AN EXTENSION OF KY FAN’S DOMINANCE THEOREM

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Abstract. We prove that for a separable Hilbert space $H$ with an orthonormal basis $\{e_i\}_{i=1}^\infty$, the equality $\|\cdot\| = \|\sum_{i=1}^\infty s_i(\cdot)e_i \otimes e_i\|$ holds for all unitarily invariant norms on $\mathcal{B}(H)$ and Ky Fan’s dominance theorem remains valid on $\mathcal{B}(H)$.

1. Introduction

There has been a great interest in studying unitarily invariant norms and symmetric norm ideals on $\mathcal{B}(H)$ in the last few decades (see, e.g., [1]-[4],[5],[7],[9]-[12] and the references therein). A norm $\|\cdot\|$ on a non-zero ideal $\mathcal{J}$ of $\mathcal{B}(H)$ is called unitarily invariant if $\|UTV\| = \|T\|$ for all unitary operators $U,V \in \mathcal{B}(H)$ and $T \in \mathcal{J}$. The $i$th $s$-number of an operator $T$ on $H$ is displayed by $s_i(T)$ and is given by

$$s_i(T) = \inf\{\|T - F\|_{op} : F \in \mathcal{B}(H) \text{ has rank } < i\},$$

where $\|\cdot\|_{op}$ denotes the usual operator norm on $\mathcal{B}(H)$. Note that every finite rank operator belongs to any non-zero ideal of $\mathcal{B}(H)$.

Typical examples of unitarily invariant norms on $\mathcal{B}(H)$ are Ky Fan $k$-norms that are defined by $N_k(\cdot) = s_1(\cdot) + \cdots + s_k(\cdot)$ [3], see also [10]. We say that a norm $\|\cdot\|$ on $\mathcal{J}$, satisfies Ky Fan’s dominance theorem, if for every $T,R \in \mathcal{J}$, with $N_k(T) \leq N_k(R)$ for all $k \in \mathbb{N}$, the inequality $\|T\| \leq \|R\|$ holds. Ky Fan’s dominance theorem holds for $\mathcal{J}$ if it holds for all unitarily invariant norms on $\mathcal{J}$.

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In the finite dimensional case, owing to the presence of the singular value
decomposition (SVD), there is a nice representation of unitarily invariant norms
as symmetric gauge functions [2, Theorem 3.5.18], which plays a major role in
solving problems and proving theorems in the finite dimensional case. In fact,
using SVD, we conclude that for every matrix \( A \in M_n(\mathbb{C}) \) the equality
\( \| A \| = \| \text{diag}(s_1(A), \ldots, s_n(A)) \| \) is satisfied for all unitarily
invariant norms on \( M_n(\mathbb{C}) \), where \( M_n(\mathbb{C}) \) is the algebra of all \( n \times n \) matrices with the entries in \( \mathbb{C} \).

In this paper we prove an alternative equality in the infinite dimensional case.
In fact, if we show if \( U \) unitary elements \( J \) for \( \| \cdot \| \) holds for all unitarily
invariant norms on \( B(\mathcal{H}) \), where \( (s_i(\cdot)e_i \otimes e_i)(h) = s_i(\cdot) < h, e_i > e_i, \) for all \( h \in \mathcal{H} \). As a
corollary we conclude that for a separable Hilbert space \( \mathcal{H} \), Ky Fan’s dominance
theorem remains valid on \( B(\mathcal{H}) \).

2. UNITARILY INVARIANT NORMS ON \( B(\mathcal{H}) \)

Though this section, when we say \( J \) is a non-zero ideal of \( B(\mathcal{H}) \), it is possible
for \( J \) to be equal the whole \( B(\mathcal{H}) \). Also \( \| \cdot \| \) will be an arbitrary unitarily invariant
norm on \( B(\mathcal{H}) \). We say that a norm \( \| \cdot \| \) on \( J \) is symmetric if \( \| T_1 S T_2 \| \leq \| T_1 \|_{op} \| S \| \| T_2 \|_{op} \) for all \( T_1, T_2 \in B(\mathcal{H}) \) and \( S \in J \).

**Lemma 2.1.** Every unitarily invariant norm \( \| \cdot \| \) on a non-zero ideal \( J \) of \( B(\mathcal{H}) \)
is symmetric.

