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ALMOST EVERYWHERE CONVERGENCE OF THE SPHERICAL PARTIAL FOURIER INTEGRALS FOR RADIAL FUNCTIONS

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This paper is dedicated to Professor Lars-Erik Persson

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Abstract. We study new conditions on a radial function f in order to have the almost everywhere convergence of the spherical partial Fourier integrals.

1. Introduction and Preliminaries

Given a function f for which the Fourier transform is well defined, the spherical partial Fourier integral is given by

$$S_R f(x) = \int_{B(0,R)} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

and it is an old and difficult open problem to show whether

$$\lim_{R\to\infty} S_R f(x) = f(x), \qquad \text{a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

whenever $f \in L^2(\mathbb{R}^n)$ with $n > 1$. The case $n = 1$ was solved positively by L. Carleson in [1] (see also [5] for the case $f \in L^p(\mathbb{R})$, $p > 1$).

Looking for conditions on a function f in order to have the convergence (1.1), it was proved in [7] that this is the case if f is a radial function belonging to

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$L^p(\mathbb{R}^n)$ with

$$\frac{2n}{n+1} < p < \frac{2n}{n-1}.$$

To prove this it was shown that, for radial functions,

$$\tilde{S}f(x) = \sup_R |S_R f(x)| \leq \frac{C(n)}{s^{(n-1)/2}}(M + L + \tilde{H} + \tilde{C})(g)(s) \tag{1.2}$$

where $s = |x|$, $g(r) = f(r)r^{(n-1)/2}\chi_{(0,\infty)}(r)$, M is the Hardy–Littlewood maximal operator, $\tilde{H}$ is the maximal Hilbert transform, $\tilde{C}$ is the maximal Carleson operator defined by

$$\tilde{C}f(x) = \sup_{y\in\mathbb{R}} \sup_{\varepsilon>0} \left| \int_{\varepsilon<|x-t|} \frac{e^{-iyt}f(t)}{x-t} dt \right|$$

and L is the Hilbert integral

$$Lf(s) = \int_0^\infty \frac{f(t)}{s+t}\, dt.$$

Using (1.2) it is proved in [9] and [2] that

$$\tilde{S} : L^{p_j,1}_{rad} \longrightarrow L^{p_j,\infty}, \qquad j = 0, 1$$

is bounded with

$$p_0 = \frac{2n}{n+1}, \qquad\qquad p_1 = \frac{2n}{n-1}, \tag{1.3}$$

and, for a space X of functions in $\mathbb{R}^n$,

$$X_{rad} = \{f \in X;\; f \text{ is radial}\}.$$

From this the almost everywhere convergence of $S_R f(x)$ at the end-point spaces $L^{p_j,1}_{rad}$ follows.

Again (1.2) is used in [8] to prove that if w is a radial weight such that $u(s) = w(s)|s|^{(n-1)(1-\frac{p}{2})}$ is in the Muckenhoupt class $A_p(\mathbb{R})$ (see [6]) then

$$||\tilde{S}f||_{L^p(w)} \leq C_{w,p}||f||_{L^p_{rad}(w)}. \tag{1.4}$$

In fact, from (1.2) we have that, if w is radial,

$$||\tilde{S}f||_{L^p(w)} \lesssim ||Tg||_{L^p\left(\mathbb{R}^+;\, w(s)s^{(n-1)(1-\frac{p}{2})}\right)} = ||Tg||_{L^p(\mathbb{R};u)}$$

where $u(s)$ is as before and

$$Tg(s) = (M + L + \tilde{H} + \tilde{C})(g)(|s|).$$

Now, if $u \in A_p(\mathbb{R})$, all the operators appearing in T are bounded on $L^p(u)$ and hence (1.4) is obtained.

