ON THE EQUIVALENCE OF HERMITIAN INNER PRODUCTS
ON TOPOLOGICAL *-ALGEBRAS

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ABSTRACT. Sufficient conditions for a topological *-algebra under which several Hermitian inner products are mutually equivalent are given.

1. INTRODUCTION

For constructing a Hermitian $K$-theory for topological *-algebras, one usually supposes that the algebra under consideration is unital and locally $m$-convex (see [7]). In this paper we obtain similar results as in [7] for the case of both unital and non-unital topological *-algebras without assuming locally $m$-convexity.

1.1. Preliminary definitions. Throughout this paper $\mathbb{K}$ denotes either the set $\mathbb{R}$ of all real numbers or the set $\mathbb{C}$ of all complex numbers. Let $A$ be a *-algebra over $\mathbb{K}$ and $M$ a left $A$-module. $\text{Hom}_A(M,A)$ stands for the set of all $A$-linear maps $f : M \to A$. Under the operations:

\[(f + g)(m) := f(m) + g(m), \quad (af)(m) := f(m)a^*, \quad (\lambda f)(m) := \overline{\lambda}[f(m)] \quad (1.1)\]

for all $f, g \in \text{Hom}_A(M,A)$, $m \in M$ and $\lambda \in \mathbb{K}$, $\text{Hom}_A(M,A)$ becomes a left $A$-module.

An $A$-valued Hermitian inner product on $M$ is a map $\alpha : M \times M \to A$ which satisfies the following properties:

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Similarly, (2) and (3) imply
$\alpha$ implies $\phi$ implies that the map $\phi$ is Hermitian, i.e., $\alpha(x, y) = \alpha(y, x)^*$ for every $x, y \in M$;
(3) $\alpha$ is Hermitian, i.e., $\alpha(x, y) = \alpha(y, x)^*$ for every $x, y \in M$;
(4) the map $\phi : M \to \text{Hom}_A(M, A)$, $x \mapsto \phi(x)$, defined by $(\phi(x))(y) := \alpha(y, x)$,
is an isomorphism of $A$-modules.

Notice, that the conditions (1) and (3) together imply
$(1') \alpha(x, \lambda y + \mu z) = \lambda \alpha(x, y) + \mu \alpha(x, z)$ for every $\lambda, \mu \in \mathbb{K}$ and $x, y, z \in M$.

Similarly, (2) and (3) imply
$(2') \alpha(x, ay) = \alpha(x, y)a^*$ for every $a \in A$ and $x, y \in M$.

Hence, $\alpha$ is also $\mathbb{K}$-sesquilinear and $A$-homogeneous on both arguments.

Moreover, the condition
$(4a) \alpha(x, x) = \theta_A$ if and only if $x = \theta_M$
implies that the map $\phi$ defined in (4) is one-to-one. (Indeed, suppose that $\phi(x) = \phi(y)$ for some $x, y \in M$. Then
\[ \alpha(x - y, x) = [\phi(x)](x - y) = [\phi(y)](x - y) = \alpha(x - y, y) \]
implies $\alpha(x - y, x - y) = \theta_A$. Hence, $x - y = \theta_M$ by $4a$ and $x = y$.)

A Hermitian inner product $\alpha$ on $M$ is said to be spectrally positive definite (for short positive definite\(^1\)), if $\text{Sp}_A(\alpha(x, x)) \subset [0, \infty)$ for every $x \in M$.

Let $B$ be a non-unital algebra. The set $\{e_1, \ldots, e_m\}$ of elements of a $B$-module $M$ is said to be a basis of $M$ if for every $m \in M$ there exist unique elements $b_1, \ldots, b_m \in B$ and unique numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ such that
\[ m = \sum_{i=1}^{m} b_i e_i + \sum_{i=1}^{m} \lambda_i e_i. \]

In case $A$ is a unital algebra, the set $\{e_1, \ldots, e_m\}$ of elements of an $A$-module $M$ is said to be a basis of $M$ if for every $m \in M$ there exist unique elements $a_1, \ldots, a_m \in A$ such that
\[ m = \sum_{i=1}^{m} a_i e_i. \]

2. On the existence of a Hermitian inner product

First, we show that under some conditions, which are automatically fulfilled for Hausdorff locally $C^*$-algebras (see [3, Remark 1.2, p. 184]), every topological $\ast$-algebra admits a positive definite Hermitian inner product.

**Lemma 2.1.** Let $A$ be a unital $\ast$-algebra for which the following conditions are fulfilled:

(a) If $a \in A$, then $aa^* = \theta_A$ if and only if $a = \theta_A$.

