator. In this case \( T_{[n]} \) is a Fredholm operator for all \( m \geq n \) ([10]). This enables one to define the index of a Fredholm as \( \text{ind} T = \text{ind} T_{[n]} \). A bounded operator \( T \in L(X) \) is said to be B-Weyl (\( T \in BW(X) \)) if for some integer \( n \geq 0 \) \( T_n(X) \) is closed and \( T_{[n]} \) is Weyl. The B-Weyl spectrum \( \sigma_{bw}(T) \) is defined

\[
\sigma_{bw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \notin BW(X) \}.
\]

Another version of Weyl’s theorem has been introduced by Berkani and Koliha ([11] as follows: \( T \in L(X) \) is said to verify generalized Weyl’s theorem, (abbreviated, \( gW \)), if

\[
\sigma(T) \setminus \sigma_{bw}(T) = E(T), \tag{2.2}
\]

where

\[
E(T) := \{ \lambda \in \sigma(T) : 0 < \alpha(I - T) \}.
\]

Note the generalized Weyl’s theorem entails Weyl’s theorem.
The following result shows that in presence of SVEP the polaroid condition entails Weyl’s type theorems.

**Theorem 2.1.** Let \( T \in L(X) \) be polaroid and suppose that either \( T \) or \( T^* \) has SVEP. Then both \( T \) and \( T^* \) satisfy generalized Weyl’s theorem.

**Proof.** If \( T \) is polaroid also \( T^* \) is polaroid, and Weyl’s theorem and generalized Weyl’s theorem for \( T \), or \( T^* \), are equivalent, see [3, Theorem 3.7]. The assertion then follows from [3, Theorem 3.3]. \( \square \)

As an immediate consequence of Theorem 2.1 we obtain that, for every algebraically paranormal operator \( T \) defined on a separable Banach space, or defined on a Hilbert space (in this case, the dual \( T^* \) may be replaced by the Hilbert adjoint \( T' \)), then both \( T \) and \( T^* \) satisfy generalized Weyl’s theorem. This result, for algebraically paranormal operators on Hilbert spaces, has been proved in [14].

It should be noted that if \( T \) is paranormal on a Banach space \( X \) then Weyl’s theorem holds for \( T \) and \( T^* \), without assuming separability on \( X \), see [12, Theorem 2.12].

Let \( H_{nc}(\sigma(T)) \) denote the set of all analytic functions, defined on an open neighborhood of \( \sigma(T) \), such that \( f \) is nonconstant on each of the components of its domain. Define, by the classical functional calculus, \( f(T) \) for every \( f \in H_{nc}(\sigma(T)) \).

The proof of the following results may be found in Lemma 1.76 and Lemma 3.101 of [1].

**Lemma 2.2.** Let \( \{\lambda_1, \ldots, \lambda_k\} \) be a finite subset of \( \mathbb{C} \), with \( \lambda_i \neq \lambda_j \) for \( i \neq j \). If \( \{\nu_1, \ldots, \nu_k\} \subset \mathbb{N} \) and \( p(\lambda) := \prod_{i=1}^k (\lambda_i - \lambda)^{\nu_i} \) then

\[
\ker p(T) = \bigoplus_{i=1}^k \ker (\lambda_i I - T)^{\nu_i}.
\]

Furthermore, if \( p(\lambda_0) \neq 0 \) for some \( \lambda_0 \in \mathbb{C} \) then \( H_0(\lambda_0 I - T) \cap \ker p(T) = \{0\} \).

**Remark 2.3.** It is easy to check from the definition of a quasi-nilpotent part the following properties:

(i) \( H_0(T) \subseteq H_0(T^k) \), for all \( k \in \mathbb{N} \).

(ii) If \( T, U \in L(X) \) commutes and \( S = TU \) then \( H_0(T) \subseteq H_0(S) \).

We are now ready for the main result of this section.

**Theorem 2.4.** For an operator \( T \in L(X) \) the following statements are equivalent.

(i) \( T \) is polaroid;

(ii) \( f(T) \) is polaroid for every \( f \in H_{nc}(\sigma(T)) \);

(iii) there exists a non-trivial polynomial \( p \) such that \( p(T) \) is polaroid;

(iv) there exists \( f \in H_{nc}(\sigma(T)) \) such that \( f(T) \) is polaroid.

