HILBERT-SCHMIDT DIFFERENCES OF COMPOSITION OPERATORS BETWEEN THE WEIGHTED BERGMAN SPACES ON THE UNIT BALL

LI ZHANG AND ZE-HUA ZHOU*

Communicated by M. A. Ragusa

ABSTRACT. Let $\varphi, \psi$ be the analytic self-maps of the unit ball $B$, we characterize the Hilbert-Schmidt differences of two composition operator $C_\varphi$ and $C_\psi$ on weighted Bergman space $A^2_\alpha$, and give some conclusions about the topological structure of $C(A^2_\alpha)$, the space of all bounded composition operators on $A^2_\alpha$ endowed with operator norm.

1. Introduction

Let $\mathbb{B}$ be the unit ball in the $N$-dimensional complex space $\mathbb{C}^N$, with $\mathbb{D}$ for the unit disk of complex plane $\mathbb{C}$, $S(\mathbb{B})$ the collection of all holomorphic self-maps of $\mathbb{B}$ and let $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Some function spaces, for instance, bounded mean oscillation class (BMO), vanishing mean oscillation class (VMO), Bergman space, Bloch space or other recent spaces, are treated by many authors (see e.g. [1, 16, 25, 26]). The inner product of $\mathbb{C}^N$ defined as

$$\langle z, w \rangle = \sum_{k=1}^{N} z_k \overline{w}_k,$$

where $z = (z_1, \cdots, z_N) \in \mathbb{C}^N$ and $w = (w_1, \cdots, w_N) \in \mathbb{C}^N$. 

---

Date: Received: 26 July 2012; Accepted: 1 September 2012.

* Corresponding author.

2010 Mathematics Subject Classification. Primary: 47B38; Secondary: 45P05, 47G10, 32A37, 47B33, 33B30, 46E15.

Key words and phrases. Hilbert-Schmidt operator, composition operator, weighted Bergman space, unit ball.
For $\alpha > -1$, the weighted Bergman space $A^2_\alpha = A^2_\alpha(\mathbb{B})$ consists of holomorphic functions $f$ on $\mathbb{B}$ satisfying

$$\|f\|_{A^2_\alpha}^2 = \int_{\mathbb{B}} |f(z)|^2 d\nu_\alpha(z),$$

where

$$d\nu_\alpha(z) = \frac{\Gamma(N + \alpha + 1)}{\Gamma(N + 1)\Gamma(\alpha + 1)} (1 - |z|^2)^\alpha d\nu(z),$$

$d\nu$ denotes the normalized Lebesgue volume measure on $\mathbb{B}$ and $\Gamma$ the usual Euler function, extension of the factorial function.

The Hardy space $H^2 = H^2(\mathbb{B})$ is the set of functions analytic on $\mathbb{B}$ such that

$$\|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta) < \infty,$$

where $\mathbb{S}$ is the unit sphere and $d\sigma$ is the normalized measure on $\mathbb{S}$.

Let $\varphi \in S(\mathbb{B})$, the composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$. When $N = 1$, the Littlewood Subordination Theorem shows that $C_\varphi$ is bounded on $A^2_\alpha(\mathbb{D})$ for any analytic self-map $\varphi$ of $\mathbb{D}$, and many other properties of $C_\varphi$ have been characterized, see, e.g. [3, 11, 13, 18, 25]. However, for $N > 1$, it is no longer the case that every composition operator is bounded on the weighted Bergman space of the ball (see Section 3.5 in [3]). We know that if $C_\varphi$ maps $A^2_\alpha$ into $A^2_\alpha$, then $C_\varphi$ is a bounded operator by the closed graph theorem. So in this paper, for $\varphi, \psi \in S(\mathbb{B})$, we always suppose $C_\varphi$ and $C_\psi$ map $A^2_\alpha$ into $A^2_\alpha$. The mapping properties of the differences of two composition operators, i.e. an operator of the form

$$T = C_\varphi - C_\psi$$

have also been studied. For related papers on the disk see [10, 12, 14, 15, 19, 23, 24], and on the unit ball [6, 7, 21, 22]. For the research of Hilbert-Schmidt operator we can see [2, 3, 4, 5, 17, 20]. The authors [2] studied the Hilbert-Schmidt differences on the weighted Bergman space $A^2_\alpha(\mathbb{D})$, the present paper continues this line of research, and characterizes the Hilbert-Schmidt differences on the unit ball. The paper is organized as follows: Section 3 is devoted to characterizing the conditions about Hilbert-Schmidt differences. Some conclusions about topological structure are given in section 4.

Throughout the remainder of this paper, $C$ will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $a \preceq b$ means that there is a positive constant $C$ such that $a \leq Cb$. We say $a \asymp b$, if both $a \preceq b$ and $b \preceq a$ hold.

2. Prerequisites

In this section, we will give some notations and well-known lemmas.