**Proof.** Let \( T \in B(\mathcal{H}), S \in J \) and consider a real number \( \alpha > 1 \). By [8], there are
unitary elements \( U_1, \ldots, U_n \) in \( B(\mathcal{H}) \) such that \( \frac{T}{\alpha \| T \|_{op}} = \frac{U_1 + \cdots + U_n}{n} \). Hence
\[
\| T S \| = \alpha \| T \|_{op} \left\| \frac{T}{\alpha \| T \|_{op}} S \right\| = \alpha \| T \|_{op} \left\| \frac{U_1 + \cdots + U_n}{n} \right\| \leq \alpha \| T \|_{op} \| S \|.
\]

Since \( \alpha > 1 \) is arbitrary, the inequality \( \| T S \| \leq \| T \|_{op} \| S \| \) holds. Similarly we can show
that \( \| S T \| \leq \| S \| \| T \|_{op} \). \( \square \)

**Corollary 2.2.** Let \( \| \cdot \| \) be a unitarily invariant norm on a non-zero ideal \( J \) of
\( B(\mathcal{H}) \) and \( T, S \in J \). Then

\( (i) \) \( \| T \| = \| \| T \| \| . \)

\( (ii) \) \( \| T \| = \| T^* \| . \)

\( (iii) \) \( \| p \| = \| q \| , \) for any equivalent projections \( p \) and \( q \) in \( J \).

\( (iv) \) If \( T \geq 0, S \geq 0 \), then \( \| T \| \geq \| S \| \).

**Proof.** The polar decomposition \( T = u | T | \) of \( T \) implies that \( | T | = u^* T, T^* = | T | u^* \)
and \( | T | = T^* u \). Also, if \( p \) and \( q \) are equivalent then \( p = vv^* \) and \( q = v^* v, \) for
some partial isometry \( v \) in \( B(\mathcal{H}) \) and hence we have \( v^* pv = q \) and \( vqv^* = p. \) If
\( T \geq S \geq 0, \) there is an operator \( R \) with \( \| R \|_{op} \leq 1 \) such that \( S = RT \). These
arguments together with Lemma 2.1 imply (i)-(iv). \( \square \)

**Lemma 2.3.** Let \( J \) be a non-zero ideal of \( B(\mathcal{H}), T \in J \) and \( P \) be a projection
of rank one. For every unitarily invariant norm \( \| \cdot \| \) on \( J \) the inequality
\( \| T \| \geq \| P \| \| T \|_{op} \) holds.
Proof. Suppose $T \neq 0$ and consider a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\|x_n\| = 1$, for all $n$ and $\lim_{n \to \infty} \|T(x_n)\| = \|T\|_{op}$. Without loss of generality we can suppose that $T(x_n) \neq 0$, for all $n \in \mathbb{N}$. Let $U_n$ be a unitary operator that $U_n(\frac{T(x_n)}{\|T(x_n)\|}) = x_n$. Setting $P_n = x_n \otimes x_n$ we have

$$
\|T\| = \|P_n\|_{op} \|T\| \geq \|TP_n\| = \left\| \frac{T(x_n)}{\|T(x_n)\|} \otimes x_n \right\| \|T(x_n)\|
= \left\| U_n \left( \frac{T(x_n)}{\|T(x_n)\|} \otimes x_n \right) \right\| \|T(x_n)\|
= \|x_n \otimes x_n\| \|T(x_n)\|
= \|P_n\| \|T(x_n)\|
= \|P\| \|T(x_n)\|,
$$

where the last equality has resulted from (iii) of Corollary 2.2. Now if $n \to \infty$ we get the desired result. 

\[ \square \]

Corollary 2.4. If $P$ is a projection of rank one in $\mathbb{B}(\mathcal{H})$ then

$$
\|P\| \|T\|_{op} \leq \|T\| \leq \|P\| \|T\|_{op} \quad (T \in \mathbb{B}(\mathcal{H})),
$$

where $I$ is the identity operator on $\mathcal{H}$. Therefore, all unitarily invariant norms on $\mathbb{B}(\mathcal{H})$ are equivalent to the operator norm.