However, no information is given in [8] about the behavior of the constant $C_{w,p}$ in (1.4). In the recent paper [3], this constant has been explicitly computed showing that for every $1 < p < \infty$ and u as before

$$C(w,p) \lesssim \max\left(||u||_{A_p}^{\frac{1}{p-1}}, ||u||_{A_p}, \inf_{r>1} \frac{1}{(r-1)^2}||u||_{A_{\frac{p}{r}}}^{\frac{r}{\frac{p}{r}-1}} \right). \tag{1.5}$$

Using this estimate, it was easy to see, for example, that if w is a radial function such that $w_0 \in A_1(\mathbb{R})$, where $w_0(r) = w(|x|)$ for $|x| = r > 0$ and $w_0(r) = w_0(-r)$ for $r < 0$ and where we recall that $w_0 \in A_1(\mathbb{R})$ if

$$Mw_0(s) \le Cw_0(s), \ a.e. \ s \in \mathbb{R}$$

and $||w_0||_{A_1}$ is the infimum of all the above constants C, then

$$C_{w,p} \lesssim ||w_0||_{A_1(\mathbb{R})} \left(\frac{1}{p - p_0} \right)^3, \tag{1.6}$$

for $p_0 < p \le 2$ and p_0 as in (1.3).

Definition 1.1. We shall say that a radial weight w defined in $\mathbb{R}^n$ is in $A_1(\mathbb{R})$ if $||w||_{A_1(\mathbb{R})} = ||w_0||_{A_1} < \infty$. and we shall write

$$w \in A_1(\mathbb{R}).$$

Using (1.4), (1.6) and a Yano's extrapolation argument (see [10]), the following result was obtained in [3].

Theorem 1.2. *If w is a radial function in $\mathbb{R}^n$ such that $w \in A_1(\mathbb{R})$ then (1.1) holds for every radial function f satisfying*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} \Big(1 + \log^+ |f(x)|\Big)^{p_0 \beta} w(x) dx < \infty$$

with $\beta > 3$.

On the other hand, in the other end-point p_1, the result obtained in [3] reads as follows.

Theorem 1.3. *If w is a radial function in $\mathbb{R}^n$ such that $w(s)|s|^{\frac{-2n}{n-1}} \in A_1(\mathbb{R})$ then (1.1) holds for every radial function f satisfying*

$$\int_{\mathbb{R}^n} |f(x)|^{p_1} \Big(1 + \log^+ |f(x)|\Big)^{p_1 \beta} w(x) dx < \infty$$

with $\beta > 3$.

It was not completely clear in [3] why the conditions on the weight w differs in p_0 and p_1 and which other condition on a radial weight we can assume in order to have the almost everywhere convergence in a space "near" $L^p_{rad}(w)$ for other values of $1 < p < \infty$. This will be clarified in the present note.

Given two quantities A and B, we shall use the symbol $A \lesssim B$ to indicate the existence of a positive universal constant C such that $A \le CB$. Also for simplicity, we write

$$\overline{\log}\, x = 1 + \log^+ x$$

with $\log^+ x = \max(\log x, 0)$.

2. Main results

Let us recall (see [4]) that a weight $v \in A_p$ if and only if $v = v_0 v_1^{1-p}$ with $v_j \in A_1$, $j = 0, 1$ and

$$||v||_{A_p} \leq ||v_0||_{A_1} ||v_1||_{A_1}^{p-1}.$$

Also, a power weight $v(x) = |x|^\alpha \in A_1(\mathbb{R})$ if and only if $-1 < \alpha \leq 0$ and ([3])

$$||v||_{A_1} \leq \frac{2}{1+\alpha}.$$

With these estimates, let us assume now that

$$w(x) = v(x)|x|^\delta$$

for some $\delta \in \mathbb{R}$ and v a radial weight in $\mathbb{R}^n$ such that $v \in A_1(\mathbb{R})$. Then if