2.1. Weighted Bergman space and Hilbert-Schmidt operator. Given $\alpha > -1$, the space $A^2_\alpha$ is a Hilbert space with inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}} f \bar{g} d\nu_\alpha.$$
for \( f, g \in A_α^2 \). The reproducing kernel for the bounded linear functional of evaluation at \( w \in \mathbb{B} \) in \( A_α^2 \) is

\[
K_w(z) = \frac{1}{(1 - \langle z, w \rangle)^{N+1+\alpha}}
\]

such that

\[
\langle f, K_w \rangle = f(w),
\]

and it has norm \((1 - |w|^2)^{-(N+1+\alpha)}\). We also have

\[
K_w(z) = \sum_n e_n(z)e_n(w), \quad z, w \in \mathbb{B}
\]

for any choice of an orthonormal basis \( \{e_n\} \) for \( A_α^2 \).

Let \( T \) be the linear operator from Banach space \( X \) to Banach space \( Y \), the operator norm define as follows:

\[
\|T\| = \sup_{\|f\|_X = 1} \|Tf\|_Y,
\]

the notions \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) denote the norm of \( X \) and \( Y \), respectively.

A linear operator \( T \) on a separable Hilbert space \( H \) is Hilbert-Schmidt if

\[
\|T\|^2_{HS} = \sum_{k=1}^\infty \|Te_k\|^2_H = \sum_{k,m=1}^\infty |\langle Te_k, e_m \rangle_H|^2 < \infty \quad (2.1)
\]

for any (or some) orthonormal basis \( \{e_k\} \) of \( H \), \( \| \cdot \|_H \) \((\langle \cdot , \cdot \rangle_H)\) is the norm (respectively, inner product) of \( H \). For an arbitrary linear operator \( T \) on \( H \) the (possibly infinite) sum on the right of (2.1) does not depend on the particular choice of \( \{e_n\} \), and \( \|T\| \leq \|T\|_{HS} \). We know that if \( T \) is Hilbert-Schmidt operator, then \( T \) is compact operator.

Let \( m = (m_1, \cdots , m_N) \) be a multi-index, since the function sequence \( \{z^m\}_{\|z^m\|_α} \) is an orthonormal basis of \( A_α^2 \), we have

\[
\|C_ϕ\|^2_{HS} = \int_{\mathbb{B}} \frac{1}{(1 - |ϕ|^2)^{N+1+α}} \, dν_α.
\]

2.2. Pseudohyperbolic distance. We will describe some automorphisms of \( \mathbb{B} \) that are analogous to the disk automorphisms \((a - z)/(1 - \bar{a}z)\), for \( a \in \mathbb{D} \). Let \( a \in \mathbb{B} \), and set

\[
P_a(z) = \frac{\langle z, a \rangle}{|a|^2}a,
\]

so \( P_a \) is projection onto the subspace \([a]\) spanned by \( a \), and \( Q_a = I - P_a \), projection onto the orthogonal complement of \([a]\). To simplify notation write \( s_a = \sqrt{1 - |a|^2} \).

Define \( ϕ_a(z) \) by

\[
ϕ_a(z) = \frac{a - P_a(z) - s_aQ_a(z)}{1 - \langle z, a \rangle}.
\]

Clearly \( ϕ_a \) is analytic in \( \mathbb{B} \), \( ϕ_a(0) = a \) and \( ϕ_a(a) = 0 \).

For \( a, b \in \mathbb{B} \), the Bergman metric defined as

\[
β(a, b) = \frac{1}{2} \log \frac{1 + |ϕ_a(b)|}{1 - |ϕ_a(b)|}.
\]
we denote by $\rho(a, b)$ the pseudohyperbolic distance between $a$ and $b$, i.e.,

$$
\rho(a, b) = |\varphi_a(b)|,
$$

and we have the following equation

$$
1 - \rho^2(a, b) = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \langle a, b \rangle|^2}.
$$

(2.2)

For $\varphi \in S(\mathbb{B})$ and $z, w \in \mathbb{B}$, by the Schwarz–Pick Theorem, we have

$$
\rho(\varphi(z), \varphi(w)) \leq \rho(z, w),
$$

thus for $z \in \mathbb{B}$, we obtain

$$
|z|^2 = \rho^2(0, z) \geq \rho^2(\varphi(0), \varphi(z)) \geq 1 - \left(1 - |\varphi(0)|^2\right)\left(1 - |\varphi(z)|^2\right)
\frac{|\varphi(z) - \varphi(0)|^2}{(1 - |\varphi(z)\varphi(0)|)^2},
$$

then

$$
|\varphi(z)| \leq \frac{|z| + |\varphi(0)|}{1 + |z||\varphi(0)|}.
$$

Using the inequity above, we easily get

$$
\frac{1 - |\varphi(z)|}{1 - |z|} \geq \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} > 0.
$$

(2.3)

Now, let us recall some lemmas.

**Lemma 2.1.** ([26, Lemma 2.20]) For each $R > 0$ there exists a positive constant $C_R$ such that

$$
C_R^{-1} \leq \frac{1 - |a|^2}{1 - |z|^2} \leq C_R
$$

and

$$
C_R^{-1} \leq \frac{1 - |a|^2}{|1 - \langle a, z \rangle|} \leq C_R
$$

for all $a$ and $z$ in $\mathbb{B}$ with $\beta(z, a) \leq R$.