**Proof.** The implication (i) \( \Rightarrow \) (ii) has been proved in [6, Theorem 2.5]. The implications (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are obvious.
(iv) $\Rightarrow$ (i) Suppose $f(T)$ polaroid for some $f \in \mathcal{H}_{nc}(\sigma(T))$ and let $\lambda_0 \in \text{iso} \sigma(T)$ be arbitrary. Then $\mu_0 := f(\lambda_0) \in f(\text{iso} \sigma(T))$. It is easily seen that $\mu_0 \in \text{iso} f(\sigma(T))$. Indeed, suppose that $\mu_0$ is not isolated in $f(\sigma(T))$. Then there exists a sequence $(\mu_n) \subset f(\sigma(T))$ of distinct scalars such that $\mu_n \to \mu_0$ as $n \to +\infty$. Let $\lambda_n \in \sigma(T)$ such that $\mu_n = f(\lambda_n)$ for all $n$. Clearly, $\lambda_n \neq \lambda_m$ for $n \neq m$, and since $\mu_n = f(\lambda_n) \to \mu_0 = p(\lambda_0)$ then $\lambda_n \to \lambda_0$, and this is impossible since, by assumption, $\lambda_0 \in \text{iso} \sigma(T)$. By the spectral mapping theorem then $\mu_0 \in \text{iso} \sigma(f(T))$. Now, since $f(T)$ is polaroid, the part (i) of Theorem 1.7 entails that there exists a natural $\nu$ such that

$$H_0(\mu I - f(T)) = \ker (\mu I - f(T))^\nu.$$  \hspace{1cm} (2.3)

Let $g(\lambda) := \mu_0 - f(\lambda)$. Trivially, $\lambda_0$ is a zero of $g$, and $g$ may have only a finite number of zeros. Let $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be the set of all zeros of $g$, with $\lambda_i \neq \lambda_j$, for all $i \neq j$. Define $p(\lambda) := \prod_{i=1}^n (\lambda - \lambda_i)^{\nu_i}$, where $\nu_i$ is the multiplicity of $\lambda_i$. Then we can write, for some $k \in \mathbb{N}$,

$$g(\lambda) = (\lambda_0 - \lambda)^k p(\lambda) h(\lambda),$$

where $h(\lambda)$ is an analytic function which does not vanish in $\sigma(T)$. Consequently,

$$g(T) = \mu_0 I - f(T) = (\lambda_0 I - T)^k p(T) h(T),$$

where $h(T)$ is invertible, and hence

$$H_0(\mu_0 I - f(T)) = H_0((\lambda_0 I - T)^k p(T) h(T)) = H_0((\lambda_0 I - T)^k p(T)).$$

By Remark 2.3, we then have

$$H_0(\lambda_0 I - T) \subseteq H_0((\lambda_0 I - T)^k) \subseteq H_0((\lambda_0 I - T)^k p(T))$$

$$= H_0(\mu_0 I - f(T)),$$

and, evidently,

$$\ker g(T) = \ker [(\lambda_0 I - T)^k p(T)].$$

By Lemma 2.2, we also have

$$\ker g(T) = \ker (\mu_0 I - f(T)) = \ker [(\lambda_0 I - T)^k \oplus \ker p(T)].$$

and hence, from (2.3),

$$H_0(\mu_0 I - f(T)) = \ker (\lambda_0 I - T)^{k\nu} \oplus \ker p(T)^k.$$

Therefore,

$$H_0(\lambda_0 I - T) \subseteq \ker (\lambda_0 I - T)^{k\nu} \oplus \ker p(T)^k.$$

Since, by Lemma 2.2, we have $H_0(\lambda_0 I - T) \cap \ker p(T)^k = \{0\}$, we then conclude that $H_0(\lambda_0 I - T) \subseteq \ker (\lambda_0 I - T)^{k\nu}$. The opposite of the latter inclusion also holds, so we have $H_0(\lambda_0 I - T) = \ker (\lambda_0 I - T)^{k\nu}$. Theorem 1.7 then entails that $T$ is polaroid.

A natural question is if the analogous of Theorem 2.4 holds for left polaroid operators. The implication

$$T \text{ left polaroid } \Rightarrow f(T) \text{ left polaroid},$$

holds for every $f \in \mathcal{H}_{nc}(\sigma(T))$, see [3, Lemma 3.11]. Denote by $\mathcal{H}'_{nc}(\sigma(T))$ the subset of all $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f$ is injective.
Theorem 2.5. For an operator \( T \in L(X) \) the following statements are equivalent.

(i) \( T \) is left polaroid;
(ii) \( f(T) \) is left polaroid for every \( f \in H_{nc}(\sigma(T)) \);
(iii) there exists \( f \in H_{nc}(\sigma(T)) \) such that \( f(T) \) is left polaroid.