$$u(s) = v(s)|s|^{\delta+(n-1)(1-\frac{p}{2})}, \qquad s \in \mathbb{R}$$

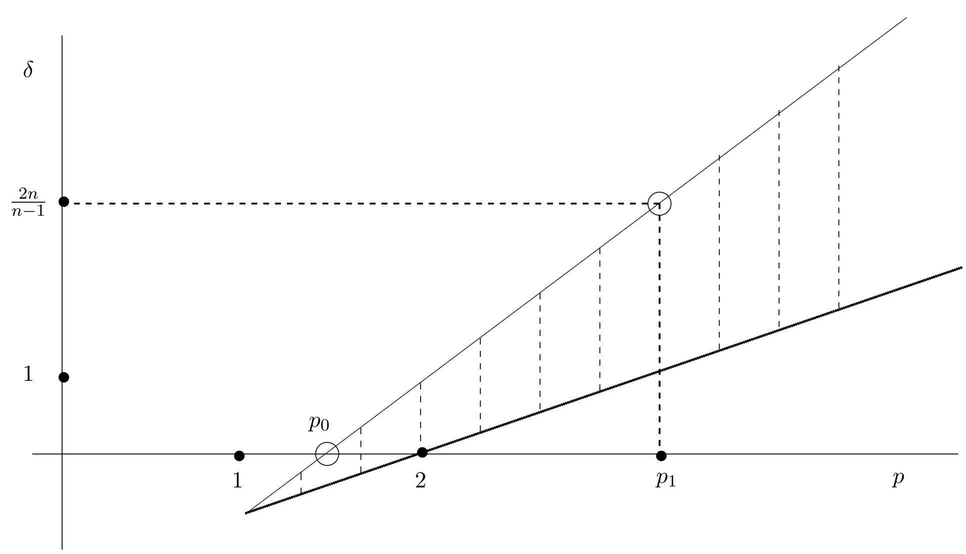
and

$$(n-1)\Big(\frac{p}{2} - 1\Big) \leq \delta < \frac{n+1}{2}p - n \tag{2.1}$$

we get that $u \in A_p(\mathbb{R})$. Moreover,

$$\|u\|_{A_p} \lesssim ||v||_{A_1} \left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^{p-1} \tag{2.2}$$

The area inside the cone together with the inferior boundary in the below picture represents the set of pairs (p, δ) satisfying (2.1) and will be called the admissible region.



Theorem 2.1. *If (p, δ) belongs to the admissible region and w is a radial function satisfying that*

$$w(x)|x|^{-\delta} \in A_1(\mathbb{R})$$

then

$$\tilde{S} : L^p_{rad}(w) \longrightarrow L^p(w)$$

is bounded. Moreover, for every f radial function,

$$||\tilde{S}f||_{L^p(w)} \lesssim \left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^{\max(3,p-1)} ||f||_{L^p(w)} \tag{2.3}$$

Consequently, if $f \in L^p_{rad}(w)$, (1.1) holds.

Proof. By (1.5) and (2.2) we have to compute

$$\begin{aligned}\inf_{r>1} \frac{1}{(r-1)^2}||u||_{A_{\frac{p}{r}}}^{\frac{r}{\frac{p}{r}-1}} &\lesssim \inf_{r>1} \frac{1}{(r-1)^2}\left(\frac{1}{\delta + n - \frac{p}{r}\frac{n+1}{2}}\right)^r \\ &\le \inf_{r>1} \frac{1}{(r-1)^2}\left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^r.\end{aligned}$$

Then, taking r such that $r-1 \approx \delta + n - p\frac{n+1}{2}$ we get that

$$\inf_{r>1} \frac{1}{(r-1)^2}||u||_{A_{\frac{p}{r}}}^{\frac{r}{\frac{p}{r}-1}} \lesssim \left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^3$$

and therefore

$$C(w,p) \lesssim \left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^{\max(3,p-1)}$$

as we wanted to see. □

Our purpose now is to use (2.3) and some extrapolation argument in order to obtain the almost everywhere convergence for a radial function in a space "near" $L^p_{rad}(w)$ with

$$w(x)|x|^{n-\frac{n+1}{2}p} \in A_1(\mathbb{R}).$$

Observe that the pair $(p, \frac{n+1}{2}p - n)$ is in the upper boundary (and hence outside) of the admissible region. Moreover, if $p = \frac{2n}{n+1}$, the above condition reads

$$w(x) \in A_1(\mathbb{R})$$

and if $p = \frac{2n}{n-1}$

$$w(x)|x|^{\frac{-2n}{n-1}} \in A_1(\mathbb{R}),$$

which are the conditions in Theorems 1.2 and 1.3 respectively. With the same proof than in those theorems, we now have the following result.