**Lemma 2.2.** ([26, Lemma 2.27]) For any $R > 0$ and any real $b$ there exists a constant $C_R > 0$ such that

$$
\left| \frac{(1 - \langle z, u \rangle)^b}{(1 - \langle z, v \rangle)^b} - 1 \right| \leq C_R \beta(u, v)
$$

for all $z, u$ and $v$ in $\mathbb{B}$ with $\beta(u, v) \leq R$.

**Remark 2.3.** Since $\beta(a, b) = \frac{1}{2} \log \frac{1 + \rho(a, b)}{1 - \rho(a, b)}$, then $\beta(a, b) \leq R \iff \rho(a, b) \leq r$, where $r = e^{R - 1}$, and we can obtain that there exists a positive constant $C_R$ such that $\beta(a, b) \leq C_R \rho(a, b)$ for $a, b \in \mathbb{B}$ with $\rho(a, b) \leq r$. So Lemma 2.1 and Lemma 2.2 also hold if $\beta(a, b)$ is replace by $\rho(a, b)$. In this paper, we always use $\rho(a, b)$ instead of $\beta(a, b)$.
If \( \rho(a, b) < r < 1 \), we can get the inequality

\[
\frac{1 - \rho(a, b)}{1 + \rho(a, b)} \leq \frac{1 - |a|^2}{1 - |b|^2} \leq \frac{1 + \rho(a, b)}{1 - \rho(a, b)}. \tag{2.4}
\]

To see this, for example, let \( b = \varphi_a(w) \), using the (2.2), we have

\[
1 - |b|^2 = \frac{(1 - |a|^2)(1 - |w|^2)}{|1 - \langle a, w \rangle|},
\]

since

\[
1 - \frac{|w|}{1 + |w|} \leq \frac{1 - |w|^2}{|1 - \langle a, w \rangle|} \leq \frac{1 + |w|}{1 - |w|},
\]

and \( w = \varphi_a(b) \), we get (2.4).

According to Lemma 2.2, we have

\[
\Re \left( \frac{1 - |a|^2}{1 - \langle a, b \rangle} \right)^b \geq 1 - C\rho(a, b) \geq 0 \tag{2.5}
\]

for all \( a, b \in \mathbb{B} \) with \( \rho(a, b) \) sufficiently small, where \( \Re z \) denote the real part of \( z \in \mathbb{C} \).

**Lemma 2.4.** Given \( s > 0 \), there is a constant \( C = C(s) > 0 \) such that

\[
\frac{1}{|1 - \langle a, b \rangle|^s} - \Re \left( \frac{1}{1 - \langle a, b \rangle} \right)^s \leq C\rho^2(a, b) \leq \frac{(1 - |a|^2)^{s/2}(1 - |b|^2)^{s/2}}{(1 - |a|^2)^{s/2}(1 - |b|^2)^{s/2}}
\]

for all \( a, b \in \mathbb{B} \).

**Proof.** Let \( s > 0 \) and fix \( a, b \in \mathbb{B} \). Choose \( \varepsilon = \varepsilon_s \in (0, 1) \) such that (2.5) is satisfied. Put \( z = (1 - \langle a, b \rangle)^{-s} = x + iy \) where \( x = \Re z \) and \( y = \Im z \). Since

\[
|z|^2 = \frac{1}{|1 - \langle a, b \rangle|^s} \leq \frac{1}{(1 - |a|)^{s/2}(1 - |b|)^{s/2}},
\]

we have

\[
|z| - x \leq 2|z| \leq \frac{\rho^2(a, b)}{\varepsilon^2(1 - |a|)^{s/2}(1 - |b|)^{s/2}}
\]

for \( \rho(a, b) \geq \varepsilon \).

When \( \rho(a, b) < \varepsilon \), we have \( x \geq 0 \), so

\[
|z| - x \leq \frac{|z|^2 - x^2}{|z|} = y^2.
\]

By Lemma 2.2, we get

\[
|y| = \frac{|z - \bar{z}|}{2} = \frac{1}{2} \left| \frac{1}{(1 - \langle a, b \rangle)^s} - \frac{1}{(1 - \langle b, a \rangle)^s} \right|
\]

\[
\leq \frac{1}{2} \left( \left| \frac{1}{(1 - \langle a, b \rangle)^s} - \frac{1}{(1 - \langle a, a \rangle)^s} \right| + \left| \frac{1}{(1 - \langle a, a \rangle)^s} - \frac{1}{(1 - \langle b, a \rangle)^s} \right| \right)
\]

\[
\leq \frac{1}{2|1 - \langle a, b \rangle|^s} \left| \frac{1}{(1 - \langle a, a \rangle)^s} - 1 \right| + \frac{1}{2|1 - \langle b, a \rangle|^s} \left| \frac{1}{(1 - \langle a, a \rangle)^s} - 1 \right|
\]

\[
\leq \frac{C}{|1 - \langle a, b \rangle|^s} \rho(a, b).
\]
Thus,
\[
y^2 \leq C \frac{\rho^2(a, b)|1 - \langle a, b \rangle|^s}{|z|} \leq C \frac{\rho^2(a, b)}{(1 - |a|)^{s/2}(1 - |b|)^{s/2}}.
\]
Since \(1 - |z|^2 \approx 1 - |z|\), the proof is complete. \(\square\)

### 3. Hilbert-Schmidt Differences

In this section we will use pseudohyperbolic distance to characterize Hilbert-Schmidt differences of composition operators on \(A^2_{\alpha}\).