Proof. We have only to show that (iii) \( \Rightarrow \) (i). Let \( \lambda_0 \) be an isolated point of \( \sigma_a(T) \) and let \( \mu_0 := f(\lambda_0) \). As in the proof of Theorem 2.4 it then follows that \( \mu_0 \in \text{iso}(\sigma_a(f(T))) \), so \( \mu_0 \) is a left pole of \( f(T) \). By Theorem 2.9 of [9] there exists a left pole \( \eta \) of \( T \) such that \( f(\eta) = \mu_0 \) and since \( f \) is injective then \( \eta = \lambda_0 \). Therefore, \( T \) is left polaroid. \( \square \)

A bounded operator \( T \in L(X) \) is said to be hereditarily polaroid, i.e. any restriction to an invariant closed subspace is polaroid. This class of operators has been first considered in [16]. Examples of hereditarily polaroid operators are \( H(p) \)-operators (i.e. operators on Banach spaces for which for every \( \lambda \in \mathbb{C} \) there exists a natural \( p := p(\lambda) \) such that \( H_0(\lambda I - T) = \ker(\lambda I - T^p) \). Property \( H(p) \) is satisfied by every generalized scalar operator, see [20] for details of this class of operators), and in particular for \( p \)-hyponormal, log-hyponormal or \( M \)-hyponormal operators on Hilbert spaces, see [21]. An example of polaroid operator which is not hereditarily polaroid may be found in [16, Example 2.6].

Corollary 2.6. Algebraically paranormal operators on Banach spaces are hereditarily polaroid.

Proof. Let \( T \in L(X) \) be algebraically paranormal and \( M \) a closed \( T \)-invariant subspace of \( X \). By assumption there exists a nontrivial polynomial \( h \) such that \( h(T) \) is paranormal. The restriction of any paranormal operator to an invariant closed subspace is also paranormal, so \( h(T|_M) = h(T)|_M \) is paranormal and hence polaroid, by Theorem 1.3. From Theorem 2.4 we then conclude that \( T|_M \) is polaroid. \( \square \)

Recall that a bounded operator \( K \in L(X) \) is said to be algebraic if there exists a non-constant polynomial \( h \) such that \( h(K) = 0 \). Trivially, every nilpotent operator is algebraic and it is well-known that if \( K^n(X) \) has finite dimension for some \( n \in \mathbb{N} \) then \( K \) is algebraic. In [4] it is shown that if \( T \) is hereditarily polaroid and has SVEP, and \( K \) is an algebraic operator which commutes with \( T \) then \( T + K \) is polaroid and \( T^* + K^* \) is \( a \)-polaroid.

Theorem 2.7. Let \( T \in L(X) \) be an algebraically paranormal operator on a separable Banach space \( X \), and let \( K \in L(X) \) be an algebraic operator commuting with \( T \). Then both \( f(T + K) \) and \( f(T^* + K^*) \) satisfies \((gW)\) for every \( f \in H_{nc}(\sigma(T + K)) \). An analogous result holds if \( T \) is an algebraically paranormal operator on a Hilbert space.

Proof. Suppose that \( T \in L(X) \) is algebraically paranormal operator, and let \( h \) be a non-trivial polynomial for which \( h(T) \) is paranormal, and hence has SVEP, since \( T \) has SVEP. From Theorem [1, Theorem 2.40] it the follows that also \( T \) has SVEP. Now, by Corollary 2.6 \( T \) is hereditarily polaroid. By Theorem 2.15
of [4] then $T + K$ is polaroid and $T^* + K^*$ is a-polaroid (and hence polaroid). By Theorem 2.4 then $f(T + K)$ is polaroid. Moreover, $T + K$ has SVEP, by [8, Theorem 2.14] and hence $f(T + K)$ has SVEP, again by [1, Theorem 2.40]. The assertions then follows by Theorem 2.1.

The last assertion is proved with the same argument, since $T$ has SVEP. □

Theorem 2.7 considerably improves the results of Theorem 2.4 of [14] proved for algebraically paranormal operators defined on a separable Hilbert spaces $H$, and also improves Theorem 2.5 of [7], proved in the case of paranormal operators on Hilbert spaces. Observe that, always in the situation of Theorem 2.7, the fact that $f(T + K)$ is polaroid entails that all Weyl type theorems (as properties $(gw)$ and $(gaW)$) hold for $f(T^* + K^*)$, see [3] for definitions and details, in particular Theorem 3.10.

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