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\(^1\)Another kind of a positive element in a $\ast$-algebra is given in [3, p.183]. This notion of positiveness agrees with spectrally positiveness for Hausdorff locally $C^*$-algebras (viz. complete locally $m$-convex $C^*$-algebras) [3, p. 184] (see also [4, Theorem 2.5, p. 205]).
(b) If $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$, then $\text{Sp}_A(\sum_{i=1}^{n} a_i a_i^*) \subseteq [0, \infty)$.
(c) The only self-adjoint element $a \in A$ with $\text{Sp}_A(a) = \{0\}$ is the zero element $\theta_A$ of $A$.

Moreover, let $M$ be an $A$-module with basis $\{e_1, \ldots, e_m\}$. Then the map $\alpha : M \times M \to A$, defined by

$$\alpha(x, y) = \alpha\left(\sum_{i=1}^{m} x_i e_i, \sum_{i=1}^{m} y_i e_i\right) := \sum_{i=1}^{m} x_i y_i^*$$

for every $x, y \in M$, defines an $A$-valued positive definite Hermitian inner product on $M$.

Proof. Let $x = \sum_{i=1}^{m} x_i e_i$, $y = \sum_{i=1}^{m} y_i e_i$, $z = \sum_{i=1}^{m} z_i e_i$ be elements of $M$, $a \in A$ and $\lambda, \mu \in \mathbb{K}$. Then

$$\alpha(\lambda x + \mu y, z) = \alpha\left(\sum_{i=1}^{m} (\lambda x_i + \mu y_i)e_i, \sum_{i=1}^{m} z_i e_i\right) = \sum_{i=1}^{m} (\lambda x_i + \mu y_i)z_i^* =$$

$$= \lambda \sum_{i=1}^{m} x_i z_i^* + \mu \sum_{i=1}^{m} y_i z_i^* = \lambda \alpha(x, z) + \mu \alpha(y, z),$$

$$\alpha(ax, y) = \alpha\left(\sum_{i=1}^{m} (ax_i)e_i, \sum_{i=1}^{m} y_i e_i\right) = \sum_{i=1}^{m} (ax_i)y_i^* = a \sum_{i=1}^{m} x_i y_i^* = a \alpha(x, y),$$

and

$$\alpha(x, y) = \sum_{i=1}^{m} x_i y_i^* = \sum_{i=1}^{m} (y_i x_i^*)^* = \left(\sum_{i=1}^{m} y_i x_i^*\right)^* = \alpha(y, x)^*.$$ 

Hence, the first 3 conditions of an $A$-valued Hermitian inner product are fulfilled. This implies that the conditions $(1')$ and $(2')$ are also fulfilled.

Clearly $\alpha(\theta_M, \theta_M) = \theta_A$. Suppose that $\alpha(x, x) = \theta_A$ for some $x \in M$. Then

$$\sum_{i=1}^{m} x_i x_i^* = \theta_A.$$ 

Hence,

$$\sum_{i=1}^{m-1} x_i x_i^* = -x_m x_m^*$$

and

$$\text{Sp}_A\left(\sum_{i=1}^{m-1} x_i x_i^*\right) = \text{Sp}_A(-x_m x_m^*) = -\text{Sp}_A(x_m x_m^*).$$

By the condition (b), we get that

$$\text{Sp}_A\left(\sum_{i=1}^{m-1} x_i x_i^*\right) \subseteq [0, \infty) \quad \text{and} \quad \text{Sp}_A(x_m x_m^*) \subseteq [0, \infty).$$

Thus,

$$\text{Sp}_A\left(\sum_{i=1}^{m-1} x_i x_i^*\right) = \{0\} = \text{Sp}_A(x_m x_m^*).$$
Condition (c) implies that \( x_m x_m^* = \theta_A \) from which by condition (a) follows that \( x_m = \theta_A \). Similarly, we get that \( x_{m-1} = \theta_A, \ldots, x_1 = \theta_A \). Hence, from \( \alpha(x, x) = \theta_A \) it follows that \( x = \theta_M \). Consequently, \( \phi : M \rightarrow \text{Hom}_A(M, A) \), defined by \( [\phi(x)](y) = \alpha(x, y) \) is one-to-one.

