ON A REVERSE OF ANDO–HIAI INEQUALITY

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This paper is dedicated to Professor Lars-Erik Persson

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Abstract. In this paper, we show a complement of Ando–Hiai inequality: Let
A and B be positive invertible operators on a Hilbert space H and α ∈ [0, 1].
If $A \sharp_\alpha B \leq I$, then
$$A^r \sharp_\alpha B^r \leq \| (A \sharp_\alpha B)^{-1} \|^{1-r} I$$
for all $0 < r \leq 1$,
where I is the identity operator and the symbol $\| \cdot \|$ stands for the operator norm.

1. Introduction

A (bounded linear) operator $A$ on a Hilbert space $H$ is said to be positive (in
symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$
is positive and invertible. For some scalars $m$ and $M$, we write $mI \leq A \leq MI$
if $m(x, x) \leq (Ax, x) \leq M(x, x)$ for all $x \in H$. The symbol $\| \cdot \|$ stands for the
operator norm. Let $A$ and $B$ be two positive operators on a Hilbert space $H$. For
each $\alpha \in [0, 1]$, the weighted geometric mean $A \sharp_\alpha B$ of $A$ and $B$ in the sense of
Kubo-Ando [6] is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\alpha A^{\frac{1}{2}}$$

if $A$ is invertible. In fact, the geometric mean $A \sharp_\frac{1}{2} B$ is a unique positive solution
of $X A^{-1} X = B$.

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To study the Golden-Thompson inequality, Ando–Hiai in [1] developed the following inequality, which is called Ando–Hiai inequality: Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ and $\alpha \in [0, 1]$. Then

$$A \underline{\alpha} B \leq I \quad \Rightarrow \quad A^r \underline{\alpha} B^r \leq I \quad \text{for all } r \geq 1,$$

or equivalently

$$\|A^r \underline{\alpha} B^r\| \leq \|A \underline{\alpha} B\|^r \quad \text{for all } r \geq 1.$$

Löwner–Heinz inequality asserts that $A \geq B \geq 0$ implies $A^r \geq B^r$ for all $0 \leq r \leq 1$. As compared with Löwner–Heinz inequality, Ando–Hiai inequality is rephased as follows: For each $\alpha \in [0, 1]$

$$\left(A^r B^r A^\alpha\right)^\alpha \leq A^r \quad \Rightarrow \quad \left(A^{1/2} B A^{1/2}\right)^\alpha \leq A \quad \text{for all } 0 < r \leq 1. \quad (1.1)$$

Now, Ando–Hiai inequality does not hold for $0 < r \leq 1$ in general. In fact, put $r = 1/2$, $\alpha = 1/3$ and

$$A = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{25} \begin{pmatrix} 45 + 14\sqrt{5} & -5 - 7\sqrt{5} \\ -5 - 7\sqrt{5} & 50 - 14\sqrt{5} \end{pmatrix}.$$

Then we have

$$A \underline{1/3} B = \frac{1}{25} \begin{pmatrix} 15 + 2\sqrt{5} & -5 - \sqrt{5} \\ -5 - \sqrt{5} & 20 - 2\sqrt{5} \end{pmatrix} \leq I$$

since $\sigma(A \underline{1/3} B) = \{1, 0.4\}$. On the other hand, since

$$A^{1/3} \underline{1/3} B^{1/3} = \begin{pmatrix} 0.866032 & -0.187030 \\ -0.187030 & 0.770683 \end{pmatrix} \quad \text{and} \quad \sigma(A^{1/3} \underline{1/3} B^{1/3}) = \{1.01137, 0.625347\},$$

we have $A^{1/3} \underline{1/3} B^{1/3} \ngeq I$.