Corollary 2.5. If $P$ is a projection of rank one in $\mathbb{B}(\mathcal{H})$ and $\|P\| = \|I\|$, then $\| \cdot \|$ is a multiple of the operator norm.

Lemma 2.6. Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$. For positive diagonal operator $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ ($\lambda_i \geq 0$), let $E = \{\lambda_i \mid i = 1, 2, \cdots \}$. Then

(i) $s_1(T) = \|T\|_{op} = \sup_{j \in \mathbb{N}} \lambda_j$.

(ii) If there exist $k - 1$ distinct positive integers $n_1, \ldots, n_{k-1}$ such that $s_i(T) = \lambda_{n_i}$ ($1 \leq i \leq k - 1$), then $s_k(T) = \sup_{j \notin \{n_1, \ldots, n_{k-1}\}} \lambda_j$. Also, if $s_k(T)$ is a limit point of $E$, then for every $i \in \mathbb{N}$, we have $s_{k+i}(T) = s_k(T)$.

(iii) If there is no s-number of $T$ that is a limit point of $E$, then there are distinct positive integers $n_1, n_2, \ldots$ such that $s_i(T) = \lambda_{n_i}$, $i \in \mathbb{N}$. Otherwise there is positive integer $k$ and $k - 1$ distinct natural numbers $n_1, \ldots, n_{k-1}$ such that $s_i(T) = \lambda_{n_i}$, $1 \leq i \leq k - 1$ and $s_k(T) = s_{k+1}(T) = \cdots$. In both cases, for every positive integer $i$, we have $s_i(T) = \sup_{\lambda_j \leq s_i(T)} \lambda_j$.

Proof. (i) This is a well known fact [6, Problem 63].

(ii) If $k = 1$ the equality is resulted from (i). Otherwise setting

$$
F = \sum_{i=1}^{k-1} \lambda_{n_i} e_{n_i} \otimes e_{n_i},
$$

we have $\text{rank}(F) < k$ and so

$$
s_k(T) \leq \|T - F\|_{op} = \sup_{j \notin \{n_1, \ldots, n_{k-1}\}} \lambda_j.
$$
On the other hand if for every \( j \in \mathbb{N} \setminus \{n_1, \ldots, n_{k-1}\} \), setting
\[
R_j = \sum_{i=1}^{k-1} \lambda_i e_{n_i} \otimes e_{n_i} + \lambda_j e_j \otimes e_j,
\]
we have \( R_j \leq T \). Since Ky Fan norms are unitarily invariant, using (iv) of Corollary 2.2, we have \( N_k(R_j) \leq N_k(T) \). Therefore \( \lambda_j \leq s_k(T) \) and hence \( s_k(T) = \sup_{j \notin \{n_1, \ldots, n_{k-1}\}} \lambda_j \).

If \( s_k(T) \) is a limit point of \( E \), then for every \( \epsilon > 0 \) and \( i \in \mathbb{N} \), there exist distinct positive integers \( m_1, \ldots, m_{i+1} \in \mathbb{N} \setminus \{n_1, \ldots, n_{k-1}\} \) such that
\[
s_k(T) - \frac{\epsilon}{i+1} < \lambda_{m_j} \leq s_k(T), \ (1 \leq j \leq i+1).
\]
Setting
\[
S = \sum_{j=1}^{i+1} \lambda_{m_j} e_{m_j} \otimes e_{m_j} + \sum_{j=1}^{k-1} \lambda_{n_j} e_{n_j} \otimes e_{n_j},
\]
we have \( 0 \leq S \leq T \) and so \( N_{k+i}(S) \leq N_{k+i}(T) \). This implies that
\[
\sum_{j=1}^{i+1} (s_k(T) - \frac{\epsilon}{i+1}) \leq s_k(T) + \cdots + s_{k+i}(T)
\]
\[
\leq s_k(T) + \cdots + s_k(T) + s_{k+i}(T).
\]

Therefore \( s_k(T) - \epsilon \leq s_{k+i}(T) \) and so \( s_k(T) \leq s_{k+i}(T) \). Hence, \( s_k(T) = s_{k+i}(T) \).