Take now any \( \psi \in \text{Hom}_A(M, A) \), define \( x_i := \psi(e_i)^* \) for every \( i \in \{1, \ldots, m\} \) and \( x := \sum_{i=1}^m x_i e_i \). Then \( x \in M \) and

\[
\psi(y) = \sum_{i=1}^m y_i \psi(e_i) = \sum_{i=1}^m y_i (\psi(e_i)^*)^* = \sum_{i=1}^m y_i x_i^* = \alpha(y, x) = [\phi(x)](y)
\]

for every \( y \in M \). Hence, \( \phi \) is also onto.

Notice, that by the properties (1), (2), (3), (1’) and (2’) of \( \alpha \) and the condition (1.1) of the operations on \( \text{Hom}_A(M, A) \), we have

\[
[\phi(ax)](y) = \alpha(y, ax) = \alpha(y, x)a^* = [a\phi(x)](y),
\]

\[
[\phi(x + z)](y) = \alpha(y, x + z) = \alpha(y, x) + \alpha(y, z) = [\phi(x)](y) + [\phi(z)](y),
\]

and

\[
[\phi(\lambda x)](y) = \alpha(y, \lambda x) = \lambda \alpha(y, x) = [\lambda \phi(x)](y)
\]

for every \( a \in A, x, y, z \in M \) and \( \lambda \in K \). Hence, \( \phi(ax) = a\phi(x) \), \( \phi(x + z) = \phi(x) + \phi(z) \) and \( \phi(\lambda x) = \lambda \phi(x) \) for every \( a \in A, \lambda \in K \) and \( x, z \in M \). Therefore, \( \phi \) is an isomorphism of \( A \)-modules. Thus, \( \alpha \) is an \( A \)-valued Hermitian inner product on \( M \). Condition (b) implies that \( \alpha \) is also positive definite. \( \square \)

**Corollary 2.2.** Let \( B \) be a non-unital \(*\)-algebra for which the following conditions are fulfilled:

(a) If \( b \in B \), then \( bb^* = \theta_B \) if and only if \( b = \theta_B \).

(b) If \( n \in \mathbb{N}, b_1, \ldots, b_n \in B \) and \( \lambda_1, \ldots, \lambda_n \in K \), then

\[
\text{Sp}_{B \times K} \left( \sum_{i=1}^n (b_i, \lambda_i)(b_i, \lambda_i)^* \right) \subset [0, \infty).
\]

(c) The only self-adjoint element \( (b, \lambda) \in B \times K \) with \( \text{Sp}_{B \times K}((b, \lambda)) = \{0\} \) is the zero element \( (\theta_B, 0) \) of \( B \times K \).

Moreover, let \( M \) be a \( B \)-module with basis \( \{e_1, \ldots, e_m\} \). Then the map \( \alpha : M \times M \rightarrow B \times K \), defined by

\[
\alpha(x, y) = \alpha \left( \sum_{i=1}^m (x_i e_i + \lambda_i e_i), \sum_{i=1}^m (y_i e_i + \mu_i e_i) \right) := \sum_{i=1}^m (x_i, \lambda_i)(y_i, \mu_i)^*
\]

for every \( x, y \in M \), defines a \( (B \times K) \)-valued positive definite Hermitian inner product on \( M \).

**Proof.** Remind, that every \( B \)-module with basis \( \{e_1, \ldots, e_m\} \) is also a \((B \times K)\)-module with the same basis and that every \( B \)-linear map is also \((B \times K)\)-linear (see [2, Proof of Corollary 3, p. 162]). Moreover, suppose that \( (b, \lambda)(b, \lambda)^* = (\theta_B, 0) \). Then we have \((bb^* + \lambda^* b + \lambda b^*, \lambda \lambda^*) = (\theta_B, 0) \) (\( \lambda^* \) stands for the conjugate of \( \lambda \in K \)). Since \( \lambda \lambda^* = |\lambda|^2 \), we get \( \lambda = 0 \). Hence, \((b, \lambda)(b, \lambda)^* = (\theta_B, 0) \) if and only if \( bb^* = \theta_B \). By condition (a) we see that
\( (b, \lambda)(b, \lambda)^\ast = (\theta_B, 0) \) if and only if \( (b, \lambda) = (\theta_B, 0) \). Thus, taking \( A := B \times \mathbb{K} \), we are in the situation of Lemma 2.1. Hence, the claim follows from Lemma 2.1. \( \Box \)

3. ON THE MATRIX ASSOCIATED WITH A HERMITIAN INNER PRODUCT

Suppose again that \( A \) is a unital algebra. With every Hermitian inner product \( \alpha \) on an \( A \)-module \( M \) with basis \( \{e_1, \ldots, e_m\} \) (i.e., \( M \) is a free \( A \)-module of rank \( m \)), we can associate its matrix \( M_\alpha \) as follows:

\[
M_\alpha := (m_{i,j}), \text{ where } m_{i,j} = \alpha(e_i, e_j) \text{ for every } i, j \in \{1, \ldots, m\}.
\]