**Theorem 3.1.** Let \(\alpha > -1\) and \(J \in \mathbb{N}\), for \(a_1, \ldots, a_J \in \mathbb{C}\) and \(\varphi_1, \ldots, \varphi_J \in S(\mathbb{B})\), the identity

\[
\left\| \sum_{j=1}^J a_j C_{\varphi_j} \right\|_{HS}^2 = \int_{\mathbb{B}} \left\| \sum_{j=1}^J a_j K_{\varphi_j(z)} \right\|_{\alpha}^2 \, d\nu_{\alpha}(z)
\]
holds.

**Proof.** The proof is similar to Proposition 3.1 in [2], we omit the detail. \(\square\)

By the theorem above, we need to consider the quantity \(\|K_z - K_w\|_{\alpha}\) in order to study Hilbert-Schmidt differences of composition operators.

**Theorem 3.2.** Let \(\alpha > -1\), for \(z, w \in \mathbb{B}\), we have

\[
\|K_z - K_w\|_{\alpha}^2 \approx (\|K_z\|_{\alpha} - \|K_w\|_{\alpha})^2 + \rho^2(z, w)\|K_z\|_{\alpha}\|K_w\|_{\alpha}.
\]

**Proof.** The reproducing property gives

\[
\|K_z - K_w\|_{\alpha}^2
= (\|K_z\|_{\alpha} - \|K_w\|_{\alpha})^2 + 2\|K_z\|_{\alpha}\|K_w\|_{\alpha} - 2\Re\langle K_z, K_w \rangle_{\alpha}

= (\|K_z\|_{\alpha} - \|K_w\|_{\alpha})^2 + 2(\|K_z\|_{\alpha}\|K_w\|_{\alpha} - |\langle K_z, K_w \rangle_{\alpha}|)
+ 2(|\langle K_z, K_w \rangle_{\alpha}| - \Re\langle K_z, K_w \rangle_{\alpha})
\]

\[
=: (\|K_z\|_{\alpha} - \|K_w\|_{\alpha})^2 + 2F_{\alpha} + 2G_{\alpha}.
\]

We estimate \(F_{\alpha}\), by (2.2) and \(1 - t^{\frac{N+1+\alpha}{2}} \approx 1 - t\) when \(0 \leq t \leq 1\), we have

\[
F_{\alpha} = \frac{1}{(1 - |z|^2)^{\frac{N+1+\alpha}{2}}} \frac{1}{(1 - |w|^2)^{\frac{N+1+\alpha}{2}}} - \frac{1}{|1 - \langle z, w \rangle|^{N+1+\alpha}}
\leq \frac{1}{(1 - |z|^2)^{\frac{N+1+\alpha}{2}}} \frac{1}{(1 - |w|^2)^{\frac{N+1+\alpha}{2}}} \left(1 - (1 - \rho^2(z, w))^{\frac{N+1+\alpha}{2}}\right)
\approx \rho^2(z, w)\|K_z\|_{\alpha}\|K_w\|_{\alpha}.
\]

By Lemma 2.4, we know \(0 \leq G_{\alpha} \leq C\rho^2(z, w)\|K_z\|_{\alpha}\|K_w\|_{\alpha}\). Consequently, we have \(F_{\alpha} + G_{\alpha} \approx \rho^2(z, w)\|K_z\|_{\alpha}\|K_w\|_{\alpha}\). The proof is complete. \(\square\)