Thus, in [7], Nakamoto and Seo showed the following complement of Ando–Hiai inequality (AH):

**Theorem A.** Let $A$ and $B$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, $h = \frac{M}{m}$ and $\alpha \in [0, 1]$. Then

$$A \underline{\alpha} B \leq I \quad \Rightarrow \quad A^r \underline{\alpha} B^r \leq K(h^2, \alpha)^{-r} I \quad \text{for all } 0 < r \leq 1,$$

where the generalized Kantorovich constant $K(h, \alpha)$ is defined by

$$K(h, p) = \frac{h^p - h}{(p - 1)(h - 1)} \left(\frac{p - 1}{h^p - h}\right)^p \quad \text{for all } p \in \mathbb{R},$$

see [5, (2.79)].

We remark that $K(h^2, \alpha)^{-r} \neq 1$ in the case of $r = 1$, though $K(h^2, \alpha)^{-r} = 1$ in the case of $\alpha = 0, 1$ in Theorem A. Thereby, in this paper, we consider another complement of Ando–Hiai inequality (AH) which differ from Theorem A.
2. Main results

First of all, we state the main result:

**Theorem 2.1.** Let $A$ and $B$ be positive invertible operators and $\alpha \in [0, 1]$. Then

$$A \sharp_\alpha B \leq I \quad \implies \quad A^r \sharp_\alpha B^r \leq \|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} I \quad \text{for all } 0 < r \leq 1,$$

or equivalently

$$\|A^r \sharp_\alpha B^r\| \leq \|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} \|A \sharp_\alpha B\|^r \quad \text{for all } 0 < r \leq 1.$$

We remark that $\|A^{-1} \sharp_\alpha B^{-1}\|^{1-r} = 1$ in the case of $r = 1$.

We need the following lemmas to give a proof of Theorem 2.1. Lemma 2.2 is regarded as a reversal of Löwner–Heinz inequality:

**Lemma 2.2.** Let $A$ and $B$ be positive invertible operators. Then

$$A \geq B \quad \implies \quad \|A^p B^q A^r \| B^p \geq A^p \quad \text{for all } 0 < p \leq 1.$$

*Proof.* This lemma follows from Löwner–Heinz inequality. In fact, $A \geq B$ implies $A^p \geq B^p$ for all $0 < p \leq 1$ and then

$$I \geq A^{-\frac{p}{2}} B^p A^{-\frac{q}{2}} \geq \|A^\frac{p}{2} B^q A^{\frac{r}{2}}\|^{-1}.$$

□

**Lemma 2.3 ([3]).** Let $A$ be a positive invertible operator and $B$ an invertible operator. For each real numbers $r$

$$(BAB^*)^r = BA^\frac{1}{2} (A^\frac{1}{2} B^* BA^\frac{1}{2})^{r-1} A^\frac{1}{2} B^*.$$

*Proof of Theorem 2.1.* If we put $C = A^{-\frac{1}{2}} BA^{-\frac{1}{2}}$, then the assumption implies $A^{-1} \geq C^\alpha$. By Lemma 2.2 and $0 < 1 - r < 1$, we have

$$A^r = A^\frac{1}{2} A^{r-1} A^\frac{1}{2} \leq \|A^\frac{1}{2} C^{\alpha(r-1)} A^{\frac{1}{2}}\|^{r-1} A^\frac{1}{2} C^{\alpha(1-r)} A^\frac{1}{2}.$$

On the other hand, it follows that $A \leq C^{-\alpha}$ implies $C^{\alpha-1} \leq (C^\frac{1}{2} AC^\frac{1}{2})^{-1}$. By Lemma 2.2, we have

$$\|(C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1} C^{(\alpha-1)(r-1)} (C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1}\| C^{(\alpha-1)(1-r)} \geq (C^\frac{1}{2} AC^\frac{1}{2})^{r-1}.$$

Furthermore, by Lemma 2.3, we have

$$B^r = (A^\frac{1}{2} CA^\frac{1}{2})^r = A^\frac{1}{2} C^\frac{1}{2} (C^\frac{1}{2} AC^\frac{1}{2})^{r-1} C^\frac{1}{2} A^\frac{1}{2}$$

$$\leq \|(C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1} C^{(\alpha-1)(r-1)} (C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1}\| A^\frac{1}{2} C^{\alpha(1-r)} A^\frac{1}{2}.$$