It is known that for a \( \ast \)-algebra \( A \) and \( A \)-valued square matrix \( M = (n_{i,j}) \), one defines \( M^\ast = (n_{j,i}) \), where \( n_{i,j} = m_{j,i}^\ast \) for every \( i \in \{1, \ldots, m\} \) and every \( j \in \{1, \ldots, m\} \). Since for a Hermitian inner product \( \alpha \) we have \( \alpha(e_i, e_j) = \alpha(e_j, e_i)^\ast \) for every \( i \in \{1, \ldots, m\} \) and every \( j \in \{1, \ldots, m\} \), then it is clear that \( M^\ast = M_\alpha \), i.e., \( M_\alpha \) is Hermitian (alias, self-adjoint). From the condition (4) of a Hermitian inner product, it follows by [5, Proposition 12, p. 385] (see also [6, Proposition 6.1, p. 465 together with Proposition 4.16, p. 456]), that \( M_\alpha \) is invertible. Moreover, for any

\[
x = \sum_{i=1}^{m} x_i e_i \quad \text{and} \quad y = \sum_{i=1}^{m} y_i e_i
\]

we have \( \alpha(x, y) = (x_1 x_2 \ldots x_m)M_\alpha(y_1^\ast y_2^\ast \ldots y_m^\ast)^T \), where \( (z_1 z_2 \ldots z_m)^T \) denotes the transpose matrix of the matrix \( (z_1 z_2 \ldots z_m) \) with one row and \( m \) columns, i.e., \( (z_1 z_2 \ldots z_m)^T \) is a matrix with \( m \) rows and 1 column.

Take any Hermitian invertible \( (m \times m) \)-matrix \( H = (h_{i,j}) \) and define a map \( \beta : M \times M \rightarrow A \) by setting

\[
\beta \left( \sum_{i=1}^{m} a_i e_i, \sum_{i=1}^{m} b_i e_i \right) := (a_1 a_2 \ldots a_m)H(b_1^\ast b_2^\ast \ldots b_m^\ast)^T.
\]

Then it is clear that \( \beta \) is \( A \)-homogeneous and \( \mathbb{K} \)-sesquilinear. Next we show that the map \( \phi : M \rightarrow \text{Hom}_A(M, A) \), defined by

\[
\left[ \phi \left( \sum_{i=1}^{m} a_i e_i \right) \right] \left( \sum_{i=1}^{m} b_i e_i \right) := \beta \left( \sum_{i=1}^{m} b_i e_i, \sum_{i=1}^{m} a_i e_i \right)
\]

is a bijection.

Suppose that

\[
\phi(m_a) = \phi \left( \sum_{i=1}^{m} a_i e_i \right) = \phi \left( \sum_{i=1}^{m} b_i e_i \right) = \phi(m_b)
\]

for some \( m_a, m_b \in M \). Then

\[
\sum_{i=1}^{m} h_{1,i} a_i^\ast = [\phi(m_a)](e_1) = [\phi(m_b)](e_1) = \sum_{i=1}^{m} h_{1,i} b_i^\ast,
\]

\[
\sum_{i=1}^{m} h_{2,i} a_i^\ast = [\phi(m_a)](e_2) = [\phi(m_b)](e_2) = \sum_{i=1}^{m} h_{2,i} b_i^\ast,
\]

\[
\ldots
\]
\[
\sum_{i=1}^{m} h_{m,i} a_i^* = [\phi(m_a)](e_m) = [\phi(m_b)](e_m) = \sum_{i=1}^{m} h_{m,i} b_i^*.
\]

Hence,
\[
\sum_{i=1}^{m} h_{ji} (a_i^* - b_i^*) = \theta_A
\]
for every \( j \in \{1, \ldots, m\} \). If we denote by \( H_i \) the \( i \)-th column of the matrix \( H \), then we get
\[
\sum_{i=1}^{m} H_i (a_i^* - b_i^*) = (\theta_A \theta_A \ldots \theta_A)^T.
\]

If \( a_i^* - b_i^* \neq \theta_A \) for at least one value of \( i \), then the columns of \( H \) are linearly dependent and \( H \) can not be invertible. Since \( H \) was assumed to be invertible, we must have \( a_i^* - b_i^* = \theta_A \) for every \( i \in \{1, \ldots, m\} \) from which \( m_a = m_b \) and \( \phi \) is one-to-one.