**Theorem 3.3.** Given \(\alpha > -1\), the estimate

\[
\|K_z - K_w\|_{\alpha} \approx \rho(z, w)(\|K_z\|_{\alpha} + \|K_w\|_{\alpha})
\]
holds for all \(z, w \in \mathbb{B}\).
Proof. For $z, w \in \mathbb{B}$, put
\[
\Phi := (\|Kz\|_\alpha^2 + \|Kw\|_\alpha^2) \rho^2(z, w),
\]
\[
\Psi_1 := (\|Kz\|_\alpha - \|Kw\|_\alpha)^2
\]
and
\[
\Psi_2 := \|Kz\|_\alpha \|Kw\|_\alpha \rho^2(z, w).
\]
We only need to establish the estimate
\[
C^{-1}\Phi \leq \Psi_1 + \Psi_2 \leq C\Phi
\]
on $\mathbb{B}^2$ by Theorem 3.2. We decompose $\mathbb{B}^2$ into three parts
\[
E := \{(a, b) \in \mathbb{B}^2 : \rho(a, b) < 1/2\},
\]
\[
Q_1 := \{(a, b) \in \mathbb{B}^2 \setminus E : 1/2 \leq \left( \frac{1 - |a|^2}{1 - |b|^2} \right)^{N+1+\alpha} \leq 2 \},
\]
\[
Q_2 := \mathbb{B}^2 \setminus (E \cup Q_1).
\]
To obtain the left inequality, that is
\[
\Phi \leq C(\Psi_1 + \Psi_2),
\]
we proceed similarly to Proposition 3.5 in [2]. Then, we only prove
\[
\Psi_1 + \Psi_2 \leq C\Phi.
\]
It is easy to see that $2\Psi_2 \leq \Phi$ on $\mathbb{B}^2$, and $\Psi_1 \leq 4\Phi$ on $\mathbb{B}^2 \setminus E$. Now, when $(z, w) \in E$, by Lemma 2.2, we have
\[
\Psi_1 = \left( \frac{1}{(1 - |z|^2)^s} - \frac{1}{(1 - |w|^2)^s} \right)^2
\]
\[
\leq \left[ \frac{1}{(1 - |z|^2)^s} \left( 1 - \left( \frac{1 - |z|^2}{1 - \langle z, w \rangle} \right)^{s} \right) + \frac{1}{(1 - |w|^2)^s} \left( 1 - \left( \frac{1 - |w|^2}{1 - \langle z, w \rangle} \right)^{s} \right) \right]^2
\]
\[
\leq \frac{C\rho(z, w)}{(1 - |z|^2)^s} + \frac{C\rho(z, w)}{(1 - |w|^2)^s}
\]
\[
\leq C\rho^2(z, w) \left[ \frac{1}{(1 - |z|^2)^{2s}} + \frac{1}{(1 - |w|^2)^{2s}} \right],
\]
here $s = \frac{N+1+\alpha}{2}$.
Thus, we obtain $\Psi_1 \leq C\Phi$ on $E$. This completes the proof. \hfill \Box

Now, we are now ready to estimate the quantity $\|C\varphi - C\psi\|_{HS}$. 

**Theorem 3.4.** Assume that $\varphi, \psi \in S(\mathbb{B})$, then the following estimate
\[
\|C\varphi - C\psi\|_{HS}^2 \asymp \int_{\mathbb{B}} \left( \frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2} \right)^{N+1+\alpha} \rho^2(\varphi(z), \psi(z)) d\nu_\alpha(z)
\]
holds.
Proof. By Theorem 3.1 and Theorem 3.3, we have

\[ \|C_\varphi - C_\psi\|_{HS}^2 = \int_B \|K_\varphi(z) - K_\psi(z)\|^2 d\nu_\alpha(z) \]

\[ \leq \int_B \left( \|K_\varphi(z)\|_\alpha + \|K_\psi(z)\|_\alpha \right)^2 \rho^2(\varphi(z), \psi(z)) d\nu_\alpha(z) \]

\[ \leq \int_B \left( \frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2} \right)^{N+1+\alpha} \rho^2(\varphi(z), \psi(z)) d\nu_\alpha(z). \]

Thus, we finish the proof.

As a corollary of Theorem 3.4, we get an equivalent condition for the differences \( C_\varphi - C_\psi \) to be Hilbert-Schmidt. This result will provide some heuristics for the proof of our theorem in section 4.

**Corollary 3.5.** Assume that \( \varphi, \psi \in S(B) \), then \( C_\varphi - C_\psi \) is Hilbert-Schmidt on \( A^\alpha_2 \) if and only if

\[ \int_B \rho^2(\varphi(z), \psi(z)) d\nu_\alpha(z) \left( \frac{1}{1 - |\varphi(z)|^2} \right)^{N+1+\alpha} \left( \frac{1}{1 - |\psi(z)|^2} \right) < \infty \]

Using (2.3), we can get another corollary of Theorem 3.4.

**Corollary 3.6.** Let \( \alpha > 1 \) and \( \varphi, \psi \in S(B) \), If \( C_\varphi - C_\psi \) is Hilbert-Schmidt on \( A^\alpha_2 \), then \( C_\varphi - C_\psi \) is Hilbert-Schmidt on \( A^\beta_2 \) for \( \beta > \alpha \).

On the disk, when composition operators \( C_\varphi \) and \( C_\psi \) are not Hilbert-Schmidt, we know that the linear combinations \( aC_\varphi + bC_\psi \) is Hilbert-Schmidt if and only if \( a + b = 0 \) and \( C_\varphi - C_\psi \) is Hilbert-Schmidt, where \( a, b \in \mathbb{C}\{0\} \). Here, we can get the same result, for the purpose, we need the following estimate, it is easy to get from Lemma 3.9 in [2] and Theorem 3.2, we omit the proof.