Theorem 2.2. *Let $1 < p < \infty$ and let w be a radial weight in $\mathbb{R}^n$ such that $w(s)|s|^{n-\frac{n+1}{2}p} \in A_1(\mathbb{R})$ then (1.1) holds for every radial function f satisfying*

$$\int_{\mathbb{R}^n} |f(x)|^{p_1}\Big(1 + \log^+|f(x)|\Big)^{p_1\beta} w(x)dx < \infty,$$

with $\beta > \max(3, p-1)$.

Observe that if $p \le p_1$, $\max(3, p-1) = 3$ and the above theorem extends Theorems 1.2 and 1.3.

Remark 2.3. In [3], it was consider the case $\delta = 0$ and the estimate at the end-point $p = p_0$ was done by a Yano's extrapolation argument applying (2.3) with $p > p_0$. Also, it was considered the end-point $p = p_1$ taking $\delta = \frac{2n}{n-1}$ and $p > p_1$ which is also inside the admissible region.

Another possibility, which is the one presented in our next theorem is to consider p fixed and move δ vertically in such a way that (p, δ) is inside the admissible region.

Theorem 2.4. *Let $p_n = \frac{n+1}{2}p - n$ and let w be a radial weight in $\mathbb{R}^n$ such that*

$$w(x)|x|^{-p_n} \in A_1(\mathbb{R}).$$

Then, for $1 < p < \infty$ and f a radial function,

$$\sup_{t>0} \frac{\left\| \min\left(1, \frac{t}{|x|}\right)^{1/p} \tilde{S}f \right\|_{L^p(w)}}{(\overline{\log}\ t)^{\max(p-1,3)}} \lesssim \left(\int_{\{|x|\geq 1/4\}} |f(x)|^p w(x)dx \right)^{1/p}$$
$$+ \sum_{i=0}^{\infty} \left(\int_{\{2^{-2^{i+1}} \leq |x| < 2^{-2^i}\}} |f(x)|^p \left(\overline{\log} \frac{1}{|x|} \right)^{p\max(p-1,3)} w(x)dx \right)^{1/p}.$$

Consequently, if f satisfies that the right term is finite, (1.1) holds.

Proof: Let us take δ in such a way that (p, δ) is inside the admissible region. Let us write $\theta = p_n - \delta$ and take δ in such a way that $0 < \theta < 1$. It is clear that, for every $t > 0$,

$$\frac{\min\left(1, \frac{t}{|x|}\right)}{t^\theta} \leq |x|^{-\theta},$$

and hence, using (2.3) we have that for every radial function f,

$$\begin{aligned}
\left\| \min\left(1, \frac{t}{|x|}\right)^{1/p} \tilde{S}f \right\|_{L^p(w)} &\leq t^{\theta/p} ||\, |x|^{-\theta/p} \tilde{S}f||_{L^p(w)} = t^{\theta/p} ||\tilde{S}f||_{L^p(w(x)|x|^{-\theta})} \\
&\lesssim \frac{t^{\theta/p}}{\theta^{\max(p-1,3)}} ||f||_{L^p(w(x)|x|^{-\theta})}.
\end{aligned}$$

Using Hölder's inequality we have that, for every $t > 0$ and every $0 < \delta < 1$,

$$\left\| \min\left(1, \frac{t}{|x|}\right)^{1/p} \tilde{S}f \right\|_{L^p(w)} \lesssim t^{\theta/p} \theta^{\max(p-1,3)} ||\, |x|^{-1/p} f||^{\theta}_{L^p(w)} ||f||^{1-\theta}_{L^p(w)}$$

and taking the infimum in θ in the right hand side, we get that

$$\begin{aligned}
\left\| \min\left(1, \frac{t}{|x|}\right)^{1/p} \tilde{S}f \right\|_{L^p(w)} &\lesssim ||f||_{L^p(w)} \left(\overline{\log}\ \frac{t^{1/p} ||\, |x|^{-1/p} f||_{L^p(w)}}{||f||_{L^p(w)}} \right)^{\max(3,p-1)} \\
&\lesssim (\overline{\log}\ t)^{\beta} ||f||_{L^p(w)} \left(\overline{\log} \frac{||\, |x|^{-1/p} f||_{L^p(w)}}{||f||_{L^p(w)}} \right)^{\max(3,p-1)}
\end{aligned}$$