Take any \( \psi \in \text{Hom}_A(M, A) \). Since \( H \) is invertible, \( H^{-1} \) exists. Take
\[
x := \sum_{i=1}^{m} x_i e_i,
\]
where \( (x_1 x_2 \ldots x_m)^T := H^{-1}(\psi(e_1)^* \psi(e_2)^* \ldots \psi(e_m)^*)^T \). Then \( [\phi(x)](y) = \psi(y) \) for every \( y \in M \). Hence, \( \phi \) is onto. Consequently, \( \phi \) is a bijection.

Thus, \( \beta \), defined above, is a Hermitian inner product. Moreover, the matrix of \( \beta \) is actually \( H \), i.e., \( M_\beta = H \).

By the facts we just obtained, we have the following result.

**Lemma 3.1.** Let \( A \) be a unital \( \ast \)-algebra and \( M \) a free \( A \)-module of rank \( m \). Then there exists a bijection between the sets of Hermitian inner products on \( M \) and \( A \)-valued Hermitian invertible \((m \times m)\)-matrices.

By Lemma 3.1, we have the following result.

**Corollary 3.2.** Let \( B \) be a non-unital \( \ast \)-algebra and \( M \) a free \( B \)-module of rank \( m \). Then there exists a bijection between the sets of Hermitian inner products on \( M \) and \((B \times \mathbb{K})\)-valued Hermitian invertible \((m \times m)\)-matrices.

**Proof.** Since every \( B \)-module is also a \((B \times \mathbb{K})\)-module with the same basis, then taking \( A := B \times \mathbb{K} \), we are in the situation of Lemma 3.1. \( \square \)

Notice, that for the Hermitian inner product \( \alpha \), defined in Lemma 2.1 or Corollary 2.2, the matrix \( M_\alpha \), associated with \( \alpha \), is an identity matrix.

**Definition 3.3.** Let \( A \) be a unital \( \ast \)-algebra and \( M \) a free \( A \)-module of rank \( m \). We say that two Hermitian inner products, \( \alpha \) and \( \beta \) on \( M \), are equivalent, if there exists an invertible \((m \times m)\)-matrix \( N \) such that \( M_\alpha = N^* M_\beta N \).

Notice, that if for any Hermitian inner product \( \beta \) there exists a Hermitian invertible matrix \( N \) such that \( M_\beta = NN = N^2 \), then \( \beta \) is equivalent to \( \alpha \) defined in Lemma 2.1.
Let $A$ be a topological algebra. A sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ is a Mackey–Cauchy sequence if there exists a bounded and balanced set $U$ in $A$ such that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $x_n - x_m \in \epsilon U$ whenever $n, m > N_\epsilon$.

The algebra $A$ is sequentially Mackey complete (one could also use the term Mackey $\sigma$-complete) if every Mackey–Cauchy sequence in $A$ converges in $A$.

**Proposition 4.1.** Let $m \in \mathbb{N}$ and $A$ be a sequentially Mackey complete topological algebra. Then the algebra $M_m(A)$ of all $(m \times m)$-matrices with elements from $A$ is also sequentially Mackey complete.

**Proof.** The topology in the algebra $M_m(A)$ of all $A$-valued $(m \times m)$-matrices is induced by a product topology, i.e., a basis of this topology consists of sets

$$U_{O_1,\ldots,O_m} = \{M = (m_{ij}) \in M_m(A) : m_{ij} \in O_{(i-1)m+j}\},$$

where $O_1,\ldots,O_m$ vary in a basis of the topology of $A$.

Take any Mackey–Cauchy sequence $(M_n)_{n \in \mathbb{N}} = ((m^n_{ij}))_{n \in \mathbb{N}}$ in $M_m(A)$. Then the sequence $(m^n_{ij})_{n \in \mathbb{N}}$ is a Mackey–Cauchy sequence in $A$ for each fixed $i, j \in \{1,\ldots,m\}$. Indeed, let $U$ be a bounded and balanced set in $M_m(A)$ such that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ with $M_k - M_l \in \epsilon U$ whenever $k, l > N_\epsilon$. For each $i, j \in \{1,\ldots,m\}$ take $V_{ij} := \{m_{ij} \in A : (m_{ij}) \in U\}$. Then all sets $V_{ij}$ are balanced and bounded in $A$ because $U$ is balanced and bounded in $M_m(A)$. Now it is clear that $m^n_{ij} - m^l_{ij} \in \epsilon V_{ij}$ whenever $k, l > N_\epsilon$. Hence, there exists balanced and bounded sets $V_{ij}$ and numbers $N_\epsilon$ for every $\epsilon > 0$ such that the conditions of Mackey–Cauchy sequence are fulfilled.