**Theorem 3.7.** For \( z, w \in \mathbb{B} \) and \( \lambda \in \mathbb{C} \), we have the following estimate

\[ (\|K_z\|_\alpha - |\lambda||K_w\|_\alpha)^2 + |\lambda|\rho^2(z, w)\|K_z\|_\alpha\|K_w\|_\alpha < \|K_z - \lambda K_w\|_\alpha^2. \]

when \( |\lambda| = 1 \), \( \|K_z - K_w\|_\alpha < \|K_z - \lambda K_w\|_\alpha^2. \)

**Theorem 3.8.** Let \( \alpha > -1 \) and \( a, b \in \mathbb{C}\{0\} \). Suppose that \( C_\varphi \) and \( C_\psi \) are not Hilbert-Schmidt on \( A^\alpha_2 \). Then \( aC_\varphi + bC_\psi \) is Hilbert-Schmidt on \( A^\alpha_2 \) if and only if \( a + b = 0 \) and \( C_\varphi - C_\psi \) is Hilbert-Schmidt on \( A^\alpha_2 \).

Proof. The proof is similar to Theorem 3.10 in [2], we also omit the proof.
4. TOPOLOGY STRUCTURE

We will give some conclusions about the topology structure in this section. Let \( C(A^2_\alpha) \) be the space of all bounded composition operators on \( A^2_\alpha \) endowed with norm topology.

Write \( C_\phi \sim C_\psi \) if \( C_\phi \) and \( C_\psi \) are in the same path component of \( C(A^2_\alpha) \). For \( t \in [0, 1] \), put \( \varphi_t = (1-t)\varphi + t\psi \), it is easy to see \( \varphi_t \in S(\mathbb{B}) \).

**Theorem 4.1.** Let \( \alpha > 1 \) and assume that \( C_\phi, C_\psi \in C(A^2_\alpha) \), \( C_\phi - C_\psi \) is Hilbert-Schmidt on \( A^2_\alpha \). Then \( C_{\varphi_s} - C_{\varphi_t} \) is Hilbert-Schmidt for any \( s, t \in [0, 1] \).

**Proof.** Since \( \|C_{\varphi_s} - C_{\varphi_t}\|_{HS} \leq \|C_\phi - C_{\varphi_t}\|_{HS} + \|C_\phi - C_{\varphi_s}\|_{HS} \), it is sufficient to prove \( C_\phi - C_{\varphi_s} \) is Hilbert-Schmidt for \( s \in [0, 1] \).

From the definition of \( \rho \), we have

\[
\rho(\varphi_s(z), \varphi(z)) = \frac{|\varphi(z) - P_{\varphi(z)}(\varphi_s(z)) - s\varphi(z)Q_{\varphi(z)}(\varphi_s(z))|}{1 - \langle \varphi_s(z), \varphi(z) \rangle}
\]

\[
= s \frac{|\varphi(z) - P_{\varphi(z)}(\psi(z)) - s\varphi(z)Q_{\varphi(z)}(\psi(z))|}{1 - \langle \varphi_s(z), \varphi(z) \rangle}
\]

\[
= \frac{s\rho(\varphi(z), \psi(z))[1 - \langle \psi(z), \varphi(z) \rangle]}{1 - \langle \varphi_s(z), \varphi(z) \rangle}.
\]

So, if \( \rho(\varphi(z), \psi(z)) \geq 1/2 \), \( \rho(\varphi_s(z), \varphi(z)) \leq 1 \leq 2\rho(\varphi(z), \psi(z)) \).

If \( \rho(\varphi(z), \psi(z)) < 1/2 \), by Lemma 2.1, we have

\[
\rho(\varphi_s(z), \varphi(z)) \leq \frac{s\rho(\varphi(z), \psi(z))[1 - \langle \psi(z), \varphi(z) \rangle]}{1 - |\varphi(z)|}
\]

\[
\leq \frac{2s\rho(\varphi(z), \psi(z))[1 - \langle \psi(z), \varphi(z) \rangle]}{1 - |\varphi(z)|^2}
\]

\[
\leq 2C\rho(\varphi(z), \psi(z)),
\]

thus, we get \( \rho(\varphi_s(z), \varphi(z)) \leq C\rho(\varphi(z), \psi(z)) \) for all \( z \in \mathbb{B} \). And

\[
|\varphi_s(z)| = |(1 - s)\varphi(z) + s\psi(z)| \leq max\{|\varphi(z)|, |\psi(z)|\},
\]

then

\[
\frac{1}{1 - |\varphi_s(z)|^2} \leq \frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2},
\]

hence we get

\[
\int_{\mathbb{B}} \frac{\rho^2(\varphi_s(z), \varphi(z))d\nu(\alpha(z))}{(1 - |\varphi_s(z)|^2)^{N+1+\alpha}} \leq C \int_{\mathbb{B}} \left(\frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2}\right)^{N+1+\alpha} \rho^2(\varphi(z), \psi(z))d\nu(\alpha(z))
\]

\[
\asymp \|C_\phi - C_\psi\|_{HS}^2 < \infty.
\]

Similarly,

\[
\int_{\mathbb{B}} \frac{\rho^2(\varphi_s(z), \varphi(z))d\nu(\alpha(z))}{(1 - |\varphi(z)|^2)^{N+1+\alpha}} < \infty,
\]
according to Corollary 3.5, we have $C_\varphi - C_{\varphi_s}$ is Hilbert-Schmidt. This completes the proof.