(iii) By (i) we have \( s_1(T) = \sup_{j \in \mathbb{N}} \lambda_j \). If \( s_1(T) \) is a limit point of \( E \), then by the second part of (ii), we have \( s_1(T) = s_2(T) = \cdots \). Otherwise there is \( n_1 \in \mathbb{N} \) such that \( s_1(T) = \lambda_{n_1} \) and by the first part of (ii) we have \( s_2(T) = \sup_{j \notin \{n_1\}} \lambda_j \).

Now if \( s_2(T) \) is a limit point of \( E \), then again by the second part of (ii), we have \( s_2(T) = s_3(T) = \cdots \). Otherwise there is \( n_2 \in \mathbb{N} - \{n_1\} \) such that \( s_2(T) = \lambda_{n_2} \) and by the first part of (ii), we have \( s_3(T) = \sup_{j \notin \{n_1, n_2\}} \lambda_j \). Continuing this process we get desired results.

In particular, the following corollary follows from the previous lemma.

**Corollary 2.7.** Suppose \( \{e_i\}_{i=1}^{\infty} \) is an orthonormal sequence in \( \mathcal{H} \). For positive diagonal operator \( T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i \), let \( s = \inf \{s_i(T) \mid i \in \mathbb{N}\} \), \( E = \{\lambda_i \mid i = 1, 2, \ldots\} \). Then

(i) for every \( \epsilon > 0 \), there exist distinct positive integers \( n_1, n_2, \ldots \) such that \( 0 \leq s_i(T) < \lambda_{n_i} + \epsilon \).

(ii) for every \( \epsilon > 0 \), \( A = \{i \mid \lambda_i > s + \epsilon\} \) is a finite set. In fact, \( A \) is empty or there exist distinct positive integers \( n_1, \ldots, n_{N_0} \), such that \( A = \{n_i \mid 1 \leq i \leq N_0\} \) and \( \lambda_{n_i} = s_i(T) \) \( (1 \leq i \leq N_0) \).

**Lemma 2.8.** Suppose \( \{e_i\}_{i=1}^{\infty} \) is an orthonormal sequence in \( \mathcal{H} \) and \( T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i (\lambda_i \geq 0) \) is a positive diagonal operator in \( \mathbb{B}(\mathcal{H}) \). Then
(i) for every $\epsilon > 0$ there exists a unitary element $U \in \mathcal{B}(\mathcal{H})$ such that

$$U T U^* \leq \sum_{i=1}^{\infty} (s_i(T) + \epsilon) e_i \otimes e_i,$$

(ii) for every $\epsilon > 0$ there exists a partial isometry $U \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{i=1}^{\infty} (s_i(T) - \epsilon) e_i \otimes e_i \leq U T U^*,

(iii) $\|T\| = \|\sum_{i=1}^{\infty} s_i(T)e_i \otimes e_i\|.$

Proof. Let $\epsilon > 0$, $s = \inf\{s_i(T) \mid i \in \mathbb{N}\}$ and $A = \{i \mid \lambda_i > s + \epsilon\}$. The previous corollary implies that $A$ is empty or there exist distinct positive integers $n_1, \cdots, n_{N_0}$, such that $A = \{n_i : 1 \leq i \leq N_0\}$ and $\lambda_i = s_i(T)$ ($1 \leq i \leq N_0$).

If $A$ is empty, we set $U = I$, otherwise we consider $U$ as a unitary operator that maps $\{e_n\}_{n=1}^{\infty}$ onto $\{e_n\}_{n=1}^{\infty}$ and $U(e_n) = e_i$ for $i = 1, \cdots, N_0$. Therefore

$$U T U^* = \sum_{i=1}^{N_0} s_i(T)e_i \otimes e_i + \sum_{i=N_0+1}^{\infty} \mu_i e_i \otimes e_i,$$

where $\mu_i \in \{\lambda_j : j \neq n_i, \text{ for all } 1 \leq i \leq N_0\}$. Since for all $i \geq N_0$ we have $\mu_i \leq s + \epsilon$, then $U T U^* \leq \sum_{i=1}^{\infty} (s_i(T) + \epsilon) e_i \otimes e_i$.