Since $(m^n_{ij})_{n \in \mathbb{N}}$ is a Mackey–Cauchy sequence in $A$ for each $i, j \in \{1,\ldots,m\}$ and $A$ is sequentially Mackey complete, then $(m^n_{ij})_{n \in \mathbb{N}}$ converges in $A$ to some element $s_{ij} \in A$ for each $i, j \in \{1,\ldots,m\}$. Take $S := (s_{ij}) \in M_m(A)$. Then $(M_n)_{n \in \mathbb{N}}$ converges to $S$ in $M_m(A)$. Hence, $M_m(A)$ is sequentially Mackey complete as well.

Let us recall, that for an element $a$ in a topological algebra $A$ its radius of boundedness is defined as

$$\beta(a) := \inf \left\{ \lambda > 0 : \left( \frac{a}{\lambda} \right)^n : n \in \mathbb{N} \right\} \text{ is bounded in } A.$$ 

We recall also that the terms ”$a$ is Hermitian” and ”$a$ is self-adjoint” are synonyms. In [1, Corollary 2.8], it was proved the following.

**Theorem 4.2.** Let $A$ be a unital sequentially Mackey complete topological algebra. If $a \in A$ satisfies the condition $\beta(a - e_A) < 1$, then there exists an element $b \in A$ such that $b^2 = a$. In particular, when $A$ is a unital sequentially Mackey complete topological $*$-algebra with continuous involution and $a$ is self-adjoint, then $b$ is also self-adjoint.

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\[2\text{It is clear that if } A \text{ is unital, then also } M_m(A) \text{ is unital because the unit element in } M_m(A) \text{ is the identity matrix.}\]
Let $A$ be a topological algebra and $m \in \mathbb{N}$. For every $i, j \in \{1, \ldots, m\}$ define the projections $p_{i,j} : M_m(A) \to A$ by $p_{i,j}(M) = m_{ij}$ for every $M = (m_{ij}) \in M_m(A)$. A map $f : M_m(A) \to M_m(A)$ is continuous if and only if all of its projections are continuous, i.e., $f$ is continuous if and only if $p_{i,j} \circ f$ is continuous for every $i, j \in \{1, \ldots, m\}$.

For the next result, see also [3, Lemma 5.3, p. 196], where the continuity of the involution of a locally $m$-convex $*$-algebra is inherited to the algebra of all infinite matrices with finite support and entries from $A$.

**Lemma 4.3.** Let $A$ be a topological $*$-algebra and $m \in \mathbb{N}$. The involution on $M_m(A)$ is continuous if and only if the involution is continuous on $A$.

**Proof.** Suppose, that the involution $i_A : A \to A$, defined by $i_A(a) = a^*$ for every $a \in A$, is continuous. Consider the involution $i_m : M_m(A) \to M_m(A)$ defined by $i_m(M) = M^*$ for every $M \in M_m(A)$. Then $(p_{i,j} \circ i_m)(M) = m_{ji}^*$ for every $M = (m_{ij}) \in M_m(A)$. Let $T : M_m(A) \to M_m(A)$ be the transpose function, i.e., $T(M) = T((m_{ij})) = (m_{ji}) = M^T$ for every $M \in M_m(A)$. Then $(i_A \circ p_{i,j} \circ T)(M) = m_{ji}^*$ for every $M = (m_{ij}) \in M_m(A)$. Hence, $p_{i,j} \circ i_m = i_A \circ p_{i,j} \circ T$.

The transpose function is continuous because for any neighbourhoods of zero $O_{i,j}$ in $A$ there exist neighbourhoods $U_{i,j} = O_{j,i}$ of zero in $A$ such that if $M \in U_{1,1} \cup U_{1,2} \cup \cdots U_{1,m} \cup U_{2,1} \cup \cdots U_{m,m}$ we get $T(M) \in UO_{1,1} \cup O_{1,2} \cup \cdots O_{1,m} \cup O_{2,1} \cup \cdots O_{m,m}$. The projections $p_{i,j}$ are also continuous. Hence, $i_A \circ p_{i,j} \circ T$ is continuous for every $i, j \in \{1, \ldots, m\}$ as a composition of continuous maps. Therefore, $p_{i,j} \circ i_m$ is continuous for every $i, j \in \{1, \ldots, m\}$. It means that $i_m$ is continuous.