Since $\|C_{\varphi_s}\| \leq \|C_{\varphi_s} - C_\varphi\| + \|C_\varphi\| \leq \|C_{\varphi_s} - C_\varphi\|_{HS} + \|C_\varphi\|$, the composition operator $C_{\varphi_s}$ belongs to $\mathcal{C}(A^2_\alpha)$ for $s \in [0, 1]$ by the theorem above.

Now, we give the sufficient condition of path connected. For the process of proof, please refer to [2, Theorem 4.2].

**Theorem 4.2.** Let $\alpha > 1$ and $\varphi, \psi \in S(\mathcal{B})$. Assume that $C_\varphi, C_\psi \in \mathcal{C}(A^2_\alpha)$, and $C_\varphi - C_\psi$ is Hilbert-Schmidt operator, then $C_\varphi \sim C_\psi$.

**Proof.** According to Theorem 4.1, we obtain $C_{\varphi_s} \in \mathcal{C}(A^2_\alpha)$, we need to show that $s \in [0, 1] \rightarrow C_{\varphi_s}$ is a continuous path in $\mathcal{C}(A^2_\alpha)$. Here, it is sufficient to consider the case $\lim_{s \to 0} \|C_\varphi - C_{\varphi_s}\| = 0$.

Given $s \in (0, 1]$, put

$$\Phi_s = \left( \frac{1}{1 - |\varphi|^2} + \frac{1}{1 - |\varphi_s|^2} \right)^{N + 1 + \alpha} \rho(\varphi, \varphi_s)$$

for short, since $\rho(\varphi, \varphi_s) \leq 2 \rho(\varphi, \psi)$ and $\frac{1}{1 - |\varphi|^2} \leq \frac{1}{1 - |\varphi|^2} + \frac{1}{1 - |\psi|^2}$, we have

$$\Phi_t \leq C \left( \frac{1}{1 - |\varphi|^2} + \frac{1}{1 - |\psi|^2} \right)^{N + 1 + \alpha} \rho(\varphi, \psi)$$

for all $s$. Because $C_\varphi - C_\psi$ is Hilbert-Schmidt operator, $\int_{\mathcal{B}} \Phi_t d\nu_\alpha$ is integrable. Since $\rho(\varphi_s, \varphi) \to 0$ as $s \to 0$, so we get

$$\lim_{s \to 0} \|C_\varphi - C_{\varphi_s}\|_{HS} = 0,$$

and

$$\lim_{s \to 0} \|C_\varphi - C_{\varphi_s}\| = 0.$$ 

This proof is complete.

By the theorem above, we can obtain the following consequences.

(1) Given $C_\varphi \in \mathcal{C}(A^2_\alpha)$, the set

$$N(\varphi) = \{C_\psi : \|C_\varphi - C_\psi\|_{HS} < \infty\}$$

is the path-connected set in $\mathcal{C}(A^2_\alpha)$ containing $C_\varphi$.

(2) Let $\mathcal{HS}(A^2_\alpha) \subset \mathcal{C}(A^2_\alpha)$ be the set of all Hilbert-Schmidt composition operators on $A^2_\alpha$, then $\mathcal{HS}(A^2_\alpha)$ belongs to a path component of $\mathcal{C}(A^2_\alpha)$.

(3) $N(\varphi)$ is ”convex” in the sense that if $C_\psi \in N(\varphi)$, then $\{C_{(1-t)\varphi + t\psi}\}_{t \in [0,1]} \in N(\varphi)$.

Next, we will study the isolation using the extreme point, for related papers see [2, 8, 9, 14]. It is easy to see that the set $S(\mathcal{B})$ is a convex set. For the set $S(\mathcal{B})$, we define the extreme point as following: If $\varphi \in S(\mathcal{B})$ is not proper convex combination of two distinct elements of $S(\mathcal{B})$, we call $\varphi$ is an extreme point. It is easy to see that $\varphi$ is an extreme point if and only if $\varphi = \frac{f + g}{2}$ for $f, g \in S(\mathcal{B})$, implies $f = g = \varphi$. 

HILBERT-SCHMIDT DIFFERENCES 169
Example 4.3. For $N = 2$, let $\varphi(z_1, z_2) = (A\varphi_1, B\varphi_2)$, where $A, B \leq 0$, $A^2 + B^2 = 1$ and $\varphi_i$ are inner functions on unit ball $\mathbb{B}_2$, then $\varphi$ is an extreme point.

To prove that this example of extreme point is correct, let us observe at first that, because of $A^2 + B^2 = 1$, it follows $|\varphi(\zeta)| = 1$ for $\zeta \in \mathbb{S}_2$ almost everywhere. If $\varphi$ is not an extreme point, then there exist two distinct maps $f$ and $g$ such that $\varphi = \frac{f + g}{2}$. When $|\varphi(\zeta)| = 1$, we have $f(\zeta) = g(\zeta) \in \mathbb{S}_2$, thus the components $\varphi_i = f_i = g_i$ on $\mathbb{S}_2$ almost everywhere for $i = 1, 2$. Since $\varphi, f, g \in S(\mathbb{B}_2)$, $\varphi_i, f_i, g_i \in H^2$, and $\|\varphi_i\|_{H^2} = \|f_i\|_{H^2} = \|g_i\|_{H^2}$, so $\varphi_i = f_i = g_i$ on $\mathbb{B}_2$, then $\varphi = f = g$ on $\mathbb{B}$, this contradicts with the choose of $f, g$. Thus, $\varphi$ is an extreme point.