For proving (ii), we recall that there exist distinct positive integers $n_1, n_2, \cdots$ such that $0 \leq s_i(T) < \lambda_n + \epsilon$. Now consider a partial isometry $U$ which satisfies

$$U(e_j) = \begin{cases} e_i & j = n_i, \text{ for some } i \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

We have

$$U T U^* = \sum_{i=1}^{\infty} \lambda_n e_i \otimes e_i$$

$$\geq \sum_{i=1}^{\infty} (s_i(T) - \epsilon) e_i \otimes e_i.$$ 

Finally (iii) follows from (i),(ii) and (iv) of Corollary 2.2.

Lemma 2.9. Let $\mathcal{H}$ be a separable Hilbert space and $T$ be a positive operator in $\mathcal{B}(\mathcal{H})$. For every $\epsilon > 0$, there exists a diagonal operator $T_\epsilon$ such that

$$|s_i(T) - s_i(T_\epsilon)| < \epsilon, \text{ for all } i \in \mathbb{N}.$$  

Proof. By [11] there exist a diagonal operator $T_\epsilon$ and a compact operator $K_\epsilon$ such that $T = T_\epsilon + K_\epsilon$ and the Hilbert-Schmidt norm of $K_\epsilon$ is less than $\epsilon$. Hence for every finite rank operator $F$, the following inequalities hold

$$\|T_\epsilon - F\|_{op} - \epsilon \leq \|T - F\|_{op} \leq \|T_\epsilon - F\|_{op} + \epsilon.$$ 

Taking infimum over $F$ with rank($F$) < $i$, we get the desired result.

Theorem 2.10. Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. The equality $\|T\| = \|\sum_{i=1}^{\infty} s_i(T)e_i \otimes e_i\|$ holds, for all orthonormal sequence $\{e_i\}_{i \in \mathbb{N}}$ in $\mathcal{H}$. 

Proof. Since \(|T| = | |T| |\) and \(s_i(T) = s_i(|T|)\), we can suppose that \(T\) is a positive operator. Let \(\epsilon > 0\) and \(T_\epsilon = \sum_{i=1}^\infty \lambda_i e_i \otimes e_i\) be the diagonal operator of the previous lemma. We have by Lemmas 2.1 and 2.8
\[
\|T\| \leq \|T_\epsilon\| + \|K_\epsilon\| \\
\leq \|T_\epsilon\| + \|I\|\|K_\epsilon\|_{op} \\
\leq \left[ \sum_{i=1}^\infty s_i(T_\epsilon)e_i \otimes e_i \right] + \epsilon\|I\| \\
\leq \left[ \sum_{i=1}^\infty s_i(T)e_i \otimes e_i \right] + 2\epsilon\|I\|.
\]
Similarly, we have
\[
\left\| \sum_{i=1}^\infty s_i(T)e_i \otimes e_i \right\| \leq \left\| \sum_{i=1}^\infty s_i(T_\epsilon)e_i \otimes e_i \right\| + \epsilon\|I\| \\
= \|T_\epsilon\| + \epsilon\|I\| \\
\leq \|T\| + 2\epsilon\|I\|.
\]

Lemma 2.11. Suppose \(\{e_i\}_{i \in \mathbb{N}}\) is an orthonormal sequence in \(\mathcal{H}\) and \(D_1 = \sum_{i=1}^\infty \lambda_i e_i \otimes e_i\), \(D_2 = \sum_{i=1}^\infty \mu_i e_i \otimes e_i\) \((\lambda_i, \mu_i \geq 0)\), are positive diagonal operators in \(\mathbb{B}(\mathcal{H})\). Assume moreover that there exists \(N \in \mathbb{N}\) such that \(\lambda_k = \mu_k\) for all \(k > N\) and \(s_k(D_1) = \lambda_k, s_k(D_2) = \mu_k\) for all \(1 \leq k \leq N\). If \(N_k(D_1) \leq N_k(D_2)\) for all \(1 \leq k \leq N\), then \(\|D_1\| \leq \|D_2\|\).