Suppose that $i_m$ is continuous. Take any neighbourhood $O$ of zero in $A$. Then $P = U_{O_{1,1}} \cup O_{1,2} \cup \cdots \cup O_{m,2} \cup O_{2,2} \cup A$ is a neighbourhood of zero in $M_m(A)$. Since the involution is continuous in $M_m(A)$, then there exists a neighbourhood $V = U_{V_{1,1}} \cup \cdots \cup U_{V_{m,m}}$ of zero in $M_m(A)$ such that $i_m(M) \in P$ for every $M \in V$. Take

$$W := \bigcap_{1 \leq i \leq m^2} V_i$$

and $Z = U_{Z_{1,1}} \cup \cdots \cup U_{Z_{m,m}}$ with $Z_1 = Z_2 = \cdots = Z_{m^2} = W$. Then $i_m(M) \in P$ also for every $M \in Z$. Now, it is clear that $i_A(a) \in O$ for every $a \in W$ because $i_A(a) = p_{i,j} \circ i_m(M_a)$, where $M_a$ is a matrix having all its elements equal to $a$. Hence, $i_A$ is continuous as well. \(\square\)

For $m \in \mathbb{N}, I_m \equiv I$ denotes the identity matrix in $M_m(A)$. Using Theorem 4.2, we get the following result.

**Theorem 4.4.** Let $A$ be a unital sequentially Mackey complete topological $*$-algebra with continuous involution, $M$ a free $A$-module of rank $m$ and $\alpha : M \times M \to A$ a Hermitian inner product on $M$. If the matrix $M_\alpha \in M_m(A)$ associated with $\alpha$ fulfils the condition $\beta(M_\alpha - I) < 1$, then there exists a Hermitian inner product $\gamma : M \times M \to A$ such that $M_\alpha = M_\gamma^2$.

**Proof.** By assumption, $m$ is a free $A$-module of rank $m$. Consider the $*$-algebra $M_m(A)$. By Proposition 4.1, $M_m(A)$ is a unital sequentially Mackey complete topological algebra. The involution in $M_m(A)$ is continuous by Lemma 4.3.
Let $\alpha : M \times M \to A$ be a Hermitian inner product on $M$ and let its matrix $M_\alpha$ fulfil the condition $\beta(M_\alpha - I) < 1$. Then, by the first part of Theorem 4.2, there exists a matrix $N \in M_m(A)$ such that $N^2 = M_\alpha$.

Since the involution on $M_m(A)$ is continuous and $M_\alpha$ is a Hermitian matrix, $N$ is Hermitian, by the second part of Theorem 4.2. Moreover, since $M_\alpha$ is invertible, $N$ must be also invertible (its inverse is $N^{-1} = M_\alpha^{-1}N$). Now, by Lemma 3.1, we get that $N$ is actually a matrix of some Hermitian inner product $\gamma : M \times M \to A$, i.e., $N = M_\gamma$. Hence, $M_\alpha = M_\gamma^2$ for some Hermitian inner product $\gamma$. \hfill $\square$

Using Lemma 2.1, we get the following result.

**Theorem 4.5.** Let $A$ be a unital sequentially Mackey complete topological $*$-algebra with continuous involution for which the following conditions are fulfilled:

- (a) If $a \in A$, then $aa^* = \theta_A$ if and only if $a = \theta_A$.
- (b) If $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$, then $S_{PA}(\sum_{i=1}^n a_i a_i^*) \subset [0, \infty)$.
- (c) The only self-adjoint element $a \in A$ with $S_{PA}(a) = \{0\}$ is the zero element $\theta_A$ of $A$.

Moreover, let $M$ be a free $A$-module of rank $m$. Then all Hermitian inner products $\delta : M \times M \to A$, with matrices $M_\delta$ such that $\beta(M_\delta - I) < 1$, are mutually equivalent.

**Proof.** Let $\delta$ be a Hermitian inner product for which $\beta(M_\delta - I) < 1$. By Theorem 4.4, there exists a Hermitian inner product $\gamma : M \times M \to A$ such that $M_\gamma^2 = M_\delta$. By Lemma 2.1, we know that there exists an inner product $\alpha : M \times M \to A$ with $M_\alpha = I$. Since $M_\gamma$ is Hermitian, then $M_\gamma^* = M_\gamma$. Therefore, $M_\delta = M_\gamma^2 = M_\gamma^* M_\gamma = M_\gamma^* I M_\gamma = M_\gamma^* M_\alpha M_\gamma$. Hence, the Hermitian inner products $\delta$ and $\alpha$ are equivalent.