Hence, by Araki-Cordes inequality [2, Theorem IX.2.10], we have

$$\|(C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1} C^{(\alpha-1)(r-1)} (C^\frac{1}{2} AC^\frac{1}{2})^{\alpha-1}\| \leq \|(C^\frac{1}{2} AC^\frac{1}{2})^{-\frac{1}{2}} C^{1-\alpha} (C^\frac{1}{2} AC^\frac{1}{2})^{-\frac{1}{2}}\|^{1-r}.$$
since $0 < 1 - r < 1$. Let $r(A)$ be the spectral radius of $A$. Then we have
\[
\| (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}} \| = r((C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}}) \\
= r((C^{-\frac{1}{2}} AC^{-\frac{1}{2}})^{-1} C^{1-\alpha}) \\
= r(A^{-1} C^{-\alpha}) \\
= r(A^{-\frac{1}{2}} C^{-\alpha} A^{-\frac{1}{2}}) \\
\leq \| A^{-\frac{1}{2}} C^{-\alpha} A^{-\frac{1}{2}} \|.
\]
Therefore, it follows that
\[
A^r \sharp_\alpha B^r \\
\leq \| A^{\frac{r-1}{r}} C^{(r-1)\alpha} A^{\frac{1}{r}} \|^{-1-\alpha} \| (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{\frac{r-1}{r}} C^{(1-\alpha)(r-1)} (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{r}} \|^{\alpha} \\
\times \left( A^{\frac{1}{r}} C^{(1-r)\alpha} A^{\frac{1}{r}} \sharp_\alpha A^{\frac{1}{r}} C^{(1-1)(1-r)+1} A^{\frac{1}{r}} \right) \\
\leq \| A^{\frac{1}{2}} C^{-\alpha} A^{\frac{1}{2}} \|^{1-(1-r)(1-\alpha)} \| (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}} AC^{\frac{1}{2}})^{-\frac{1}{2}} \|^{1-r\alpha} \\
\times A^{\frac{1}{r}} (C^{(1-r)\alpha} \sharp_\alpha C^{(1-1)(1-r)+1}) A^{\frac{1}{r}} \\
= \| A^{\frac{1}{2}} C^{-\alpha} A^{\frac{1}{2}} \|^{1-r} A^{\sharp_\alpha} B \leq \| (A^{\sharp_\alpha} B)^{-1} \|^{-1-r} I
\]
by $C^{(1-r)\alpha} \sharp_\alpha C^{(1-1)(1-r)+1} = C^\alpha$ and the assumption of $A^{\sharp_\alpha} B \leq I$. Hence the proof is complete. □

By Theorem 2.1, we immediately have the following corollary in the case of $r \geq 1$.

**Corollary 2.4.** Let $A$ and $B$ be positive invertible operators on $H$. Then
\[
\| A^{-r} \sharp_\alpha B^{-r} \|^{1-r} \| A^{\sharp_\alpha} B^r \| \leq \| A^{r} \sharp_\alpha B^r \| \text{ for all } r \geq 1.
\]

Finally, Furuta [4] showed the following Kantorovich type operator inequality in terms of the condition number: Let $A$ and $B$ be positive invertible operators. Then
\[
B \leq A \quad \Rightarrow \quad B^r \leq (\| B \| \| B^{-1} \|)^{r-1} A^r \quad \text{for all } r \geq 1. \tag{2.1}
\]

By Theorem 2.1, we have the following Kantorovich type inequality of (1.1) which corresponds to (2.1):

**Theorem 2.5.** Let $A$ and $B$ be positive invertible operators and $\alpha \in [0, 1]$. Then
\[
\left( A^{\frac{1}{r}} B^r A^{\frac{1}{r}} \right)^{\alpha} \leq A^r \quad \Rightarrow \quad \left( A^{\frac{1}{r}} B A^{\frac{1}{r}} \right)^{\alpha} \leq \| A^r \sharp_\alpha B^{-r} \|^{1-\frac{1}{r}} A
\]
for all $r \geq 1$.

**References**


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