Using the similar method, we can prove that every automorphisms $\varphi_a$ is an extreme point on $\mathbb{B}$. When $\varphi$ is a linear-fractional self-map of $\mathbb{B}$, $C_\varphi$ is bounded on $A^2$, moreover, $\varphi_a$ is linear-fractional, so $C_{\varphi_a}$ belongs to $\mathcal{C}(A^2)$, thus there is composition operator induce by extreme point of $S(\mathbb{B})$ in $\mathcal{C}(A^2)$.

Now, we give an equivalent condition for extreme point, and research the non-isolation using the extreme point.

**Theorem 4.4.** Let $\varphi \in S(\mathbb{B})$, then $\varphi$ is not an extreme point if and only if there exists some $\omega \in S(\mathbb{B})$ such that $\omega \neq 0$ and $|\varphi| + |\omega| \leq 1$ on $\mathbb{B}$.

**Proof.** If $\varphi$ is not an extreme point, then there are two distinct functions $f, g \in S(\mathbb{B})$ such that $\varphi = \frac{f + g}{2}$. Put $\psi = \frac{f - g}{2}$, $\psi = (\psi_1, \cdots, \psi_N)$, it is obvious that $|\varphi|^2 + |\psi|^2 \leq 1$, so $\psi \in S(\mathbb{B})$, let $\omega = \frac{(\psi_1^2, \cdots, \psi_N^2)}{2}$, then $\omega \in S(\mathbb{B})$, and

$$|\omega| + |\varphi| \leq \sqrt{\frac{\sum_{i=1}^{N} |\psi_i|^4}{2}} + |\varphi| \leq \frac{|\psi|^2}{2} + |\varphi| \leq 1.$$

If there is a non-zero function $\omega$, such that $|\varphi| + |\omega| \leq 1$, then $|\varphi \pm \omega| \leq |\omega| + |\varphi| \leq 1$, so $\varphi \pm \omega \in S(\mathbb{B})$, and $\varphi = \frac{\varphi \pm \omega}{2} + \frac{\varphi - \omega}{2}$, thus $\varphi$ is not an extreme point of $S(\mathbb{B})$.

**Theorem 4.5.** Let $\varphi \in S(\mathbb{B})$ and $C_\varphi \in \mathcal{C}(A^2)$. If $\varphi$ is not the extreme point of $S(\mathbb{B})$, then $C_\varphi$ is not isolated in $\mathcal{C}(A^2)$.

**Proof.** Suppose that $\varphi$ is not extreme point, we know that there is some non-zero element $\omega \in S(\mathbb{B})$, such that $|\varphi| + |\omega| \leq 1$. Let $s = \frac{N + 3 + \alpha}{2}$ and $\psi = \varphi + \frac{(\omega_1^2, \cdots, \omega_N^2)}{2}$, since

$$1 - |\psi| > 1 - |\varphi| - \frac{|(\omega_1^2, \cdots, \omega_N^2)|}{2} \geq 1 - |\varphi| - \frac{|\omega|^2}{2} \geq 1 - |\varphi| \geq 0,$$
we have $\psi \in S(\mathbb{B})$. Moreover, it is easy to observe that
\[
\rho^2(\varphi(z), \psi(z)) = 1 - \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)}{|1 - \langle \varphi(z), \psi(z) \rangle|^2}
\]
\[
= \frac{|\langle \varphi(z), \psi(z) \rangle|^2 - 2\Re \langle \varphi(z), \psi(z) \rangle + |\varphi(z)|^2 + |\psi(z)|^2 - |\varphi(z)|^2|\psi(z)|^2}{|1 - \langle \varphi(z), \psi(z) \rangle|^2}
\]
\[
\leq \frac{|\psi(z) - \varphi(z)|^2}{(1 - |\varphi(z)|)^2} \leq \frac{2s}{4(1 - |\varphi(z)|)^2} \leq (1 - |\varphi(z)|)^{2s - 2},
\]
so we have
\[
\left(\frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2}\right)^{N+1+\alpha} \rho^2(\varphi(z), \psi(z)) \leq 3^{N+1+\alpha},
\]
then $C_\varphi - C_\psi$ is Hilbert–Schmidt by Theorem 3.4, and $C_\psi \in \mathcal{C}(A^2_n)$, thus $C_\varphi$ is not isolated. The proof is complete. \qed

Acknowledgment. The authors would like to thank the referees for the useful comments and suggestions which improved the presentation of this paper. This research is supported in part by the National Natural Science Foundation of China (Grant Nos. 10971153, 10671141).

References


Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China.

E-mail address: zhangli0977@126.com
E-mail address: zehuazhou2003@yahoo.com.cn