Let $\kappa : M \times M \to A$ be another Hermitian inner product with $\beta(M_\kappa - I) < 1$. As before, we can now show that $\kappa$ and $\alpha$ are equivalent. Hence, $\kappa$ is equivalent to $\delta$. Therefore, all such Hermitian inner products $\delta$ with $\beta(M_\delta - I) < 1$ are mutually equivalent. \hfill $\square$

Let $B$ be a non-unital algebra, $m \in \mathbb{N}$ and $J$ denote the identity matrix in the algebra $M_m(B \times \mathbb{K})$. Suppose that the involution $i_B : B \to B$, defined by $i_B(b) := b^*$ for every $b \in B$, is continuous on $B$. Take any neighbourhood $O$ of zero in $B \times \mathbb{K}$. Then there exist neighbourhoods of zero $U$ in $B$ and $V$ in $\mathbb{K}$ such that $U \times V \subset O$. Since involution is continuous on $B$ and $\mathbb{K}$, there exist neighbourhoods of zero $W$ in $B$ and $Z$ in $\mathbb{K}$ such that $i_B(b) \in U$ for every $b \in W$ and $i_{\mathbb{K}}(\lambda) \in V$ for every $\lambda \in Z$ (here $i_{\mathbb{K}}$ denotes the involution on $\mathbb{K}$). Denote the involution in $B \times \mathbb{K}$ by $i_{B \times \mathbb{K}}$. Since $P := U \times V$ is a neighbourhood of zero in $B \times \mathbb{K}$ and since $i_{B \times \mathbb{K}}((b, \lambda)) \in O$ for every $(b, \lambda) \in P$, then the involution $i_{B \times \mathbb{K}}$ in $B \times \mathbb{K}$ is also continuous.

From the last two Theorems we can have the following results in nonunital case.

**Corollary 4.6.** Let $B$ be a non-unital sequentially Mackey complete topological $*$-algebra with continuous involution, $M$ a free $B$-module of rank $m$ and
\( \alpha : M \times M \to B \times \mathbb{K} \) a Hermitian inner product on \( M \). If the matrix \( M_\alpha \in M_m(B \times \mathbb{K}) \), associated with \( \alpha \), fulfils the condition \( \beta(M_\alpha - J) < 1 \), then there exists a Hermitian inner product \( \gamma : M \times M \to B \times \mathbb{K} \) such that \( M_\alpha = M_\gamma \).

**Proof.** Since \( \mathbb{K} \) is complete, it is also Mackey complete. By assumption, \( B \) is sequentially Mackey complete, so \( B \times \mathbb{K} \), endowed with the product topology, turns to be Mackey complete. For the latter, one can argue as in the proof of Proposition 4.1, that \( B \times \mathbb{K} \) is Mackey complete. Since every \( B \)-module with \( m \) elements in its basis is also a \( (B \times \mathbb{K}) \)-module with the same basis, then we are in the context of Theorem 4.4, if we take \( A := B \times \mathbb{K} \). Hence, the claim follows by Theorem 4.4. \( \square \)

**Corollary 4.7.** Let \( B \) be a non-unital sequentially Mackey complete topological \( \ast \)-algebra with continuous involution for which the following conditions are satisfied:

(a) If \( b \in B \), then \( bb^* = \theta_B \) if and only if \( b = \theta_B \).
(b) If \( n \in \mathbb{N} \), \( b_1, \ldots, b_n \in B \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{K} \), then
\[
\mathrm{Sp}_{B \times \mathbb{K}} \left( \sum_{i=1}^{n} (b_i, \lambda_i)(b_i, \lambda_i)^* \right) \subset [0, \infty).
\]
(c) The only self-adjoint element \( (b, \lambda) \in B \times \mathbb{K} \) with \( \mathrm{Sp}_{B \times \mathbb{K}}((b, \lambda)) = \{0\} \) is the zero element \( (\theta_B, 0) \) of \( B \times \mathbb{K} \).

Moreover, let \( M \) be a free \( B \)-module of rank \( m \). Then all Hermitian inner products \( \delta : M \times M \to B \times \mathbb{K} \) with matrices \( M_\delta \) such that \( \beta(M_\delta - J) < 1 \) are mutually equivalent.

**Proof.** Using the same argumentation as in the proofs of Corollaries 2.2 and 4.6, we see that by taking \( A := B \times \mathbb{K} \), we are in the situation of Theorem 4.5, thus the assertion follows. \( \square \)

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