Let us decompose

$$f = f_0 + \sum_{i=1}^{\infty} f_i$$

with $f_i = f\chi_{\{2^{-2^{i+1}}\le|x|<2^{-2^i}\}}$, $i\ge 1$. Then, by sublinearity, $\tilde S f \le \sum_{i=0}^\infty \tilde S f_i$ and since f_i is also radial, we have that

$$\begin{aligned}
&\Big\| \min\Big(1, \frac{t}{|x|}\Big)^{1/p}\tilde S f_i\Big\|_{L^p(w)}\\
\lesssim\ & (\overline{\log}\ t)^{\max(3,p-1)}||f_i||_{L^p(w)}\left(\overline{\log}\frac{||\,|x|^{-1/p}f_i||_{L^p(w)}}{||f_i||_{L^p(w)}}\right)^{\max(3,p-1)}\\
\lesssim\ & (\overline{\log}\ t)^{\max(3,p-1)}||f_i||_{L^p}2^{i\max(3,p-1)}.
\end{aligned}$$

Summing in i we obtain the result. $\square$

As an immediate consequence of the previous theorem, we have the following.

Corollary 2.5. *Under the condition of Theorem 2.4 we have that if f is a radial function satisfying that*

$$\int_{\mathbb{R}^n}|f(x)|^p\Big(\overline{\log}\frac{1}{|x|}\Big)^{p\max(p-1,3)}\Big(\overline{\log}\,\overline{\log}\,\frac{1}{|x|}\Big)^q w(x)dx<\infty,$$

with $q>p-1$, the almost everywhere convergence (1.1) holds.

Proof. The proof follows easily by observing that if $I_i=\{2^{-2^{i+1}}\le|x|<2^{-2^i}\}$ with $i\ge 1$, then $\overline{\log}\,\overline{\log}\,\frac{1}{|x|}\approx 1+i$ for every $x\in I_i$ and hence, since $q>p-1$,

$$\begin{aligned}
&\sum_{i=0}^\infty\left(\int_{I_i}|f(x)|^p\Big(\overline{\log}\frac{1}{|x|}\Big)^{p\max(p-1,3)}w(x)dx\right)^{1/p}\\
\approx\ & \sum_{i=0}^\infty\frac{1}{(1+i)^{q/p}}\left(\int_{I_i}|f(x)|^p\Big(\overline{\log}\frac{1}{|x|}\Big)^{p\max(p-1,3)}\Big(\overline{\log}\,\overline{\log}\,\frac{1}{|x|}\Big)^q w(x)dx\right)^{1/p}\\
\lesssim\ & \left(\sum_{i\ge1}\frac{1}{(1+i)^{q/(p-1)}}\right)^{1/p'}\\
\times\ & \left(\int_{\mathbb{R}^n}|f(x)|^p\Big(\overline{\log}\frac{1}{|x|}\Big)^{p\max(p-1,3)}\Big(\overline{\log}\ \overline{\log}\ \frac{1}{|x|}\Big)^q w(x)dx\right)^{1/p}<\infty.
\end{aligned}$$

$\square$

Finally, as a consequence of Theorems 2.2 and 2.4 we can conclude our last result.

Corollary 2.6. *Let $1<p<\infty$ and let w satisfy the hypothesis of Theorem 2.4. Then, for every $f\in L^p_{rad}(w)$ such that for some constants $A,B>0$,*

$$\sup_{|x|\le A}|f(x)|\le B,$$

condition (1.1) holds.

Proof. The proof reduces to decompose the function in the sum of two functions $f=f_0+f_1$ such that $f_0(x)=f(x)\chi_{B(0,A)}(x)$ and apply Theorem 2.2 to f_0 and Theorem 2.4 to f_1. $\square$

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