Proof. Let \(X_1 = \sum_{i=1}^N \lambda_i e_i \otimes e_i\) and \(X_2 = \sum_{i=1}^N \mu_i e_i \otimes e_i\). We have \(\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i\), for every \(1 \leq k \leq N\) and so, there are unitary matrices \(U_1, \cdots, U_{2^N N!}\) in \(M_N(\mathbb{C})\) and non-negative numbers \(c_1, \cdots, c_{2^N N!}\) such that \(X_1 = \sum_{j=1}^{2^N N!} c_j U_j X_2 U_j^*\) and \(\sum_{j=1}^{2^N N!} c_j = 1\) [2, II.2.10]. Now, we can choose unitary operators \(\tilde{U}_1, \cdots, \tilde{U}_{2^N N!}\) in \(\mathbb{B}(\mathcal{H})\) such that
\[
\sum_{i=1}^N \lambda_i e_i \otimes e_i = \sum_{j=1}^{2^N N!} c_j \tilde{U}_j \left( \sum_{i=1}^N \mu_i e_i \otimes e_i \right) \tilde{U}_j^*,
\]
and \(\tilde{U}_j(e_i) = e_i\), for every \(1 \leq j \leq 2^N N!\) and \(i > N\). A direct computation shows that \(D_1 = \sum_{j=1}^{2^N N!} c_j \tilde{U}_j D_2 \tilde{U}_j^*\) and so \(\|D_1\| \leq \|D_2\|\).

Now, we can show that Ky Fan’s dominance theorem is valid on \(\mathbb{B}(\mathcal{H})\), where \(\mathcal{H}\) is a separable Hilbert space.

Theorem 2.12. Let \(\mathcal{H}\) be a separable Hilbert space and \(T, R \in \mathbb{B}(\mathcal{H})\). If \(N_k(T) \leq N_k(R)\) for all \(k \in \mathbb{N}\), then \(\|T\| \leq \|R\|\).

Proof. For every \(\epsilon > 0\), we can choose \(N \in \mathbb{N}\) such that:
1) \(s_N(T) \leq s_N(R) + \epsilon\),
ii) $s_N(R) \leq s_i(R) + \epsilon$, for every $i \geq N$.

Using (iv) of Corollary 2.2 and Lemma 2.10 together with Lemma 2.11, we have

\[
\|T\| = \left\| \sum_{i=1}^{\infty} s_i(T)e_i \otimes e_i \right\| \\
\leq \left\| \sum_{i=1}^{N} s_i(T)e_i \otimes e_i + \sum_{i=N+1}^{\infty} s_N(T)e_i \otimes e_i \right\| \\
\leq \left\| \sum_{i=1}^{N} s_i(R)e_i \otimes e_i + \sum_{i=N+1}^{\infty} s_N(T)e_i \otimes e_i \right\| \\
\leq \left\| \sum_{i=1}^{N} s_i(R)e_i \otimes e_i + \sum_{i=N+1}^{\infty} (s_N(R) + \epsilon)e_i \otimes e_i \right\| \\
\leq \left\| \sum_{i=1}^{N} s_i(R)e_i \otimes e_i + \sum_{i=N+1}^{\infty} (s_i(R) + 2\epsilon)e_i \otimes e_i \right\| \\
\leq \left\| \sum_{i=1}^{\infty} s_i(R)e_i \otimes e_i \right\| + 2\epsilon \|I\| = \|R\| + 2\epsilon \|I\|.
\]

\[\Box\]

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References

1. J. Alaminos, J. Extremera and A.R. Villena, 
   Uniqueness of rotation invariant norms, 
2. R. Bhatia, 
   Matrix Analysis, 
3. J.T. Chan, C.K. Li and C.N. Tu, 
   A class of unitarily invariant norms on $\mathcal{B}(\mathcal{H})$, 
4. J. Fang, D. Hadwin, E. Nordgren and J. Shen, 
   Tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property, 
5. I.C. Gohberg and M.G. Krein, 
   Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space, 
6. P.R. Halmos, 
   A Hilbert Space Problem Book, 
7. R.A. Horn and C.R. Johnson, 
   Topics in Matrix Analysis, 
8. R.V. Kadison and G.K. Pedersen, 
   Means and convex combinations of unitary operators, 
9. C.K. Li, 
   Some aspects of the theory of norms, 
10. M. S. Moslehian, 
    Ky Fan inequalities, 
    Linear Multilinear Algebra (to appear), Available online at 
11. D. Voiculescu, 
    Some results on norm-ideal perturbations of Hilbert space operators, 
12. J. Von Neumann, 
    Some matrix-inequalities and metrization of matrix-space, 
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